Optimal Risk Sharing

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Introduction

In this talk we discuss

- the optimal approach to share risks (in general) among different stake holders: individuals, insurers and reinsurers.

We assume that

- there are n individuals who are exposed to some risk
- there is an insurer who write insurance policies to the individuals
- there is a reinsurer who whites reinsurance policies to the insurer

Introduction

Individual Decision Making:

- X_i: losses to individual i.
- $f(X_i)$: loss coverage to individual i by the insurer .
- P_{f_i}: insurance premium
- $X_i - f(X_i)$: Total retained loss for individual i.

Question: Given a premium principle, what is the optimal form of $f(X_i)$?

Introduction

Insurer Decision Making:

- $Y = \sum_{i=1}^{n} f(X_i)$: Total losses for the insurer (diversification happens here).
- P_I: Reinsurance premium determined by some premium principle.
- $I(Y)$: insurer's ceded losses
- $C_I = Y - I(Y) + P_I$: Insurer's total loss.
- $R_I = I(Y) - P_I$: Reinsurer's total loss.

Question: Given a reinsurance premium principle, what is the optimal form of $I(Y)$?

Introduction

We consider the following set of feasible functions for f and I:

$$\mathcal{C} := \{ f : f(x) \text{ and } x - f(x) \text{ are non-decreasing, and } 0 \leq f(x) \leq x \text{ for all x} \}.$$

- The Principle of Indemnity
- Moral Hazard

Optimal Insurance Policies

Maximizing utility (Arrow (1963)) Arrow

- Individuals are risk averse with the utility function $U(\cdot)$
- $\max_{f \in \mathcal{C}} \mathbb{E}[U(w - X + f(X) - P)]$
- if $P = \mathbb{E}[f(X)]$, then $f^*(x) = x$ for all x.
- if $P = (1+\theta)\mathbb{E}[f(X)]$, where $\theta > 0$ is a risk loading, then $f^*(x) = (x-d)_+$.
- this is something like buying a financial option when you are exposed to stock price risk–hedging by changing the distribution of losses.

Optimal Insurance Policies

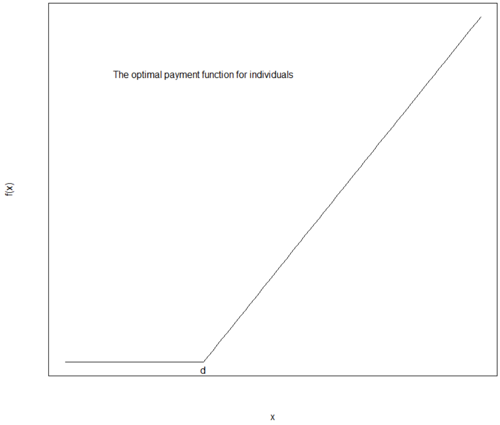

Figure 1: The optimal payout function.

Optimal Reinsurance Policies-EU

Maximizing utility (Raviv (1979))

- $\max_{I \in \mathcal{C}} \mathbb{E}[u(w - Y + I(Y) - P)]$ subject to $\mathbb{E}[v(w - I(Y) + P)] \geq k$
- For fixed P, $I^*(y)$ takes one of the two forms

$$\begin{aligned} I^*(y) &= 0 \quad if \quad y < d \\ 0 < I^*(y) &< y \quad if \quad y > d \end{aligned}$$

$$\begin{aligned} I^*(y) &= y \quad if \quad y < d \\ 0 < I^*(y) &< y \quad if \quad y > d \end{aligned}$$

- In both cases, when $0 < I^*(y) < y$, $I^*(y)$ satisfies a differential equation.

Optimal Reinsurance Policies-EU

Maximizing utility (Raviv (1979))

- When $k \to -\infty$, the same as before
- Whether the insurer pay small or large losses depends on the value of k,
- For quadratic utility functions, $I^*(y)$ is linear above d.

Optimal Reinsurance Policies-EU

Golubin (2006) and Jiang et al. (2019) consider the set of Pareto optimal reinsurance policies, in the sense that one party's EU cannot be increased without decreasing that of the other parties.

Optimal Reinsurance Policies-EU

Jiang et al. (2019) studied the problem:

Problem 1

For $k \geq 0$,

$$\max_{I \in \mathcal{C}, P \in [0, \overline{P}]} J(I, P) = \mathbb{E}_1[u(w_1 - Y + I(Y) - P)] + k\mathbb{E}_2[v(w_2 - I(Y) + P)],$$

Subject to

$$\begin{cases} \mathbb{E}_1[u(w_1 - Y + I(Y) - P)] \geq \mathbb{E}_1\left[u(w_1 - Y)\right] \\ \mathbb{E}_2[v(w_2 - I(Y) + P)] \geq v(w_2) \end{cases}.$$

Optimal Reinsurance Policies-EU

- The solution is similar to those obtained in Raviv (1979).
- The policy form depends on the value of k.
- By changing the value of k, one gets the set of Pareto optimal policies (efficient frontier).

Optimal Reinsurance Policies-Nash Equilibrium

To obtain the best policy, one can apply the classical game-theoretical equilibrium introduced in Nash (1950), where the reinsurance policy (I_N, P_N) solves

$$\max_{\substack{I \in \mathcal{C}, \\ P \in [0, \min\{w_1, \overline{P}\}]}} \{\mathbb{E}_1 [u(w_1 - Y + I(Y) - P)] - \mathbb{E}_1 [u(w_1 - Y)]\} \times \{\mathbb{E}_2 [v(w_2 - I(Y) + P)] - v(w_2)\}.$$

Optimal reinsurance contracts-Nash Equilibrium

The Nash solution lies on the efficient frontier. It can be identified numerically.

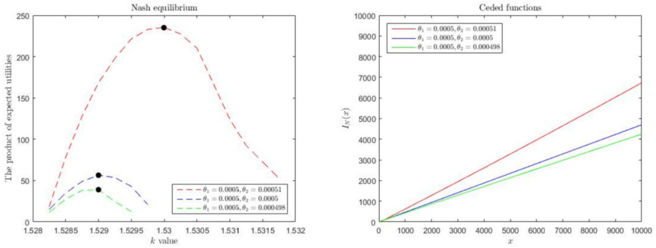

Figure 2: The Nash Solution.

Optimal reinsurance contracts-Risk minimizing

Minimizing Risks (Borch (1960))

- $\min_{I \in \mathcal{C}} Var[(w - Y + I(Y) - P)]$
- $I^*(y) = (y - d)_+$
- This is in fact the same as maximizing a quadratic utility.

Optimal reinsurance contracts-Risk minimizing

Minimizing Risks (Cai et al. (2008))

- value-at-risk (VaR)
  $VaR_\alpha(Y) = F_Y^{-1}(\alpha) = \inf\{y : F_Y(y) \geq \alpha\}$.
- $\min_{I \in \mathcal{C}} VaR_\alpha[(w - Y + I(Y) - P)]$
- $P = (1 + \theta)\mathbb{E}[I(Y)]$
- Roughly, $I^*(y) = (y - S_Y^{-1}(\theta^*))_+ \wedge l$, where $d^* = \frac{1}{1+\theta}$.

Optimal reinsurance contracts-VaR minimizing for insurer

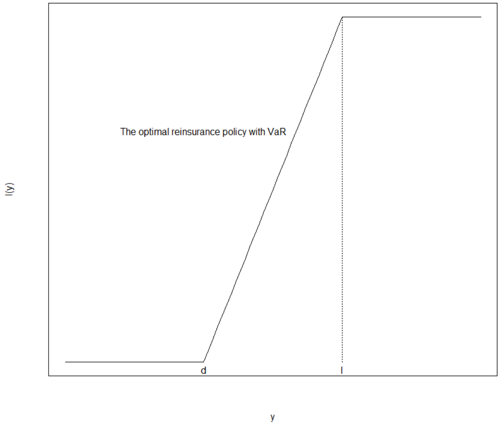

Figure 3: Optimal reinsurance contracts-VaR minimizing for insurer.

Optimal reinsurance contracts-Risk minimizing

Jiang et al. (2017) considered the problem

$$\min_{I\in\mathcal{C}} \quad \beta VaR_{\alpha_c}(Y - I(Y) + P) + (1-\beta)VaR_{\alpha_r}(I(Y) - P), \qquad \beta \in [0,$$

$$\text{s.t} \quad VaR_{\alpha_c}(Y - I(Y) + P) \leq L_1$$

$$VaR_{\alpha_r}(I(Y) + P) \leq L_2$$

Optimal reinsurance contracts-Risk minimizing

The form of optimal reinsurance policies depends on the value of β.

- With $\beta > \frac{1}{2}$, $I^*(y) = (y - S_Y^{-1}(\theta^*))_+ \wedge l$
- With $\beta < \frac{1}{2}$, $I^*(y) = y \wedge d + (y - l)_+$

Optimal reinsurance contracts-Risk minimizing for both

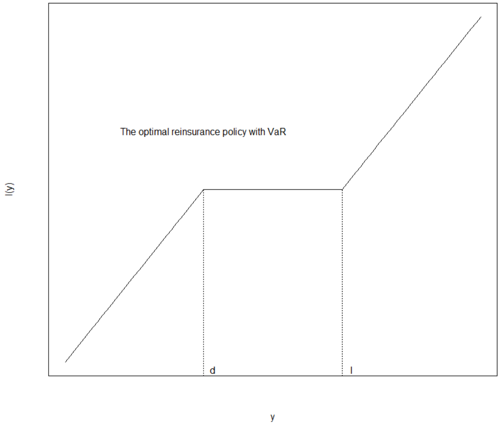

Figure 4: Optimal reinsurance contracts-VaR minimizing for both.

Future Research

- Individual: combination of loss mitigation and insurance
- Insurer: Maximizing EU under risk constraints.
- Financial market
- Government

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