The Spectral Point of View in Geometry

Masoud Khalkhali
Western University, London, Canada
Colloqium, Windsor, Feb. 2013

A brief history of spectrum

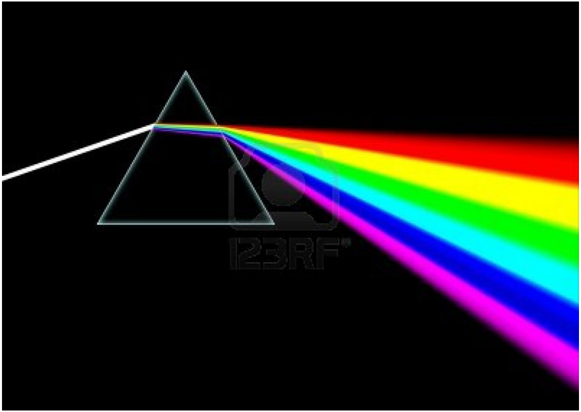

Figure: Newton first used the word spectrum in print in 1671 in describing his experiments in optics

Frauenhoffer lines, 1812

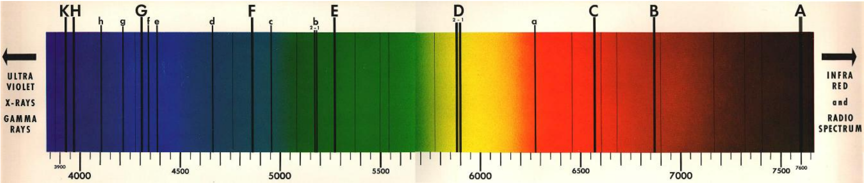

Figure: Sun's absorption spectrum; notice the black lines

Hydrogen lines

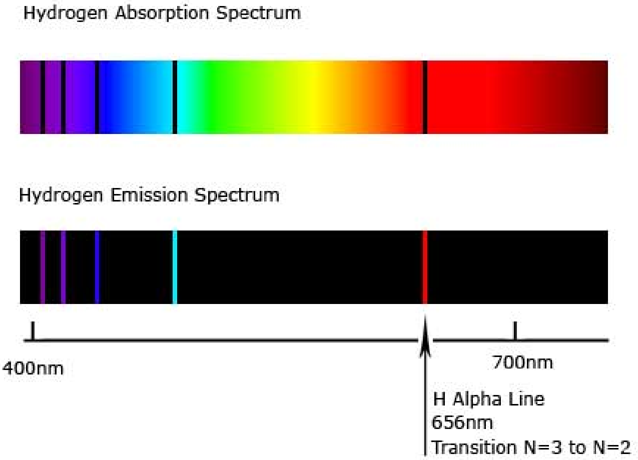

Figure: Hydrogen spectral lines in the visible range

Black body radiation

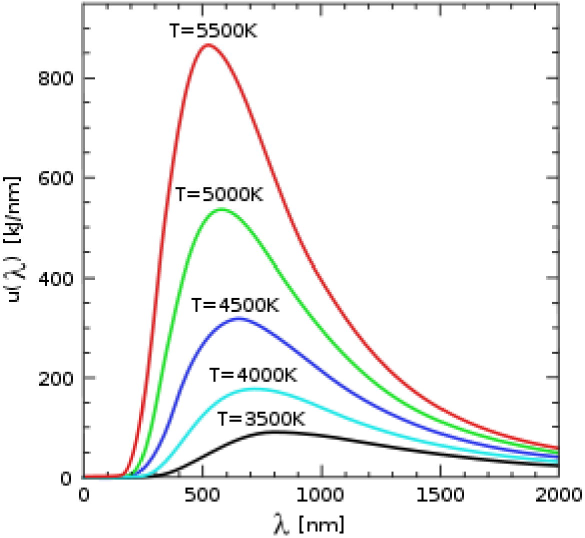

Figure: Black body spectrum

Balmer's formula; Planck's radiation law

- Balmer's formula for hydrogen lines (1885):

$$\frac{1}{\lambda} = R(\frac{1}{m^2} - \frac{1}{n^2}), \quad m = 2,\ n = 3, 4, 5, 6.$$

  Ritz-Rydberg combination principle; spectral terms and spectral lines.

- Physicists understood that the *spectral energy density function* $\rho(\nu, T)$ will be independent of the shape of the cavity and should only depend on its *volume*.

- Planck's formula (1900):

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

Weyl's law: asymptotic distribution of eigenvalues

- In 1910 H. A. Lorentz gave a series of lectures in Göttingen under the title "old and new problems of physics". Weyl and Hilbert were in attendance. In particular he mentioned attempts to drive Planck's radiation formula in a mathematically satisfactory way and remarked:

'It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume. ....There is no doubt that it holds in general even for multiply connected spaces'.

- Hilbert was not very optimistic to see a solution in his lifetime. His bright student Hermann Weyl solved this conjecture of Lorentz and Sommerfeld within a year and announced a proof in 1911! All he needed was Hilbert's theory of integral equations and compact operators developed by Hilbert and his students in 1900-1910.

Figure: Hermann Weyl in Göttingen

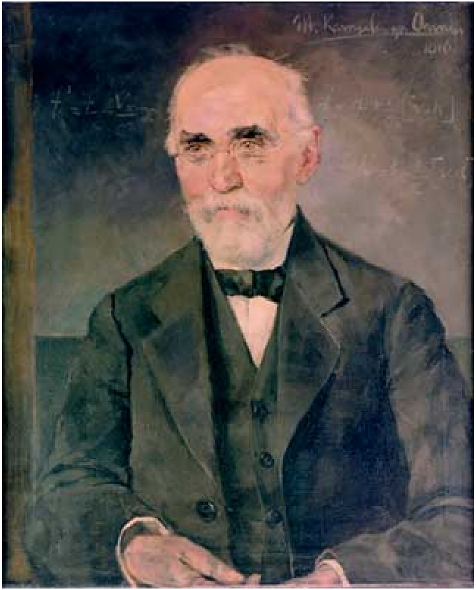

Figure: H. A. Lorentz

Dirichlet eigenvalues and Weyl law

- Let $\Omega \subset \mathbb{R}^2$ be a compact connected domain with a piecewise smooth boundary.

$$\begin{cases} \Delta u = \lambda u \\ u|\partial\Omega = 0 \end{cases}$$

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

$$\langle u_i, u_j \rangle = \delta_{ij} \quad \text{o.n. basis for } L^2(\Omega)$$

- Weyl Law for planar domains $\Omega \subset \mathbb{R}^2$

$$N(\lambda) \sim \frac{\text{Area}(\Omega)}{4\pi}\lambda \qquad \lambda \to \infty$$

where $N(\lambda)$ is the eigenvalue counting function.

- In general, for $\Omega \subset \mathbb{R}^n$

$$N(\lambda) \sim \frac{\omega_n \text{Vol}(\Omega)}{(2\pi)^n}\lambda^{\frac{n}{2}} \qquad \lambda \to \infty$$

One can hear the scalar curvature

- (M, g) = closed Riemannian manifold. Laplacian on functions

$$\triangle = d^* d : C^\infty(M) \to C^\infty(M)$$

is an unbounded positive operator with pure point spectrum

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- The spectrum contains a lot of geometric and topological information on M. In particular the dimension, volume, total scalar curvature, of M are fully determined by the spectrum of Δ. To see this we need:

The heat engine

- Let $k(t,x,y) =$ kernel of $e^{-t\triangle}$. Restrict to the diagonal: as $t \to 0$, we have (Minakshisundaram-Plejel; Seeley, MacKean-Singer, Gilkey,...)

$$k(t,x,x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x,\triangle) + a_1(x,\triangle)t + a_2(x,\Delta)t^2 + \cdots)$$

- $a_i(x,\triangle)$: the Seeley-De Witt-Gilkey coefficients.

- Functions $a_i(x, \triangle)$: expressed by universal polynomials in curvature tensor R and its covariant derivatives:

$$\begin{aligned} a_0(x,\triangle) &= 1 \\ a_1(x,\triangle) &= \frac{1}{6}S(x) \qquad \text{scalar curvature} \\ a_2(x,\triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\ a_3(x,\triangle) &= \cdots\cdots \end{aligned}$$

Short time asymptotics of the heat trace

$$\begin{aligned}\mathrm{Trace}(e^{-t\triangle}) &= \sum e^{-t\lambda_i} = \int_M k(t,x,x)d\mathrm{vol}_x \\ &\sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)\end{aligned}$$

So

$$a_j = \int_M a_j(x,\triangle)d\mathrm{vol}_x,$$

are manifestly spectral invariants.

$$\begin{aligned}a_0 &= \int_M d\mathrm{vol}_x = \mathrm{Vol}(M), & \Longrightarrow \text{Weyl's law} \\ a_1 &= \frac{1}{6}\int_M S(x)d\mathrm{vol}_x, & \text{total scalar curvature}\end{aligned}$$

Abelian-Tauberian Theorem

Assume $\sum_1^\infty e^{-\lambda_n t}$ is convergent for all $t > 0$. TFAE:

$$\lim_{t \to 0^+} t^r \sum_1^\infty e^{-\lambda_n t} = a,$$

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^r} = \frac{a}{\Gamma(r+1)}$$

Spectral Triples: $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A}$= involutive unital algebra, $\mathcal{H}$ = Hilbert space,

  $$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \qquad D : \mathcal{H} \to \mathcal{H}$$

  D has compact resolvent and all commutators $[D, \pi(a)]$ are bounded.

- An asymptotic expansion holds

  $$\text{Trace}\,(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

The metric dimension and dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$

- Metric dimension $= n$ (need not be an integer) if

$$|D|^{-n} \in \mathcal{L}^{1,\infty}(\mathcal{H})$$

- Let $\triangle = D^2$. Spectral zeta function

$$\zeta_D(s) = \mathrm{Tr}\,(|D|^{-s}) = \mathrm{Tr}\,(\Delta^{-s/2}), \quad \mathrm{Re}(s) >> 0.$$

The set of poles of $\zeta_D(s)$ is the dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$. For classical spectral triples defined on M, the top pole of $\zeta_D(s)$ is integer and is exactly equal to dim (M); also the dimension spectrum is inside $\mathbb{R}$. In general both of these restrictions can fail.

The Dixmier trace

Let

$$\mathcal{L}^{1,\infty}(\mathcal{H}) := \{T \in \mathcal{K}(\mathcal{H});\ \sum_{1}^{N} \mu_n(T) = O(\log N)\}.$$

The Dixmier trace of an operator $T \in \mathcal{L}^{1,\infty}(\mathcal{H})$ measures the *logarithmic divergence* of its ordinary trace. Let

$$\sigma_N(T) = \frac{\sum_1^N \mu_n(T)}{\log N}$$

$\lim \sigma_N(T)$ may not exist and must be replaced by a carefully chosen 'regularized limit' ω :

$$\mathrm{Tr}_\omega(T) := \lim_\omega \sigma_N(T)$$

For operators T for which $\mathrm{Lim}_{N\to\infty}\,\sigma_N(T)$ exit, the Dixmier trace is independent of the choice of ω and is equal to $\mathrm{Lim}_{N\to\infty}\,\sigma_N(T)$.

Weyl's law implies that for any elliptic s.a. differential operator of order 1 on M, $|D|^{-}n \in \mathcal{L}^{1,\infty}(\mathcal{H})$ and

$$\mathrm{Tr}_\omega(|D|^{-n}) = c_n \mathrm{Vol}(M)$$

Commutative examples of spectral triples

Natural first order elliptic PDE's on M define spectral triples on $\mathcal{A} = C^\infty(M)$:

1. $D = d + d^*$, $\mathcal{H} = L^2(\wedge T^* M)$, $\mathcal{A} = C^\infty(M)$ acting on $\mathcal{H}$ by left multiplication. Index (D) is the Euler characteristic of M. Signature of M is the index of a closely related spectral triple.

2. $D =$ Dirac operator on a compact Riemannian Spin^c manifold, $\mathcal{H} = L^2(M, S)$, L^2-spinors on M. $A = C^\infty(M)$ acts on $L^2(M, S)$ by multiplication. One checks that for any smooth function f, the commutator $[D, f]$ extends to a bounded operator on $L^2(M, S)$.

Geodesic distance from spectral triples

Now the geodesic distance d on M can be recovered from the following beautiful *distance formula* of Connes:

$$d(p,q) = \text{Sup}\{|f(p) - f(q)|; \| [D,f] \| \leq 1\}, \qquad \forall p,q \in M.$$

Compare with Riemannian formula:

$$d(p,q) = \text{Inf}\{\int_0^1 \sqrt{g_{\mu\nu} dx^\mu dx^\nu}; c(0) = p,\ c(1) = q\}, \qquad \forall p,q \in M$$

Example: a spectral triple on the Cantor set

Let $\mathcal{A} = C(\Lambda)$ be the commutative algebra of continuous functions on a Cantor set $\Lambda \subset \mathbb{R}$. Let J_k be the collection of bounded open intervals in $\mathbb{R} \setminus \Lambda$ with lengths $L = \{\ell_k\}_{k \geq 1}$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k \cdots > 0. \tag{1}$$

Let $E = \{x_{k,\pm}\}$ be the set of the endpoints of J_k. Consider the Hilbert space

$$\mathcal{H} := \ell^2(E) \tag{2}$$

There is an action of $\mathrm{C}(\Lambda)$ on $\mathcal{H}$ given by

$$f \cdot \xi(x) = f(x)\xi(x), \quad \forall f \in \mathrm{C}(\Lambda),\ \forall \xi \in \mathcal{H},\ \forall x \in E.$$

A sign operator F is defined by choosing the closed subspace $\hat{\mathcal{H}} \subset \mathcal{H}$ given by

$$\hat{\mathcal{H}} = \{\xi \in \mathcal{H} : \xi(x_{k,-}) = \xi(x_{k,+}),\ \forall k\}.$$

Then F has eigenspaces $\hat{\mathcal{H}}$ with eigenvalue $+1$ and $\hat{\mathcal{H}}^{\perp}$ with eigenvalue -1, so that

$$F|_{\mathcal{H}_k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Dirac operator $D = |D|F$ is defined as:

$$D|_{\mathcal{H}_k} \left(\begin{array}{c} \xi(x_{k,+}) \\ \xi(x_{k,-}) \end{array} \right) = \ell_k^{-1} \cdot \left(\begin{array}{c} \xi(x_{k,-}) \\ \xi(x_{k,+}) \end{array} \right).$$

The data $(\mathcal{A}, \mathcal{H}, D)$ form a spectral triple. The zeta function satisfies

$$\mathrm{Tr}(|D|^{-s}) = 2\zeta_L(s),$$

where $\zeta_L(s)$ is the geometric zeta function of $L = \{\ell_k\}_{k\geq 1}$, defined as

$$\zeta_L(s) := \sum_k \ell_k^s.$$

For the classical middle-third Cantor set, we have $\ell_k = 3^{-k}$ with multiplicities $m_k = 2^{k-1}$, so that

$$\mathrm{Tr}(|D|^{-s}) = 2\zeta_L(s) = \sum_{k\geq 1} 2^k 3^{-sk} = \frac{2\cdot 3^{-s}}{1 - 2\cdot 3^{-s}}.$$

Thus the dimension spectrum of the spectral triple of a Cantor set has complex points! In fact, the set of poles of (26) is

$$\left\{ \frac{\log 2}{\log 3} + \frac{2\pi i n}{\log 3} \right\}_{n\in\mathbb{Z}}.$$

Thus the dimension spectrum lies on a vertical line and it intersects the real axis in the point $D = \frac{\log 2}{\log 3}$ which is the Hausdoff dimension of the ternary Cantor set.

Noncommutative Torus

- Fix $\theta \in \mathbb{R}$. $A_\theta =$ C^*-algebra generated by unitaries U and V satisfying
$$VU = e^{2\pi i\theta} UV.$$
- Dense subalgebra of *'smooth functions'*:
$$A_\theta^\infty \subset A_\theta,$$
$a \in A_\theta^\infty$ iff
$$a = \sum a_{mn} U^m V^n$$
where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is rapidly decreasing:
$$\sup_{m,n} (1 + m^2 + n^2)^k \, |a_{mn}| < \infty$$
for all $k \in \mathbb{N}$.

- Differential operators on A_θ

$$\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty,$$

Infinitesimal generators of the action

$$\alpha_s(U^m V^n) = e^{is.(m,n)} U^m V^n \quad s \in \mathbb{R}^2.$$

Analogues of $\frac{1}{i}\frac{\partial}{\partial x}$, $\frac{1}{i}\frac{\partial}{\partial y}$ on 2-torus.

- Canonical trace $\mathfrak{t} : A_\theta \to \mathbb{C}$ on smooth elements:

$$\mathfrak{t}\Big(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\Big) = a_{0,0}.$$

Complex structures on A_θ

- Let $\mathcal{H}_0 = L^2(A_\theta)$= GNS completion of A_θ w.r.t. $\mathfrak{t}$.
- Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 = \Im(\tau) > 0$, and define

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

- Hilbert space of $(1,0)$-forms:

  $\mathcal{H}^{(1,0)}$:= completion of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, w.r.t.

$$\langle a\partial b, a'\partial b'\rangle := \mathfrak{t}((a'\partial b')^* a\partial b).$$

- Flat Dolbeault Laplacian: $\partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$.

Conformal perturbation of metric

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form $\mathfrak{t}$ by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \mathfrak{t}(ae^{-h}), \quad a \in A_\theta.$$

- We have

$$\varphi(ab) = \varphi(b\Delta(a)), \quad \forall a, b \in A_\theta.$$

  where

$$\Delta(x) = e^{-h}xe^{h}.$$

- Warning: $\triangle$ and Δ are very different operators!

Connes-Tretkoff spectral triple

- Hilbert space $\mathcal{H}_\varphi :=$ *GNS* completion of A_θ w.r.t. $\langle , \rangle_\varphi$,

$$\langle a, b\rangle_\varphi := \varphi(b^* a), \quad a, b \in A_\theta$$

- View $\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$. and let

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi$$

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Full perturbed Laplacian:

$$\triangle := D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Lemma: $\partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$, and $\partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}$ are anti-unitarily equivalent to

$$k\partial^* \partial k : \mathcal{H}_0 \to \mathcal{H}_0,$$

$$\partial^* k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)},$$

where $k = e^{h/2}$.

Scalar curvature for A_θ

- The scalar curvature of the curved nc torus $(\mathbb{T}^2_\theta, \tau, k)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\mathrm{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \mathrm{Trace}\,(aP) = \mathfrak{t}\,(aR), \qquad \forall a \in A_\theta^\infty,$$

  where P is the projection onto the kernel of $\triangle$.

Local expression for the scalar curvature

- Cauchy integral formula:

$$e^{-t\triangle} = \frac{1}{2\pi i}\int_C e^{-t\lambda}(\triangle-\lambda)^{-1}\,d\lambda.$$

- $B_\lambda \approx (\triangle-\lambda)^{-1}$:

$$\sigma(B_\lambda) \sim b_0(\xi,\lambda) + b_1(\xi,\lambda) + b_2(\xi,\lambda) + \cdots ,$$

each $b_j(\xi,\lambda)$ is a symbol of order $-2-j$, and

$$\sigma(B_\lambda(\triangle-\lambda)) \sim 1.$$

(Note: λ is considered of order 2.)

Proposition: The scalar curvature of the spectral triple attached to (A_θ, τ, k) is equal to

$$\frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_C e^{-\lambda} b_2(\xi, \lambda)\, d\lambda\, d\xi,$$

where b_2 is defined as above.

Final formula for the scalar curvature (Connes-Moscovici, Fathizadeh-K, Oct. 2011)

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log\Delta)\big(\triangle_0(\log k)\big)+$$

$$R_2(\log\Delta_{(1)},\log\Delta_{(2)})\Big(\delta_1(\log k)^2+|\tau|^2\delta_2(\log k)^2+\tau_1\big\{\delta_1(\log k),\delta_2(\log k)\big\}\Big)+$$

$$iW(\log\Delta_{(1)},\log\Delta_{(2)})\Big(\tau_2\big[\delta_1(\log k),\delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1+\cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s+\sinh t-\sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

The limiting case

In the commutative case, $\theta = 0$, the above modular curvature reduces to a constant multiple of the formula of Gauss:

$$\frac{1}{\tau_2}\delta_1^2(\log k) + \frac{|\tau|^2}{\tau_2}\delta_2^2(\log k) + 2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$

First application: the Gauss-Bonnet theorem for A_θ

- How to relate geometry (short term asymptotics) to topology (long term asymptotics)? MacKean-Singer formula:

$$\sum_{p=0}^{m}(-1)^p \operatorname{Tr}\left(e^{-t\Delta_p}\right) = \sum_{p=0}^{m}(-1)^p \beta_p = \chi(M) \qquad \forall t > 0$$

- Spectral formulation of the Gauss-Bonnet theorem:

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R \operatorname{vol}(g) = \frac{1}{6}\chi(\Sigma)$$

Theorem (Connes-Tretkoff; Fathizadeh-K.): Let $\theta \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, $k \in A_\theta^\infty$ be a positive invertible element. Then

$$\mathrm{Trace}(\triangle^{-s})_{|_{s=0}} + 2 = \mathfrak{t}(R) = 0,$$

where $\triangle$ is the Laplacian and R is the scalar curvature of the spectral triple attached to (A_θ, τ, k).

Weyl law for noncommutative two tori (Fathizadeh and M.K.)

$$N(\lambda) \sim \frac{\pi}{\Im(\tau)}\varphi(1)\lambda \qquad \text{as} \qquad \lambda \to \infty.$$

Equivalently:

$$\lambda_j \sim \frac{\Im(\tau)}{\pi\varphi(1)} j \qquad \text{as} \qquad j \to \infty.$$

- This suggests:

$$\text{Vol}(\mathbb{T}_\theta^2) := \frac{4\pi^2}{\Im(\tau)}\varphi(1) = \frac{4\pi^2}{\Im(\tau)}\varphi_0(k^{-2}).$$

Summary

- According to quantum mechanics any observation will end up with finding a point in the spectrum of a selfadjoint operator. Moreover, as a consequence of Heisenberg's uncertainty principle, the algebra of observables is definitely a noncommutative algebra.

- One can do geometry and topology on certain classes of noncommutative algebras that come equipped with *spectral triples*. Metric aspects of *noncommutative geometry* are informed by *spectral geometry*. Spectral invariants are the only means by which we can formulate metric ideas of noncommutative geometry.