Holomorphic Structures on the Quantum Projective Line

Masoud Khalkhali

University of Western Ontario

January 2010

Joint work with Gianni Landi and Walter Van Suijlekom

available at: arXiv:0907.0154

- Pre-complex structres on $*$-algebras
- Holomorphic structures on NC vector bundles
- A holomorphic structure on the quantum projective line $\mathbb{C}P_q^1$
- Canonical line bundles on $\mathbb{C}P_q^1$ and their holomorphic structure
- The quantum homogeneous ring of $\mathbb{C}P_q^1$
- Positive Hoschschild cocycles and uniqueness

Noncommutative pre-complex structures

Initial data: $\mathcal{A}$ an $*$-algebra over $\mathbb{C}$, $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$ an *involutive differential calculus* over $\mathcal{A}$.

$$\Omega^0(\mathcal{A}) = \mathcal{A}, \qquad d(a^*) = (da)^*$$

Definition: A *pre-complex structure* on $\mathcal{A}$ for the differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$ is a bigraded differential $*$- algebra $\Omega^{(\bullet,\bullet)}(\mathcal{A})$ with differentials (derivations)

$$\partial : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{p+1,q}(\mathcal{A}),$$

$$\bar{\partial} : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p,q+1)}(\mathcal{A})$$

s.t.

$$\Omega^n(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{(p,q)}(\mathcal{A})$$

$$\partial(a)^* = \bar{\partial}(a^*), \quad \mathsf{d} = \partial + \bar{\partial}$$

Also, $*$ maps $\Omega^{(p,q)}(\mathcal{A})$ to $\Omega^{(q,p)}(\mathcal{A})$.

Motivating example: the de Rham complex of a complex manifold.

NC examples: Let L be a *real* Lie algebra with a *complex structure*:

$$L^{\mathbb{C}} = L_0 \oplus \overline{L_0}$$

Given $L \to \mathrm{Der}(\mathcal{A}, \mathcal{A})$, an action of L by $*$-derivations on $\mathcal{A}$, then

$$\Omega^{\bullet}\mathcal{A} = \mathrm{Hom}_{\mathbb{C}}(\Lambda^{\bullet} L^{\mathbb{C}}, \mathcal{A})$$

is a differential calculus for $\mathcal{A}$, and

$$\Omega^{(p,q)}\mathcal{A} = \mathrm{Hom}_{\mathbb{C}}(\Lambda^p L_0 \otimes \Lambda^q \overline{L_0}, \mathcal{A})$$

defines a pre-complex structure.

NC torus $\mathcal{A}_\theta$: Generators U_1, U_2 with

$$U_1 U_2 = e^{2\pi \mathrm{i}\theta} U_2 U_1$$

Basic derivations:

$$\delta_j(U_k) = 2\pi \mathrm{i}\, \delta_{jk} U_k, \quad j,k = 1,2$$

define an action of $\mathbb{R}^2$ on $\mathcal{A}_\theta$. Any $\tau \in \mathbb{C} \setminus \mathbb{R}$ defines a complex structure on $\mathcal{A}_\theta$:

$$\mathbb{R}^2 \otimes \mathbb{C} = L_0 \oplus \overline{L_0}$$

with $L_0 := e_1 + \tau e_2$.

The complex structure on $\mathcal{A}_\theta$ is then given by

$$\partial_{(\tau)} = \frac{1}{(\tau - \overline{\tau})}\,(-\overline{\tau}\delta_1 + \delta_2),$$

$$\overline{\partial}_{(\tau)} = \frac{1}{(\tau - \overline{\tau})}\,(\tau\delta_1 - \delta_2).$$

The fundamental cyclic 2-cocycle on $\mathcal{A}_\theta$:

$$\Psi(a_0, a_1, a_2) = \frac{i}{2\pi} \operatorname{tr}_\theta \left(a_0(\delta_1 a_1 \delta_2 a_2 - \delta_2 a_1 \delta_1 a_2) \right)$$

The positive Hochschild 2-cocycle Φ associated with Ψ:

$$\Phi(a_0, a_1, a_2) = \frac{2}{\pi} \operatorname{tr}_\theta \left(a_0 \partial_{(\tau)} a_1 \overline{\partial}_{(\tau)} a_2 \right).$$

This cocycle gives the conformal class of a general constant metric on the torus.

Holomorphic functions:

$$\mathcal{O}(\mathcal{A}) := \ker \left\{ \bar{\partial} : \mathcal{A} \to \Omega^{(0,1)}(\mathcal{A}) \right\}.$$

Holomorphic structures on modules

Definition: Let $(A, \bar{\partial})$ be an algebra with a pre-complex structure and $\mathcal{E}$ a left $\mathcal{A}$-module. A holomorphic structure on $\mathcal{E}$ with respect to $(\mathcal{A}, \bar{\partial})$ is a flat $\bar{\partial}$-connection, i.e. a connection

$$\overline{\nabla} : \mathcal{E} \to \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$

s.t.

$$F(\overline{\nabla}) = \overline{\nabla}^2 = 0$$

If in addition $\mathcal{E}$ is a finitely generated projective $\mathcal{A}$-module, we call the pair $(\mathcal{E}, \overline{\nabla})$ a holomorphic vector bundle.

Since $\overline{\nabla}$ is a flat connection, we have a complex of vector spaces:

$$\mathcal{E} \xrightarrow{\overline{\nabla}} \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\overline{\nabla}} \Omega^{(0,2)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \cdots$$

Definition: The zeroth cohomology group of the above complex is the space of holomorphic sections of $\mathcal{E}$ and denoted by $H^0(\mathcal{E}, \overline{\nabla})$. It is a left $\mathcal{O}(\mathcal{A})$-module.

Holomorphic structures on bimodules

Let $\mathcal{E}$ be an $\mathcal{A}$-bimodule.

Definition: A bimodule connection on $\mathcal{E}$ is a *left* connection $\nabla : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ for which there is a bimodule isomorphism

$$\sigma(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E},$$

such that

$$\nabla(\xi a) = \nabla(\xi)a + \sigma(\nabla)\,(\xi \otimes \mathrm{d}a)$$

for all $\xi \in \mathcal{E}, a \in \mathcal{A}$.

In particular, this definition applies to the differential calculus $(\Omega^{(0,\bullet)}(\mathcal{A}), \bar{\partial})$ thus giving a notion of *holomorphic structures on bimodules*.

Tensor products of holomorphic vector bundles

Suppose we are given two $\mathcal{A}$-bimodules $\mathcal{E}_1, \mathcal{E}_2$ with two bimodule connections ∇_1, ∇_2, respectively. Let

$$\sigma := (\sigma_1 \otimes 1) \circ (1 \otimes \sigma_2) : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to$$

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$$

Lemma: The map

$$\nabla : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \mapsto \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$$

defined by

$$\nabla = \nabla_1 \otimes 1 + (\sigma_1 \otimes \mathrm{id})(1 \otimes \nabla_2)$$

defines a σ-compatible connection on the $\mathcal{A}$-bimodule $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$.

Note: We would like the flatness condition on holomorphic structures to survive under taking this tensor product. This is not the case in higher dimensions in the NC world! A possible way out might be the use of a more exotic tensor product.

The quantum Hopf fibration

$$S^1 \hookrightarrow S_q^3 \hookrightarrow S_q^2$$

A quantum homogeneous space:

$$\mathcal{A}(S_q^2) \to \mathcal{A}(S_q^3) \to \mathcal{A}(S^1)$$

$$\mathsf{S}_q^3 = \mathsf{SU}_q(2), \quad 0 < q \leq 1$$

$\mathcal{A}(\mathsf{SU}_q(2))$:= $*$-algebra generated by a and c, with relations

$$UU^* = U^*U = 1$$

$$U = \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix}$$

Hopf algebra structure on $\mathcal{A}(\mathsf{SU}_q(2))$:

$$\Delta U = U \otimes U$$

$$S(U) = U^*$$

$$\varepsilon(U) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The quantum enveloping algebra $\mathcal{U}_q(\mathrm{su}(2))$: It is the Hopf dual of $\mathrm{SU}_q(2)$. Generators: K, K^{-1}, E, F

Hopf pairing:

$$\mathcal{U}_q(\mathrm{su}(2)) \otimes \mathcal{A}(\mathrm{SU}_q(2)) \to \mathbb{C}$$

$$\langle K, a\rangle = q^{-1/2}, \quad \langle K^{-1}, a\rangle = q^{1/2}$$

$$\langle K, a^*\rangle = q^{1/2}, \quad \langle K^{-1}, a^*\rangle = q^{-1/2}$$

$$\langle E, c\rangle = 1, \quad \langle F, c^*\rangle = -q^{-1}$$

Left and right actions (infinitesimal symmetries):

$$\mathcal{U}_q(\mathrm{su}(2)) \otimes \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2)),$$

$$(X, f) \mapsto X \triangleright f.$$

$$\mathcal{A}(\mathsf{SU}_q(2)) \otimes \mathcal{U}_q(\mathsf{su}(2)) \to \mathcal{A}(\mathsf{SU}_q(2)),$$

$$(f, X) \mapsto f \triangleleft X$$

Uniquely fixed by:

$$\langle X, Y \triangleright f \rangle = \langle XY, f \rangle \,, \quad \langle X, f \triangleleft Y \rangle = \langle YX, f \rangle \,,$$

These right and left actions are mutually commuting.

The quantum projective line

There is a *quantum principal $U(1)$-bundle*:

$$\rho : \mathcal{A}(\mathrm{SU}_q(2)) \mapsto \mathcal{A}(\mathrm{SU}_q(2)) \otimes \mathcal{A}(\mathrm{U}(1))$$

$$\rho = (\mathrm{id} \otimes \pi) \circ \Delta$$

$$\pi : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{U}(1)),$$

where

$$\pi \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix}$$

is a surjective Hopf algebra homomorphism, so that $\mathcal{A}(\mathrm{U}(1))$ becomes a quantum subgroup of $SU_q(2)$.

Coinvariants: A subalgebra of $\mathcal{A}(\mathrm{SU}_q(2))$:

$$\mathcal{A}(\mathrm{S}_q^2) := \{a \in \mathcal{A}(\mathrm{SU}_q(2));\ \rho(a) = a \otimes 1\}$$

The coordinate algebra of the Podleś sphere S^2_q

= the underlying topological space of the quantum projective line $\mathbb{C}\mathsf{P}^1_q$.

The canonical line bundles on $\mathbb{C}P^1_q$

The action of the group-like element $K \Longrightarrow$ a decomposition:

$$\mathcal{A}(\mathsf{SU}_q(2)) = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$$

where,

$$\mathcal{L}_n := \{f \in \mathcal{A}(\mathsf{SU}_q(2)) \ : \ K \triangleright f = q^{n/2} f\}$$

Notice:

$$\mathcal{A}(\mathsf{S}^2_q) = \mathcal{L}_0, \quad \mathcal{L}_n^* \subset \mathcal{L}_{-n}, \quad \mathcal{L}_n \mathcal{L}_m \subset \mathcal{L}_{n+m}$$

$\mathcal{L}_n$: $\mathcal{A}(\mathsf{S}^2_q)$-bimodule; finite projective as a left module; analogues of canonical line bundles $\mathcal{O}(n)$ on $\mathbb{C}P^1$ of degree (monopole charge) $-n$.

A covariant differential calculus for $\mathsf{SU}_q(2)$

Left covariant calculus: $(\mathcal{A}, \Omega, H)$

- $\Omega = \bigoplus_{i \geq 0} \Omega^i$ is a DGA with $\Omega_0 = \mathcal{A}$
- Ω is a left DG H-comodule algebra, i.e. there

is a morphism of DGA's

$$\rho : \Omega \longrightarrow H \otimes \Omega$$

s.t. Ω is a left DG H-comodule under ρ.

Example (Woronowicz): Let $H = \mathcal{A}(\mathsf{SU}_q(2))$ and

$$\Omega^i = \mathcal{A}(\mathsf{SU}_q(2)) \otimes \bigwedge^i \{\omega_+, \omega_-, \omega_z\} \quad 0 \leq i \leq 3$$

$\bigoplus \bigwedge^i \{\omega_+, \omega_-, \omega_z\}$ = The q-Grassmann algebra:

$$\omega_+ \wedge \omega_+ = \omega_- \wedge \omega_- = \omega_z \wedge \omega_z = 0$$

$$\omega_- \wedge \omega_+ + q^{-2} \omega_+ \wedge \omega_- = 0$$

$$\omega_z \wedge \omega_- + q^4 \omega_- \wedge \omega_z = 0,$$

$$\omega_z \wedge \omega_+ + q^{-4} \omega_+ \wedge \omega_z = 0.$$

unique top form: $\omega_- \wedge \omega_+ \wedge \omega_z$.

differential $\mathrm{d} : \mathcal{A}(\mathrm{SU}_q(2)) \to \Omega^1(\mathrm{SU}_q(2))$:

$$\mathrm{d}f = (X_+ \triangleright f)\, \omega_+ + (X_- \triangleright f)\, \omega_- + (X_z \triangleright f)\, \omega_z,$$

where

$$X_z = \frac{1 - K^4}{1 - q^{-2}}, \qquad X_- = q^{-1/2} F K$$

$$X_+ = q^{1/2} E K$$

The holomorphic calculus on $\mathbb{C}P_q^1$

The 'cotangent bundle'

$$\Omega^1(\mathsf{S}_q^2) : \mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_2\omega_+$$

The differential d:

$$\mathrm{d}f = (X_- \triangleright f)\,\omega_- + (X_+ \triangleright f)\,\omega_+$$

where $X_- = q^{-1/2}F$ and $X_+ = q^{1/2}E$.

Break d into a holomorphic and an anti-holomorphic part, $\mathrm{d} = \bar{\partial} + \partial$, with:

$$\bar{\partial}f = (X_- \triangleright f)\,\omega_-, \quad \partial f = \left(X_+ \triangleright f\right)\omega_+$$

The above shows that:

$$\Omega^1(\mathsf{S}_q^2) = \Omega^{(0,1)}(\mathsf{S}_q^2) \oplus \Omega^{(1,0)}(\mathsf{S}_q^2)$$

where

$$\Omega^{(0,1)}(\mathsf{S}^2_q) \simeq \mathcal{L}_{-2} \simeq \bar\partial(\mathcal{A}(\mathsf{S}^2_q)),$$

$$\Omega^{(1,0)}(\mathsf{S}^2_q) \simeq \mathcal{L}_{+2} \simeq \partial(\mathcal{A}(\mathsf{S}^2_q))$$

These modules are not free.

2-forms: Let $\omega = \omega_- \wedge \omega_+$. We have $\omega f = f\omega$, for all $f \in \mathcal{A}(\mathsf{S}^2_q)$.

$$\Omega^2(\mathsf{S}^2_q) := \omega\mathcal{A}(\mathsf{S}^2_q) = \mathcal{A}(\mathsf{S}^2_q)\omega$$

Proposition: The 2D differential calculus on the sphere S^2_q is given by:

$$\Omega^\bullet(\mathsf{S}^2_q) = \mathcal{A}(\mathsf{S}^2_q) \oplus \Big(\mathcal{L}_{-2} \oplus \mathcal{L}_{+2}\Big) \oplus \mathcal{A}(\mathsf{S}^2_q)\omega_+ \wedge \omega_-$$

with the exterior differential $\mathsf{d} = \bar\partial + \partial$:

$$f \mapsto (q^{-1/2} F \triangleright f,\ q^{1/2} E \triangleright f)$$

$$(x,y) \mapsto q^{-1/2}(E \triangleright x - q^{-1} F \triangleright y)$$

for $f \in \mathcal{A}(\mathsf{S}^2_q)$, $(x,y) \in \mathcal{L}_{-2} \oplus \mathcal{L}_{+2}$.

Holomorphic functions on $\mathbb{C}\mathsf{P}^1_q$

$$\bar\partial : \mathcal{A}(\mathbb{C}\mathsf{P}^1_q) \to \Omega^{(0,1)}(\mathbb{C}\mathsf{P}^1_q)$$

We shall use the q-*number notation*:

$$[s] = [s]_q := \frac{q^s - q^{-s}}{q - q^{-1}}$$

Proposition: There are no non-trivial holomorphic *polynomial* functions on $\mathbb{C}\mathsf{P}^1_q$.

Proof:

$\bar\partial f = 0$ iff $F \triangleright f = 0$. Write f in PBW-basis $\{a^m c^k c^{*l}\}$ of $\mathcal{A}(SU_q(2))$,

$$f = \sum_{k,l \geq 0} f_{kl}\ a^{l-k} c^k c^{*l},$$

where $a^{-m} := a^{*m}$. The monomials $a^{l-k} c^k c^{*l}$ are the only K-invariant elements in the PBW-basis.

The vanishing of $F \triangleright f$ implies the following relations between f_{kl} with $0 \leq l < k$:

$$f_{kl} q^{-l} [k] = f_{k+1,l+1} q^{-k-1} [l+1]$$

the solutions of which are given by

$$f_{kl} = \frac{[k-1][k-2] \cdots [k-l] q^k q^{k-1} \ldots q^{k-l+1}}{[l]! q^{l-1} q^{l-2} \ldots q^0} \tilde{f}_{k-l}$$

$$= q^{(k-l+1)l} \begin{bmatrix} k-1 \\ l \end{bmatrix}_q \tilde{f}_{k-l}$$

where $\tilde{f}_{k-l}$ are arbitrary. Clearly, the only polynomial solution is when $f_{kl} = 0$ for $(k,l) \neq (0,0)$. $\square$

Remark: In fact we prove a stronger result by looking at holomorphic functions among smooth functions and show that the analogue of the GAGA principle holds.

Holomorphic vector bundles on $\mathbb{C}\mathrm{P}^1_q$

The 'line bundle' $\mathcal{L}_n$ is represented by a projection $\mathfrak{p}_n$ in $M_{|n|+1}(\mathcal{A}(\mathrm{S}^2_q))$. So we have a Grassmannian connection on $\mathcal{L}_n = (\mathcal{A}(\mathrm{S}^2_q))^{|n|+1}\mathfrak{p}_n$.

Equivalently, a connection is defined by a covariant splitting

$$\Omega^1(\mathrm{SU}_q(2)) = \Omega^1_{\mathrm{ver}}(\mathrm{SU}_q(2)) \oplus \Omega^1_{\mathrm{hor}}(\mathrm{SU}_q(2))$$

Let: ω_z to be vertical, and $\omega_\pm$ to be horizontal.

Now let $\mathcal{E} = \mathcal{L}_n$. We have:

$$\nabla\phi = \left(X_+ \triangleright \phi\right)\omega_+ + \left(X_- \triangleright \phi\right)\omega_-$$

$$= q^{-n-2}\omega_+\left(X_+ \triangleright \phi\right) + q^{-n+2}\omega_-\left(X_- \triangleright \phi\right)$$

Split ∇ into holomorphic and anti-holomorphic parts:

$$\nabla = \nabla^{\partial} + \nabla^{\bar{\partial}}$$

with

$$\nabla^{\partial}\phi = q^{-n-2}\omega_+ \left(X_+ \triangleright \phi\right)$$

$$\nabla^{\bar{\partial}}\phi = q^{-n+2}\omega_- \left(X_- \triangleright \phi\right)$$

Definition: The *standard holomorphic structure* on $\mathcal{L}_n$ is given by

$$\overline{\nabla} := \nabla^{\bar{\partial}} = q^{-n+2}\omega_- \left(X_- \triangleright -\right)$$

the anti-holomorphic part of ∇.

Theorem: With notation as above,

1. For n positive, $H^0(\mathcal{L}_n, \overline{\nabla})$ is an $\mathcal{O}(\mathbb{C}\mathrm{P}^1_q)$-module of rank 0.

2. For n negative, $H^0(\mathcal{L}_n, \overline{\nabla})$ is an $\mathcal{O}(\mathbb{C}\mathsf{P}^1_q)$-module of rank $|n| + 1$. $\square$

Remark: In fact we prove a stronger result by looking at holomorphic sections among smooth sections and show that the analogue of the GAGA principle holds.

We next study the tensor product of two noncommutative holomorphic line bundles.

Proposition: For any integer n there is a 'twisted flip' isomorphism

$$\Phi_{(n)} : \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)} \Omega^{(0,1)} \xrightarrow{\sim} \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)} \mathcal{L}_n$$

as $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$-bimodules.

Proof: $\Omega^{(0,1)}$ is generated (as a $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$-module) by $a^2\omega_-$, $ac\omega_-$ and $c^2\omega_-$. Define

$$\Phi_{(n)}(\phi_1 \otimes \ a^2\omega_- + \phi_2 \otimes ac\omega_- + \phi_3 \otimes c^2\omega_-)$$

$$= q^{-n}\left(a^2\omega_- \otimes \tilde{\phi}_1 + ac\omega_- \otimes \tilde{\phi}_2 + c^2\omega_- \otimes \tilde{\phi}_3\right)$$

with $\tilde{\phi}_1$ satisfying $\phi_1 a^2 = a^2\tilde{\phi}_1$ as elements of $\mathcal{A}(\mathsf{SU}_q(2))$ and similarly for $\tilde{\phi}_2, \tilde{\phi}_3$.

Proposition: The holomorphic structure $\overline{\nabla}$ on $\mathcal{L}_n$ is a bimodule connection with $\sigma(\overline{\nabla}) = \Phi_{(n)}$, i.e. it satisfies the left Leibniz rule and the *twisted right Leibniz rule*:

$$\overline{\nabla}(\xi f) = \overline{\nabla}(\xi)f + \Phi_{(n)}(\xi \otimes \bar{\partial} f)$$

for all $\xi \in \mathcal{L}_n$, $f \in \mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$).

So now we can consider the tensor product of these holomorphic line bundles $(\mathcal{L}_{n_i}, \overline{\nabla}_{n_i}), i = 1, 2.$

Proposition: The tensor product connection

$$\overline{\nabla}_{n_1} \otimes 1 + (\Phi_{(n_1)} \otimes 1)(1 \otimes \overline{\nabla}_{n_2})$$

coincides with the standard holomorphic structure on $\mathcal{L}_{n_1} \otimes_{\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)} \mathcal{L}_{n_2}$ when identified with $\mathcal{L}_{n_1+n_2}$.

The quantum homogeneous coordinate ring

Classical situation: X a projective variety and L a *very ample line bundle* on X. The *homogeneous coordinate ring* of (X, L) is the graded algebra

$$R = \bigoplus_{n \geq 0} H^0(X, L^{\otimes n})$$

For the quantum projective line $\mathbb{C}P^1_q$, using the line bundles $\mathcal{L}_n$, we define

$$\mathcal{R} := \bigoplus_{n \geq 0} H^0(\mathcal{L}_{-n}, \overline{\nabla})$$

Where now the n-th component has dimension $n+1$. Notice that thanks to the twisting maps $\phi_{(n)}$, $\mathcal{R}$ is an algebra. What is the structure of this algebra?

Describe $\mathcal{L}_{-n}$: right $\mathcal{A}(S_q^2)$-module basis:

$$a^{-n-\mu}c^{\mu}, \qquad \mu = 0, 1, \cdots, n$$

Describe $H^0(\mathcal{L}_{-n}, \overline{\nabla})$: $\{a^{-n-\mu}c^{\mu}\}$ form a basis over $\mathbb{C}$

$\mathcal{R}$ is generated by a, c in degree one with one relation

$$ac = qca$$

which is one of the defining relation of the quantum group $\mathrm{SU}_q(2)$

Corollary: The homogeneous coordinate ring of $\mathbb{C}P_q^1$ is isomorphic to the coordinate ring of the quantum plane.

Twisted positivity

An approach to NC complex geometry suggested by Alain Connes [Book, 1994]: Let $\mathcal{A}$ be an $*$-algebra, A Hochschild $2m$-cocycle $\varphi \in Z^{2m}(\mathcal{A}, \mathcal{A}^*)$ is called positive if

$$\langle \omega, \, \eta \rangle := \int_\varphi \omega \eta^*$$

is a positive sesquilinear form on $\Omega^m \mathcal{A}$. Let

$$Z_+^{2m}(\mathcal{A}, \mathcal{A}^*) \subset Z^{2m}(\mathcal{A}, \mathcal{A}^*)$$

denote the set of positive $2m$-Hochschild cocycles on $\mathcal{A}$. It is a convex cone.

Let M= 2-dimensional compact oriented manifold, $\mathcal{A} = C^\infty(M)$, and define a 2-current $\mathbf{C}$ on M by

$$\mathbf{C}(f^0 df^1 df^2) = \frac{-1}{2\pi i} \int f^0 df^1 df^2$$

Let

$$C \subset C^2(\mathcal{A}, \mathcal{A}^*)$$

denote the *Hochschild class* representing the current $\mathbf{C}$. It is an affine subspace of $C^2(\mathcal{A}, \mathcal{A}^*)$.

Theorem (Connes; Book, 1994): There is a 1-1 correspondence between conformal structures on M and the extreme points of $Z^2_+ \cap C$ defined by $g \mapsto \varphi_g$, where

$$\varphi_g(f^0, f^1, f^2) = \frac{-1}{\pi i} \int_M f^0 \partial f^1 \bar{\partial} f^2$$

How can we extend all this to our $\mathbb{C}P^1_q$? There are no interesting 2-dimensional Hochschild classes on $\mathcal{A}(S^2_q)$ (dimension drop in quantization), but there are interesting *twisted cocycles*. In general Let σ :

$A \to A$ be an automorphism of A. Twisted n-cochains on (A, σ):

$$\varphi : A^{\otimes(n+1)} \to \mathbb{C}$$

$$\varphi(a_0, \cdots, a_n) = \varphi(\sigma(a_0), \cdots, \sigma(a_n))$$

Twisted Hochschild coboundary

$$b_\sigma : C_\sigma^n(A) \to C_\sigma^{n+1}(A)$$

$$b_\sigma \varphi(a_0, \cdots, a_{n+1}) =$$

$$\sum_{i=0}^{n} (-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1})$$

$$+(-1)^{n+1} \varphi(\sigma(a_{n+1}) a_n, a_0, \cdots, a_n).$$

Let us now go back to the quantum projective line. Let $h : \mathcal{A}(\mathsf{SU}_q(2)) \to \mathbb{C}$ denote the normalized Haar state of $\mathsf{SU}_q(2)$. It is a positive twisted trace obeying

$$h(xy) = h(\sigma(y)x), \qquad \text{for} \quad x, y \in \mathcal{A}(\mathsf{SU}_q(2)),$$

with (modular) automorphism $\sigma : \mathcal{A}(\mathsf{SU}_q(2)) \to \mathcal{A}(\mathsf{SU}_q(2))$ given by

$$\sigma(x) = K^{-2} \triangleright x \triangleleft K^2.$$

When restricted to $\mathbb{C}P^1_q$, it induces the automorphism

$$\sigma : \mathbb{C}P^1_q \to \mathbb{C}P^1_q, \qquad \sigma(x) = x \triangleleft K^2.$$

The bi-invariance of h on $\mathcal{A}(\mathsf{SU}_q(2))$ reduces to left invariance on $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$. Dually, there is invariance for the right action of $\mathcal{U}_q(\mathrm{su}(2))$ on $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$:

$$h(x \triangleleft v) = \varepsilon(v) h(x),$$

$$\text{for} \quad x \in \mathcal{A}(\mathbb{C}\mathrm{P}^1_q), \; v \in \mathcal{U}_q(\mathrm{su}(2)).$$

With $\omega_- \wedge \omega_+$ the central generator of $\Omega^2(\mathbb{C}\mathrm{P}^1_q)$, h the Haar state on $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$ and σ its above modular automorphism, the linear functional

$$\int_h : \;\; \Omega^2(\mathbb{C}\mathrm{P}^1_q) \to \mathbb{C}, \qquad \int_h f\, \omega_- \wedge \omega_+ := h(f),$$

defines a non-trivial twisted cyclic 2-cocycle τ on $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$ by

$$\tau(f_0, f_1, f_2) := \int_h f_0 \, \mathrm{d} f_1 \wedge \mathrm{d} f_2.$$

The non-triviality means that there is no 1-cochain α on $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$ such that $b_\sigma \alpha = \tau$ and $\lambda_\sigma \alpha = \alpha$. Thus τ is a non-trivial class in $\mathrm{HC}^2_\sigma(\mathbb{C}\mathrm{P}^1_q)$.

Proposition .1. *The cochain $\varphi \in C^2(\mathcal{A}(\mathbb{C}\mathrm{P}^1_q))$ defined by*

$$\varphi(a_0, a_1, a_2) = \int_h a_0 \, \partial a_1 \, \bar{\partial} a_2$$

is a twisted Hochschild 2-cocycle on $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$, that is to say $b_\sigma \varphi = 0$ and $\lambda_\sigma^3 \varphi = \varphi$; it is also positive, with positivity expressed as:

$$\int_h a_0 \, \partial a_1 (a_0 \, \partial a_1)^* \geq 0$$

for all $a_0, a_1 \in \mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$.

Before giving the proof we prove a preliminary result.

Lemma .2. *For any $a_0, a_1, a_2, a_3 \in \mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$ it holds that:*

$$\int_h a_0 (\partial a_1 \bar{\partial} a_2) a_3 = \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2.$$

Proof. Write $\partial a_1 \bar{\partial} a_2 = y\,\omega_- \wedge \omega_+$, for some $y \in \mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$. Using the fact that $\omega_- \wedge \omega_+$ commutes with elements in $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$, we have

$$\int_h a_0(\partial a_1 \bar{\partial} a_2)a_3 - \int_h \sigma(a_3)a_0 \partial a_1 \bar{\partial} a_2 =$$

$$\int_h a_0 y\,\omega_- \wedge \omega_+ a_3 - \int_h \sigma(a_3)a_0 y\,\omega_- \wedge \omega_+ =$$

$$\int_h a_0 y a_3\,\omega_- \wedge \omega_+ - \int_h \sigma(a_3)a_0 y\,\omega_- \wedge \omega_+ =$$

$$h(a_0 y a_3) - h(\sigma(a_3)a_0 y) = 0.$$

from the twisted property of the Haar state. □

Proof. of Proposition **??**.

Using the derivation property of ∂ and $\bar{\partial}$ we have

$$(b_\sigma \varphi)(a_0, a_1, a_2, a_3) = \int_h a_0 a_1 \partial a_2 \bar{\partial} a_3 - \int_h a_0 \partial(a_1 a_2) \bar{\partial} a_3$$

$$+\int_h a_0 \partial a_1 \bar\partial (a_2 a_3) - \int_h \sigma(a_3) a_0 \partial a_1 \bar\partial a_2 =$$

$$\int_h a_0 (\partial a_1 \bar\partial a_2) a_3 - \int_h \sigma(a_3) a_0 \partial a_1 \bar\partial a_2 = 0.$$

using the previous Lemma.

Next, the cyclic condition follows from invariance of the Haar state and of the calculus. Indeed, from the commutativity of the left and right $\mathcal{U}_q(\mathrm{su}(2))$-actions it holds that:

$$\varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2)) = \int_h \sigma(a_0)\, \partial\sigma(a_1)\, \bar\partial\sigma(a_2) =$$

$$\int_h \sigma\left(a_0\, \partial a_1\, \bar\partial a_2\right);$$

writing $a_0\, \partial a_1\, \bar\partial a_2 = y\, \omega_- \wedge \omega_+$, for some $y \in \mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$, left $\mathcal{U}_q(\mathrm{su}(2))$invariance of the forms $\omega_\pm$, yields

$$\sigma\left(a_0\, \partial a_1\, \bar\partial a_2\right) = \sigma(y)\, \omega_- \wedge \omega_+$$

and in turn,

$$\varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2)) =$$

$$\int_h \sigma(y)\, \omega_- \wedge \omega_+ = h(\sigma(y) = h(y \triangleleft K^2)$$

$$= h(y) = \int_h y\, \omega_- \wedge \omega_+$$

$$= \int_h a_0\, \partial a_1\, \bar{\partial} a_2 = \varphi(a_0, a_1, a_2).$$

Finally, for the twisted positivity of φ, the hermitian scalar product on $\Omega^{(1,0)}(\mathbb{C}\mathsf{P}^1_q)$,

$$\langle a_0 \partial a_1, b^0 \partial b^1 \rangle := \varphi(\sigma(b_0^*) a_0, a_1, b_1^*)$$

$$= \int_h \sigma(b_0^*) a_0\, \partial a_1\, \bar{\partial} b_1^*,$$

determines a positive sesquilinear form if for all $a_0, a_1 \in A(\mathbb{C}\mathsf{P}^1_q)$ it holds that

$$\int_h \sigma(a_0^*) a_0 \partial a_1\, \bar{\partial} a_1^* = \int_h a_0\, \partial a_1 (a_0\, \partial a_1)^* \geq 0.$$

The first equality follows again from the Lemma. Indeed,

$$\int_h a_0 \partial a_1 (a_0 \partial a_1)^* = \int_h a_0 \partial a_1 (\partial a_1)^* a_0^*$$

$$= \int_h \sigma(a_0^*) a_0 \partial a_1 \bar{\partial} a_1^*.$$

Then, if $\partial a_1 = y\omega_+$ it follows that $\bar{\partial} a_1^* = (\partial a_1)^* = -\omega_- y^*$; then

$$\int_h \sigma(a_0^*) a_0 \, \partial a_1 \, \bar{\partial} a_1^* = -\int_h \sigma(a_0^*) a_0 \, y \, \omega_+ \wedge \omega_- y^*$$

$$= q^2 \int_h \sigma(a_0^*) a_0 \, y \, y^* \, \omega_- \wedge \omega_+$$

$$= q^2 h(\sigma(a_0^*) a_0 \, y y^*) = q^2 h(a_0 y y^* (a_0)^*)$$

$$= q^2 h(a_0 y (a_0 y^*)^*) \geq 0,$$

the positivity being evident. □