Spectral Zeta Functions and the Gauss-Bonnet Theorem for the Noncommutative Two Torus

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(joint work with F. Fathizadeh)
Canadian Operator Symposium (COSY)
Victoria BC, May 2011

The geometry in noncommutative geometry

- Geometry starts with metric and curvature. While there are a good number of 'soft' topological tools in NCG, like cyclic cohomology, K and KK-theory, and index theory, a truly noncommutative theory of curvature is still illusive. The situation is better with scalar curvature, but computations are quite tough at the moment.

- Metric aspects of NCG are informed by Spectral Geometry. Spectral invariants are the only means by which we can formulate metric ideas of NCG.

A little bit of spectral geometry

- (M, g)= closed, oriented, Riemannian manifold,

$$\Delta_p = dd^* + d^*d : \Omega^p M \subset H \to \Omega^p M \quad \text{Laplacian on } p\text{-forms.}$$

is an unbounded positive operator with discrete spectrum

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

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- The spectrum contains a wealth of geometric and topological informations about M. In particular the dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of M are fully determined by the spectrum of Δ.

- The classical (commutative!) Gauss-Bonnet-Chern theorem gives an expression for $\chi(M)$ in terms of curvature tensor of (M, g):

$$\chi(M) = c \int_M \mathrm{Pf}\,(\Omega).$$

For surfaces:

$$\chi(M) = \frac{1}{2\pi} \int_M K(x) dvol_x.$$

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- An ideal analytic tool to relate these two aspects is the mighty Heat Operator

$$e^{-t\Delta} : H \to H, \quad t > 0$$

is a trace class (in fact smoothing) operator. Solves the heat equation

$$(\frac{\partial}{\partial t} + \Delta)\omega = 0.$$

- Long term behavior ($t \to \infty$) of $e^{-t\Delta}$ is rather simple and just depends on the topology of M, while its short term behavior is complicated and depends on the geometry (metric g) of M:

$$e^{-t\Delta} \to P_{\mathcal{H}} \quad (t \to \infty), \quad \mathcal{H} = \text{Harmonic p-forms}$$

So, by Hodge theory,

$$\mathrm{Tr}(e^{-t\Delta}) \to \beta_p, \quad (t \to \infty)$$

You can hear the cohomology of M.

- Near $t = 0$, we have an asymptotic expansion of the trace of the heat operator:

$$\text{Trace}\left(e^{-t\Delta_p}\right) \sim \sum_{0}^{\infty} a_n(\Delta_p) t^{\frac{n-m}{2}} \qquad (t \to 0)$$

where

$$a_n(\Delta_p) = \int_M a_n(x, \Delta_p) d\text{vol}_x,$$

and the local invariants $a_n(x, \Delta_p)$ (Seeley-DeWitt-Gilkey coefficients) can, in principle, be computed in any local coordinate system using expressions for g and its derivatives.

- For example: $a_n(x, \Delta_p) = 0$, if n is odd;

$$a_0(x, \Delta_p) = (4\pi)^{-m/2} \mathrm{Tr}\ (\mathrm{id});$$

and

$$a_2(x, \Delta_p) = (4\pi)^{-m/2} \mathrm{Tr}\ (-\frac{R}{6}).$$

Higher terms $a_4(x, \Delta_p), \cdots$, are way more complicated!

- In particular,

$$a_0(\Delta_0) = \int_M a_0(x, \Delta_0) dVol = (4\pi)^{-m/2} \mathrm{Vol}(M)$$

Tauberian theorems $\Rightarrow$ Weyl's law:

$$N(\lambda) = \frac{\mathrm{Vol}\ (M)}{(4\pi)^{m/2}\Gamma(1 + m/2)} \lambda^{m/2} + \ \mathrm{O}(\lambda^{m/2})$$

One can hear the Volume and Dimension of a Riemannian manifold.

A beautiful observation

- How to relate geometry (short term asymptotics) to topology (long term asymptotics)? Easy to see that the alternating sum

$$\sum_{p=0}^{m}(-1)^p \mathrm{Tr}\,(e^{-t\Delta_p}) = \sum_{p=0}^{m}(-1)^p \beta_p = \chi(M)$$

is independent of t! Let

$$a_n(x,\Delta) = \sum_{p=0}^{m}(-1)^p a_n(x,\Delta_p), \quad n \geq 0$$

$$\chi(M) \sim \sum_{0}^{\infty} a_n(\Delta) t^{\frac{n-m}{2}} \qquad (t \to 0).$$

Since asymptotic expansion is unique, we obtain a local expression for the Euler characteristic

$$\int_M a_m(x, \Delta) dvol_x = \chi(M)$$

$$\int_M a_n(x, \Delta) dvol_x = 0, \quad n \neq m.$$

This will give a proof of the Gauss-Bonnet theorem, provided one can show that, by a magical cancelation, the alternating sum $a_m(x, \Delta)$ simplify to Pf (Ω). This is indeed the case (Gilkey, Patodi, Getzler,....). And, we shall see that it is also the case for the NC two torus by some fabulous and magical cancelations!

- Spectral zeta function:

$$\zeta_{\Delta_p}(s) = \sum_{\lambda_i \neq 0} \frac{1}{\lambda_i^s}, \quad \mathrm{Re} s > \frac{m}{2}$$

is a holomorphic function and has a meromorphic continuation to all of $\mathbb{C}$ with simple poles inside the set

Spectral Triples

- $(\mathcal{A}, \mathcal{H}, D)$, $\mathcal{A}=$ involutive unital algebra, acting by bounded operators on a Hilbert space $\mathcal{H}$, D = a s.a. operator on $\mathcal{H}$ with compact resolvent such that all commutators $[D, a]$ are bounded.

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  holds.

- Let $\Delta = D^2$. Spectral zeta function

$$\zeta_D(s) = \operatorname{Tr}\left(|D|^{-s}\right) = \operatorname{Tr}\left(\Delta^{-s/2}\right), \quad \operatorname{Re}(s) \gg 0.$$

Spectral Triples

- Using the Mellin transform and the asymptotic expansion, easy to show that: ζ_D has a meromorphic extension to all of $\mathbb{C}$ and non-zero terms a_α, $\alpha < 0$, give a pole of ζ_D at -2α with

$$\mathrm{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2a_\alpha}{\Gamma(-\alpha)}.$$

Also, $\zeta_D(s)$ is holomorphic at $s = 0$ and

$$\zeta_D(0) + \dim \ker D = a_0$$

Gauss-Bonnet for Noncommutative Torus

- Fix $\theta \in \mathbb{R}$. $A_\theta = C^*$-algebra generated by unitaries U and V satisfying

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- Dense subalgebra of *'smooth functions'*:

$$A_\theta^\infty \subset A_\theta,$$

$a \in A_\theta^\infty$ iff

$$a = \sum a_{mn} U^m V^n$$

where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is rapidly decreasing:

$$\sup_{m,n} (1 + m^2 + n^2)^k \, |a_{mn}| < \infty$$

for all $k \in \mathbb{N}$.

- A_θ has a normalized, faithful, and positive trace (unique if θ is irrational):

$$\tau_0 : A_\theta \to \mathbb{C}$$

$$\tau_0\left(\sum a_{mn} U^m V^n\right) = a_{00}.$$

- Derivations $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$; uniquely defined by:

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V.$$

  We have

$$\delta_1\delta_2 = \delta_2\delta_1, \qquad \delta_i(a^*) = -\delta_i(a)^*,$$

- Invariance property:

$$\tau_0(\delta_i(a)) = 0.$$

- The Hilbert space
$$\mathcal{H}_0 = L^2(A_\theta, \tau_0),$$
completion of A_θ w.r.t. inner product
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- The derivations

$$\delta_1, \delta_2 : \mathcal{H}_0 \to \mathcal{H}_0$$

are formally selfadjoint unbounded operators (analogues of $\frac{1}{i}\frac{d}{dx}, \frac{1}{i}\frac{d}{dy}$).

Complex structure

- Metrics on A_θ will be defined through their conformal class. Fix

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0,$$

and define

$$\partial = \delta_1 + \tau\delta_2, \qquad \partial^* = \delta_1 + \bar{\tau}\delta_2.$$

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- Define the Hilbert space (analogue of $(1,0)$-forms)

$$\mathcal{H}^{(1,0)} \subset \mathcal{H}_0$$

as the completion of the subspace spanned by finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$. Connes and Tretkoff consider $\tau = i$.

Laplacian

- View
$$\partial = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$$
as an unbounded operator with the adjoint given by
$$\partial^* = \delta_1 + \bar{\tau}\delta_2.$$
- Define the Laplacian
$$\triangle := \partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

Conformal perturbation of the metric

- To investigate the Gauss-Bonnet theorem for general metrics, vary the metric by a Weyl factor e^h, $h = h^* \in A_\theta^\infty$: Define a positive linear functional $\varphi : A_\theta \to \mathbb{C}$ by

$$\varphi(a) = \tau_0(ae^{-h}), \qquad a \in A_\theta.$$

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- It is a twisted trace

$$\varphi(ba) = \varphi(a\sigma_i(b))$$

which is the KMS condition at $\beta = 1$ for the automorphisms $\sigma_t : A_\theta \to A_\theta, \; t \in \mathbb{R}$,

$$\sigma_t(x) = e^{ith} x e^{-ith}.$$

In fact

$$\sigma_t = \Delta^{-it}$$

with the modular operator

$$\Delta(x) = e^{-h} x e^{h}.$$

The perturbed Laplacian

- Let $\mathcal{H}_\varphi$ = completion of A_θ w.r.t. $\langle , \rangle_\varphi$, where

$$\langle a, b\rangle_\varphi = \varphi(b^* a), \qquad a, b \in A_\theta.$$

Let

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

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- It has a formal adjoint ∂_φ^* given by

$$\partial_\varphi^* = R(e^h)\partial^*$$

where $R(e^h)$ is the right multiplication operator by e^h $(R(e^h)(x) = e^h x)$.

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Lemma (Connes-Tretkoff; continues to hold for general τ)

$\triangle'$ is anti-unitarily equivalent to the positive unbounded operator $k\Delta k$ acting on $\mathcal{H}_0$, where $k = e^{h/2}$.

Spectral Zeta Function

$$\zeta(s) = \sum \lambda_i^{-s} = \text{Trace}\,(\triangle'^{-s}), \qquad \text{Re}(s) > 1.$$

Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt$$

gives us

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Trace}^+(e^{-t\triangle'}) t^{s-1} dt,$$

where

$$\text{Trace}^+(e^{-t\Delta'}) = \text{Trace}\,(e^{-t\Delta'}) - \text{Dim Ker}(\Delta').$$

ζ has a moromorphic extension to $\mathbb{C} \setminus 1$ with a simple pole at $s = 1$.

The Gauss-Bonnet theorem

Theorem (Gauss-Bonnet for classical Riemann surfaces)

Let Σ = compact connected oriented Riemann surface with metric g. Then

$$\zeta(0) + 1 = \frac{1}{12\pi} \int_\Sigma R = \frac{1}{6}\chi(\Sigma),$$

where ζ is the zeta function associated to the Laplacian $\triangle_g = d^* d$, and R is the (scalar) curvature. In particular $\zeta(0)$ is a topological invariant; e.g. is invariant under conformal perturbations of the metric $g \mapsto e^f g$.

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ be an invertible positive element. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is independent of k.

Pseudodifferential calculus

Recall: Connes (1980; C^*-algebras and Noncommutative Differential Geometry)
Differential operators of order n:

$$P : A_\theta^\infty \to A_\theta^\infty, \quad P = \sum_j a_j \delta_1^{j_1} \delta_2^{j_2}$$

with $a_j \in A_\theta^\infty$, $j = (j_1, j_2)$, $|j| \leq n$.

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Operator valued symbols of order $n \in \mathbb{Z}$: smooth maps

$$\rho : \mathbb{R}^2 \to A_\theta^\infty$$

s.t.

$$||\delta_1^{i_1}\delta_2^{i_2}\,(\partial_1^{j_1}\partial_2^{j_2}\,\rho(\xi))|| \leq c(1+|\xi|)^{n-|j|},$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$, and ρ is homogeneous of order n at infinity:

$$\lim \lambda^{-n}\rho(\lambda\xi_1, \lambda\xi_2), \qquad \lambda \to \infty$$

exists and is smooth.

Given a symbol ρ, define a pseudodifferential operator

$$P_\rho : A_\theta^\infty \to A_\theta^\infty$$

by

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi,$$

where

$$\alpha_s(U^n V^m) = e^{is.(n,m)} U^n V^m.$$

For pseudodifferential operators P, Q, with symbols $\sigma(P) = \rho, \sigma(Q) = \rho'$:

$$\sigma(PQ) \sim \sum \frac{1}{\ell_1! \ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).$$

Elliptic Symbols: A symbol $\rho(\xi)$ of order n is called elliptic if $\rho(\xi)$ is invertible for $\xi \neq 0$, and, for $|\xi|$ large enough,

$$||\rho(\xi)^{-1}|| \leq c(1+|\xi|)^{-n}.$$

Example:

$$\triangle = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$$

is an elliptic operator with an elliptic symbol

$$\sigma(\Delta) = \xi_1^2 + 2\tau_1\xi_1\xi_2 + |\tau|^2\xi_2^2.$$

Computing $\zeta(0)$

Recall:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle'})t^{s-1} - 1)dt,$$

$1 = \mathrm{Dim}\ \mathrm{Ker}(\triangle')$.
Cauchy integral formula:

$$e^{-t\Delta'} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(\Delta' - \lambda 1)^{-1} d\lambda$$

gives the asymptotic expansion as $t \to 0^+$:

$$\mathrm{Trace}(e^{-t\triangle'}) \sim t^{-1} \sum_0^\infty B_{2n}(\triangle')t^n.$$

It follows that:

$$\zeta(0) = B_2(\triangle'),$$

$$B_2(\Delta') = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_C e^{-\lambda} \tau_0(b_2(\xi, \lambda)) d\lambda d\xi$$

where

$$(b_0(\xi,\lambda) + b_1(\xi,\lambda) + b_2(\xi,\lambda) + \cdots)\sigma(\Delta' - \lambda) \sim 1,$$

$$b_j(\xi,\lambda) \text{ is a symbol of order } -2-j.$$

Can assume $\lambda = -1$:

$$\zeta(0) = -\int \tau_0(b_2(\xi,-1))d\xi.$$

$$\sigma(\Delta' + 1) = \sigma(k\Delta k + 1) = (a_2 + 1) + a_1 + a_0$$

where

$$a_2 = k^2\xi_1^2 + 2\tau_1 k^2\xi_1\xi_2 + |\tau|^2 k^2\xi_2^2$$

$$a_1 = (2k\delta_1(k) + 2\tau_1 k\delta_2(k))\xi_1 +$$

$$(2\tau_1 k\delta_1(k) + 2|\tau|^2 k\delta_2(k))\xi_2$$

$$a_0 = k\delta_1^2(k) + 2\tau_1 k\delta_1\delta_2(k) + |\tau|^2 k\delta_2^2(k).$$

Using the calculus for symbols:

$$b_0 = (a_2 + 1)^{-1}$$

$$b_1 = -(b_0 a_1 b_0 + \partial_i(b_0)\delta_i(a_2)b_0)$$

$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_i(b_0)\delta_i(a_1)b_0$$

$$+\partial_i(b_1)\delta_i(a_2)b_0 + (1/2)\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0).$$

Integrating $b_2(\xi, -1)$ over the plane

Pass to these coordinates:

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta$$

$$\xi_2 = \frac{r}{\tau_2}\sin\theta$$

where θ ranges from 0 to 2π and r ranges from 0 to ∞. After integrating $\int_0^{2\pi} \bullet \, d\theta$ we have terms such as

$$4\tau_1 r^3 b_0^3 k^2 \delta_2(k)\delta_1(k),$$

$$2r^3 b_0^2 k^2 \delta_1(k) b_0 \delta_1(k),$$

$$-2r^5 b_0^2 k^2 \delta_1(k) b_0^2 k^2 \delta_1(k),$$

where

$$b_0 = (1 + r^2 k^2)^{-1}.$$

Lemma (Connes-Tretkoff)

For $\rho \in A_\theta^\infty$ and every non-negative integer m:

$$\int_0^\infty \frac{k^{2m+2}u^m}{(k^2u+1)^{m+1}}\rho\frac{1}{(k^2u+1)}du = \mathcal{D}_m(\rho)$$

where

$$\mathcal{D}_m = \mathcal{L}_m(\Delta),$$

$$\Delta = \textit{the modular automorphism},$$

$$\mathcal{L}_m(u) = \int_0^\infty \frac{x^m}{(x+1)^{m+1}}\frac{1}{(xu+1)}dx =$$

$$(-1)^m(u-1)^{-(m+1)}\left(\log u - \sum_{j=1}^{m}(-1)^{j+1}\frac{(u-1)^j}{j}\right)$$

(modified logarithm).

Lemma

Let k be an invertible positive element of A_θ^∞. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is given by

$$\zeta(0)+1=\frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k))+\frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k))+$$

$$\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_2(k))+\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_1(k)),$$

where $\varphi(x)=\tau_0(xk^{-2})$, τ_0 is the unique trace on A_θ, Δ is the modular automorphism, and

$$f(u)=\frac{1}{6}u^{-1/2}-\frac{1}{3}+\mathcal{L}_1(u)-2(1+u^{1/2})\mathcal{L}_2(u)+(1+u^{1/2})^2\mathcal{L}_3(u).$$

($\mathcal{L}_m$ is the modified logarithm.)

The following theorem was proved by Alain Connes and Paula Tretkoff for conformal parameter $\tau = i$, and then for all conformal parameters by Farzad Fathizadeh and M.K.

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ be an invertible positive element. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is independent of k.

Proof.

$$\varphi(f(\Delta)(\delta_j(k))\delta_j(k)) = 0 \quad \text{for } j = 1, 2,$$

$$\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) = -\varphi(f(\Delta)(\delta_2(k))\delta_1(k)).$$

Therefore

$$\zeta(0) + 1 = \frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k)) + \frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k)) +$$

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$$= 0$$

. $\square$

An argument of Moscovici: variational method. A technique of Branson-Orsted in the commutative case can be extended to the NC case, when there is a good pseudodifferential calculus and good resolvent approximation. Write, for $P=$ a NC polynomial in D and elements of $\mathcal{A}$,

$$\mathrm{Tr}\,(Pe^{-tD_{sh}^2}) \sim \sum_{j=0}^{\infty} a_j(P,s)t^{\frac{j-n-p}{2}} \quad (t \to 0)$$

Term by term differentiate w.r.t. s and observe that $\frac{d}{ds}a_p(s) = 0$. This brings you back to $h = 0$ (still you have to evaluate a zeta value using the spectrum of $\triangle'$ on A_θ).

Back to the Scalar Curvature

But: we are really interested in computing the scalar curvature as a variable function on $\mathbb{T}^2_\theta$. Gauss-Bonnet computes its total integral. Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable regular spectral triple. Consider the zeta function

$$\zeta_D(P, z) = \mathrm{Tr}\,(P|D|^{-z}), \quad P \in \Psi(\mathcal{A}, \mathcal{H}, D)$$

For the NC torus, the scalar curvature can be defined as the functional on the NC torus:

$$a \mapsto \zeta_{\triangle'}(a, 0)$$

Challenge: compute the density of this functional!