Spectral Zeta Functions and Gauss-Bonnet Theorems in Noncommutative Geometry

Masoud Khalkhali (joint work with Farzad Fathizadeh)

- One of the backbones of Alain Connes' program of NCG, specially its metric and spectral aspects, is Spectral Geometry and the Correspondence Principle which relates QM to CM. The correspondence principle has its roots in Planck's derivation of his celebrated Radiation Law and in Bohr-Sommerfeld Quantization Rules.

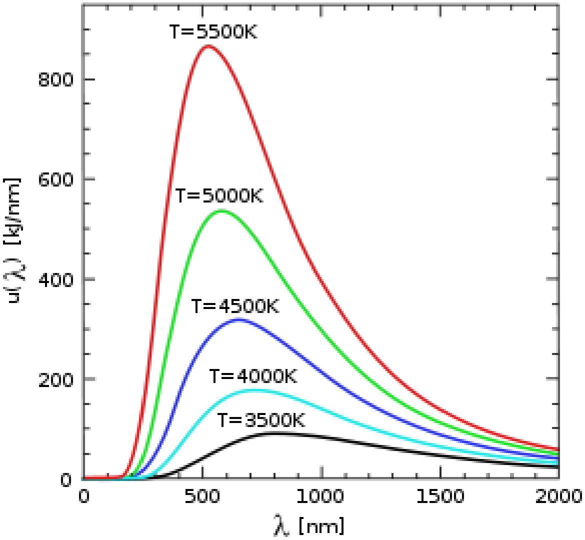

Figure: Black body spectrum

Planck's Radiation Law

- From 1859 (Kirchhoff) till 1900 (Planck) a great effort went into finding the right formula for spectral energy density function of a radiating black body (T = temperature, ν = frequency, h= Planck's constant, k= Boltzmann's constant, c = speed of light):

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

Planck's Radiation Law

- From 1859 (Kirchhoff) till 1900 (Planck) a great effort went into finding the right formula for spectral energy density function of a radiating black body (T = temperature, ν = frequency, h= Planck's constant, k= Boltzmann's constant, c = speed of light):

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

- Kirchhoff predicted: ρ will be independent of the shape of the cavity and should only depend on its volume.

Limits of Planck's Law

- Quantum Limit (high-frequency or low temperature regime; $h\nu/kT \gg 1$)

$$\rho(\nu, T) \sim A\nu^3 e^{-B\nu/T} \quad (T \to 0)$$

Limits of Planck's Law

- Quantum Limit (high-frequency or low temperature regime; $h\nu/kT \gg 1$)

$$\rho(\nu, T) \sim A\nu^3 e^{-B\nu/T} \quad (T \to 0)$$

- Semiclassical Limit (low frequency or high temperature; $h\nu/kT \ll 1$)

$$\rho(\nu, T) = \frac{8\pi\nu^2}{c^3}(kT)(1 + O(h)) \quad (T \to \infty)$$

RHS is the Rayleigh-Jeans-Einstein radiation formula. It can be established, assuming the Weyl's Law: *" For high frequencies there are approximately $V(8\pi\nu^3 d\nu/c^3)$ modes of oscillations in the frequency interval $v, \nu + d\nu$."*

- Moral: To relate classical and quantum worlds, Weyl's law is needed:

One can hear the volume of a cavity.

- Moral: To relate classical and quantum worlds, Weyl's law is needed:

One can hear the volume of a cavity.

- The conjecture of Lorentz (1910; proved by Weyl in 1911): ' *It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between* ν *and* $\nu + d\nu$ *is independent of the shape of the enclosure and is simply proportional to its volume. ....There is no doubt that it holds in general even for multiply connected spaces'.*

- Moral: To relate classical and quantum worlds, Weyl's law is needed:

One can hear the volume of a cavity.

- The conjecture of Lorentz (1910; proved by Weyl in 1911): *' It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume. ....There is no doubt that it holds in general even for multiply connected spaces'.*

- But the ultimate question is

What else can one hear about the shape of a cavity?

Bohr-Sommerfeld Quantization

- Stationary Schrodinger equation:

$$(\frac{\hbar^2}{2m}\Delta + V)\varphi(x) = \lambda\varphi(x)$$

- WKB approximation ansatz

$$\varphi(x) = Ae^{\frac{i}{\hbar}}B(x)$$

- $S = \oint pdq$, the total action; the Bohr-Sommerfeld quantization rule:

$$\exp\{\frac{i}{\hbar}S\} = 1 \quad \text{or} \quad \oint pdq = 2\pi n\hbar, \quad n = 1, 2, \ldots$$

It relates classical periodic orbits to energy levels in the corresponding quantum system.

Can one hear the periodic orbits of a classical system?

Yes, for Riemann surfaces, In general, trace formula (Selberg, Connes) relates lengths of periodic orbits in chaotic systems to the spectrum.

How to Quantize?

- Weyl's law imposes some constraints.

How to Quantize?

- Weyl's law imposes some constraints.
- Consider a Classical System (X, h);
  $X =$ symplectic manifold, $h : X \to \mathbb{R}$, Hamiltonian. Assume

$$\{x \in X;\ h(x) \leq \lambda\}$$

are compact for all λ (confined system).

How to Quantize?

- Weyl's law imposes some constraints.
- Consider a Classical System (X, h);
  X = symplectic manifold, $h : X \to \mathbb{R}$, Hamiltonian. Assume

  $$\{x \in X;\ h(x) \leq \lambda\}$$

  are compact for all λ (confined system).
- Typical example: $X = T^*M$, (M, g) = compact Riemannian manifold, $h = T + V$.
  T= kinetic energy, V= potential energy.

- How to quantize this?

$$(X, h) \rightsquigarrow (\mathcal{H}, H),$$

where $\mathcal{H}$ = Hilbert space, H = self-adjoint operator on $\mathcal{H}$. No one knows! No functor!, but ....Dirac rules, geometric quantization, deformation quantization, ...and the correspondence principle:

- Looking for a pair $(\mathcal{H}, H)$, $\mathcal{H}$ = Hilbert space, H = self-adjoint operator on H, Hamiltonian, with discrete spectrum

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

s.t.

$$N(\lambda) \sim c \,\text{Volume}\,(h \leq \lambda) \qquad \lambda \to \infty$$

$$N(\lambda) = \#\{\lambda_i \leq \lambda\} \quad \text{Eigenvalue Counting Function}$$

Thus:quantized energy levels are approximated by phase space volumes.

- Apply this to $X = T^*M$, (M, g) = compact Riemannian manifold, $h(q,p) = ||p||^2$; set

$$\mathcal{H} = L^2(M), \quad H = \Delta \quad \text{Laplacian}$$

obtain Weyl's Law:

$$N(\lambda) \sim c\,\text{Vol (M)}\lambda^{m/2} \qquad (\lambda \to \infty)$$

A Heuristic, Physical 'Proof' of Weyl's Law

- Classical partition function from Gibbs equilibrium state at inverse temperature $\beta = 1/kT$

$$Z = \int_X e^{-h/\beta} dvol = \int_0^\infty e^{-x/\beta} d\mu(x)$$

$$\mu[0,\lambda] = \text{Vol}\,(h \leq \lambda)$$

- Quantum partition function

$$Z_q = \text{Trace}\,(e^{-H/\beta}) = \int_0^\infty e^{-x/\beta} d\mu_q(x)$$

Eigenvalue counting measure

$$\mu_q[0,\lambda] = \#\{\lambda_i \leq \lambda\}$$

- Experimental fact: classical statistical mechanics gives good results at high temperatures; in particular specific heat obtained from Z should converge to its quantum value from Z_q.

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{1}{kT^2} \frac{\partial^2 \ln Z}{\partial^2 \beta}$$

- In particular, the measures $\mu[0, \lambda]$ and $\mu_q[0, \lambda]$ are asymptotically proportional:

$$\frac{\mu[0, \lambda]}{\mu_q[0, \lambda]} \to (2\pi\hbar)^N$$

$(\dim(X) = 2N)$

Weyl's Law

- (M, g) = compact Riemannian manifold

$$\Delta = d^* d : L^2(M) \to L^2(M), \qquad \text{Laplacian}$$

- Is a s. a. positive operator. In local coordinates:

$$\Delta = -g^{\mu\nu}\partial_\mu \partial_\nu + A^\mu \partial_\mu + B$$

- Spectrum of Δ (counting multiplicities):

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Eigenvalue counting function:

$$N(\lambda) := \#\{\lambda_i \leq \lambda\}$$

- Weyl's Law:

$$N(\lambda) = \frac{\text{Vol } (M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} + \text{ O}(\lambda^{m/2})$$

One can hear the Volume and Dimension of a Riemannian manifold.

- Weyl's Law:

$$N(\lambda) = \frac{\text{Vol }(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} + \ O(\lambda^{m/2})$$

One can hear the Volume and Dimension of a Riemannian manifold.

- Asymptotic expansion of the trace of the heat kernel:

$$\text{Trace}\,(e^{-t\Delta}) \sim \sum_{0}^{\infty} a_n t^{\frac{n-m}{2}} \qquad (t \to 0)$$

$$a_n = \int_M a_n(x,\Delta) d\text{Vol}_x \qquad \text{local invariants}$$

- Seeley-DeWitt coefficients $a_n(x,\Delta),\ n \geq 0$

$$a_0(x,\Delta) = (4\pi)^{-m/2}$$

$$a_0 = \int_M a_0(x,\Delta) dVol = (4\pi)^{-m/2} \mathrm{Vol}(M)$$

Tauberian theorems $\Rightarrow$ Weyl's law.

Spectral Triples

- $(\mathcal{A}, \mathcal{H}, D)$, $\mathcal{A}=$ involutive unital algebra, acting by bounded operators on a Hilbert space $\mathcal{H}$, $D =$ a s.a. operator on $\mathcal{H}$ with compact resolvent such that all commutators $[D, a]$ are bounded.

Spectral Triples

- $(\mathcal{A}, \mathcal{H}, D)$, $\mathcal{A}=$ involutive unital algebra, acting by bounded operators on a Hilbert space $\mathcal{H}$, $D =$ a s.a. operator on $\mathcal{H}$ with compact resolvent such that all commutators $[D, a]$ are bounded.
- Assume: an asymptotic expansion of the form

$$\text{Trace}\,(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

holds.

Spectral Triples

- $(\mathcal{A}, \mathcal{H}, D)$, $\mathcal{A}=$ involutive unital algebra, acting by bounded operators on a Hilbert space $\mathcal{H}$, $D =$ a s.a. operator on $\mathcal{H}$ with compact resolvent such that all commutators $[D, a]$ are bounded.

- Assume: an asymptotic expansion of the form

$$\mathrm{Trace}\,(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

holds.

- Let $\Delta = D^2$. Spectral zeta function

$$\zeta_D(s) = \mathrm{Tr}\,(|D|^{-s}) = \mathrm{Tr}\,(\Delta^{-s/2}), \quad \mathrm{Re}(s) \gg 0.$$

Spectral Triples

- Using the Mellin transform and the asymptotic expansion, easy to show that: ζ_D has a meromorphic extension to all of $\mathbb{C}$ and non-zero terms a_α, $\alpha < 0$, give a pole of ζ_D at -2α with

$$\text{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2a_\alpha}{\Gamma(-\alpha)}.$$

Also, $\zeta_D(s)$ is holomorphic at $s = 0$ and

$$\zeta_D(0) + \dim \ker D = a_0$$

Gauss-Bonnet for Noncommutative Torus

- Fix $\theta \in \mathbb{R}$. $A_\theta = C^*$-algebra generated by unitaries U and V satisfying

$$VU = e^{2\pi i\theta} UV.$$

Gauss-Bonnet for Noncommutative Torus

- Fix $\theta \in \mathbb{R}$. $A_\theta = C^*$-algebra generated by unitaries U and V satisfying

$$VU = e^{2\pi i\theta} UV.$$

- Dense subalgebra of *'smooth functions'*:

$$A_\theta^\infty \subset A_\theta,$$

$a \in A_\theta^\infty$ iff

$$a = \sum a_{mn} U^m V^n$$

where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is rapidly decreasing:

$$\sup_{m,n} (1 + m^2 + n^2)^k \, |a_{mn}| < \infty$$

for all $k \in \mathbb{N}$.

- A_θ has a normalized, faithful, and positive trace (unique if θ is irrational):

$$\tau_0 : A_\theta \to \mathbb{C}$$

$$\tau_0(\sum a_{mn} U^m V^n) = a_{00}.$$

- Derivations $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$; uniquely defined by:

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V.$$

  We have

$$\delta_1\delta_2 = \delta_2\delta_1, \qquad \delta_i(a^*) = -\delta_i(a)^*,$$

- Invariance property:

$$\tau_0(\delta_i(a)) = 0.$$

- The Hilbert space

$$\mathcal{H}_0 = L^2(A_\theta, \tau_0),$$

completion of A_θ w.r.t. inner product

$$\langle a, b \rangle = \tau_0(b^* a).$$

- The Hilbert space

$$\mathcal{H}_0 = L^2(A_\theta, \tau_0),$$

completion of A_θ w.r.t. inner product

$$\langle a, b \rangle = \tau_0(b^* a).$$

- The derivations

$$\delta_1, \delta_2 : \mathcal{H}_0 \to \mathcal{H}_0$$

are formally selfadjoint unbounded operators (analogues of $\frac{1}{i}\frac{d}{dx}, \frac{1}{i}\frac{d}{dy}$).

Complex structure

- Metrics on A_θ will be defined through their conformal class. Fix

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0,$$

and define

$$\partial = \delta_1 + \tau\delta_2, \qquad \partial^* = \delta_1 + \bar{\tau}\delta_2.$$

Complex structure

- Metrics on A_θ will be defined through their conformal class. Fix

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0,$$

and define

$$\partial = \delta_1 + \tau\delta_2, \qquad \partial^* = \delta_1 + \bar{\tau}\delta_2.$$

- Define the Hilbert space (analogue of $(1,0)$-forms)

$$\mathcal{H}^{(1,0)} \subset \mathcal{H}_0$$

as the completion of the subspace spanned by finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$. Connes and Tretkoff consider $\tau = i$.

Laplacian

- View

$$\partial = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$$

as an unbounded operator with the adjoint given by

$$\partial^* = \delta_1 + \bar{\tau}\delta_2.$$

- Define the Laplacian

$$\triangle := \partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

Conformal perturbation of the metric

- To investigate the Gauss-Bonnet theorem for general metrics, vary the metric by a Weyl factor e^h, $h = h^* \in A_\theta^\infty$: Define a positive linear functional $\varphi : A_\theta \to \mathbb{C}$ by

$$\varphi(a) = \tau_0(ae^{-h}), \qquad a \in A_\theta.$$

Conformal perturbation of the metric

- To investigate the Gauss-Bonnet theorem for general metrics, vary the metric by a Weyl factor e^h, $h = h^* \in A_\theta^\infty$: Define a positive linear functional $\varphi : A_\theta \to \mathbb{C}$ by

$$\varphi(a) = \tau_0(ae^{-h}), \qquad a \in A_\theta.$$

- It is a twisted trace

$$\varphi(ba) = \varphi(a\sigma_i(b))$$

which is the KMS condition at $\beta = 1$ for the automorphisms $\sigma_t : A_\theta \to A_\theta, \; t \in \mathbb{R}$,

$$\sigma_t(x) = e^{ith} x e^{-ith}.$$

In fact

$$\sigma_t = \Delta^{-it}$$

with the modular operator

$$\Delta(x) = e^{-h} x e^h.$$

The perturbed Laplacian

- Let $\mathcal{H}_\varphi =$ completion of A_θ w.r.t. $\langle , \rangle_\varphi$, where

$$\langle a, b \rangle_\varphi = \varphi(b^* a), \qquad a, b \in A_\theta.$$

Let

$$\partial_\varphi = \partial = \delta_1 + \tau \delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

The perturbed Laplacian

- Let $\mathcal{H}_\varphi =$ completion of A_θ w.r.t. $\langle , \rangle_\varphi$, where

$$\langle a, b \rangle_\varphi = \varphi(b^* a), \qquad a, b \in A_\theta.$$

  Let

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

- It has a formal adjoint ∂_φ^* given by

$$\partial_\varphi^* = R(e^h)\partial^*$$

  where $R(e^h)$ is the right multiplication operator by e^h ($R(e^h)(x) = e^h x$).

- Define the new Laplacian:

$$\triangle' = \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi.$$

- Define the new Laplacian:

$$\triangle' = \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi.$$

Lemma (Connes-Tretkoff; continues to hold for general τ)

$\triangle'$ is anti-unitarily equivalent to the positive unbounded operator $k\Delta k$ acting on $\mathcal{H}_0$, where $k = e^{h/2}$.

Spectral Zeta Function

$$\zeta(s) = \sum \lambda_i^{-s} = \text{Trace}\,(\triangle'^{-s}), \qquad \text{Re}(s) > 1.$$

Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt$$

gives us

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Trace}^+(e^{-t\triangle'}) t^{s-1} dt,$$

where

$$\text{Trace}^+(e^{-t\Delta'}) = \text{Trace}\,(e^{-t\Delta'}) - \text{Dim Ker}(\Delta').$$

ζ has a moromorphic extension to $\mathbb{C} \setminus 1$ with a simple pole at $s = 1$.

The Gauss-Bonnet theorem

Theorem (Gauss-Bonnet for classical Riemann surfaces)

Let Σ = compact connected oriented Riemann surface with metric g. Then

$$\zeta(0) + 1 = \frac{1}{12\pi} \int_\Sigma R = \frac{1}{6}\chi(\Sigma),$$

where ζ is the zeta function associated to the Laplacian $\triangle_g = d^* d$, and R is the (scalar) curvature. In particular $\zeta(0)$ is a topological invariant; e.g. is invariant under conformal perturbations of the metric $g \mapsto e^f g$.

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ be an invertible positive element. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is independent of k.

Pseudodifferential calculus

Recall: Connes (1980; C^*-algebras and Noncommutative Differential Geometry)

Differential operators of order n:

$$P : A_\theta^\infty \to A_\theta^\infty, \quad P = \sum_j a_j \delta_1^{j_1} \delta_2^{j_2}$$

with $a_j \in A_\theta^\infty,\ j = (j_1, j_2),\ |j| \leq n.$

Pseudodifferential calculus

Recall: Connes (1980; C^*-algebras and Noncommutative Differential Geometry)

Differential operators of order n:

$$P : A_\theta^\infty \to A_\theta^\infty, \quad P = \sum_j a_j \delta_1^{j_1} \delta_2^{j_2}$$

with $a_j \in A_\theta^\infty$, $j = (j_1, j_2)$, $|j| \leq n$.

Operator valued symbols of order $n \in \mathbb{Z}$: smooth maps

$$\rho : \mathbb{R}^2 \to A_\theta^\infty$$

s.t.

$$||\delta_1^{i_1}\delta_2^{i_2}(\partial_1^{j_1}\partial_2^{j_2}\rho(\xi))|| \leq c(1+|\xi|)^{n-|j|},$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$, and ρ is homogeneous of order n at infinity:

$$\lim \lambda^{-n}\rho(\lambda\xi_1, \lambda\xi_2), \qquad \lambda \to \infty$$

exists and is smooth.

Given a symbol ρ, define a pseudodifferential operator

$$P_\rho : A_\theta^\infty \to A_\theta^\infty$$

by

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi,$$

where

$$\alpha_s(U^n V^m) = e^{is.(n,m)} U^n V^m.$$

For pseudodifferential operators P, Q, with symbols $\sigma(P) = \rho, \sigma(Q) = \rho'$:

$$\sigma(PQ) \sim \sum \frac{1}{\ell_1! \ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2}(\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2}(\rho'(\xi)).$$

Elliptic Symbols: A symbol $\rho(\xi)$ of order n is called elliptic if $\rho(\xi)$ is invertible for $\xi \neq 0$, and, for $|\xi|$ large enough,

$$||\rho(\xi)^{-1}|| \leq c(1+|\xi|)^{-n}.$$

Example:

$$\triangle = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$$

is an elliptic operator with an elliptic symbol

$$\sigma(\Delta) = \xi_1^2 + 2\tau_1\xi_1\xi_2 + |\tau|^2\xi_2^2.$$

Computing $\zeta(0)$

Recall:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle'})t^{s-1} - 1)dt,$$

$1 = \mathrm{Dim\ Ker}(\triangle')$.
Cauchy integral formula:

$$e^{-t\Delta'} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(\Delta' - \lambda 1)^{-1} d\lambda$$

gives the asymptotic expansion as $t \to 0^+$:

$$\mathrm{Trace}(e^{-t\triangle'}) \sim t^{-1} \sum_0^\infty B_{2n}(\triangle')t^n.$$

It follows that:

$$\zeta(0) = B_2(\triangle'),$$

$$B_2(\Delta') = \frac{1}{2\pi i}\int_{\mathbb{R}^2}\int_C e^{-\lambda}\tau_0(b_2(\xi,\lambda))d\lambda d\xi$$

where

$$(b_0(\xi,\lambda)+b_1(\xi,\lambda)+b_2(\xi,\lambda)+\cdots)\sigma(\Delta'-\lambda)\sim 1,$$

$$b_j(\xi,\lambda)\,\text{is a symbol of order } -2-j.$$

Can assume $\lambda = -1$:

$$\zeta(0) = -\int \tau_0(b_2(\xi,-1))d\xi.$$

$$\sigma(\Delta' + 1) = \sigma(k\Delta k + 1) = (a_2 + 1) + a_1 + a_0$$

where

$$a_2 = k^2\xi_1^2 + 2\tau_1 k^2 \xi_1 \xi_2 + |\tau|^2 k^2 \xi_2^2$$

$$a_1 = (2k\delta_1(k) + 2\tau_1 k\delta_2(k))\xi_1 +$$

$$(2\tau_1 k\delta_1(k) + 2|\tau|^2 k\delta_2(k))\xi_2$$

$$a_0 = k\delta_1^2(k) + 2\tau_1 k\delta_1\delta_2(k) + |\tau|^2 k\delta_2^2(k).$$

Using the calculus for symbols:

$$b_0 = (a_2 + 1)^{-1}$$

$$b_1 = -(b_0 a_1 b_0 + \partial_i(b_0)\delta_i(a_2)b_0)$$

$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_i(b_0)\delta_i(a_1)b_0$$

$$+\partial_i(b_1)\delta_i(a_2)b_0 + (1/2)\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0).$$

Integrating $b_2(\xi, -1)$ over the plane

Pass to these coordinates:

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta$$

$$\xi_2 = \frac{r}{\tau_2}\sin\theta$$

where θ ranges from 0 to 2π and r ranges from 0 to ∞.
After integrating $\int_0^{2\pi} \bullet \, d\theta$ we have terms such as

$$4\tau_1 r^3 b_0^3 k^2 \delta_2(k)\delta_1(k),$$

$$2r^3 b_0^2 k^2 \delta_1(k) b_0 \delta_1(k),$$

$$-2r^5 b_0^2 k^2 \delta_1(k) b_0^2 k^2 \delta_1(k),$$

where

$$b_0 = (1 + r^2 k^2)^{-1}.$$

Lemma (Connes-Tretkoff)

For $\rho \in A_\theta^\infty$ and every non-negative integer m:

$$\int_0^\infty \frac{k^{2m+2}u^m}{(k^2u+1)^{m+1}}\rho\frac{1}{(k^2u+1)}du = \mathcal{D}_m(\rho)$$

where

$$\mathcal{D}_m = \mathcal{L}_m(\Delta),$$

$$\Delta = \textit{the modular automorphism},$$

$$\mathcal{L}_m(u) = \int_0^\infty \frac{x^m}{(x+1)^{m+1}}\frac{1}{(xu+1)}dx =$$

$$(-1)^m(u-1)^{-(m+1)}\left(\log u - \sum_{j=1}^{m}(-1)^{j+1}\frac{(u-1)^j}{j}\right)$$

(modified logarithm).

Lemma

Let k be an invertible positive element of A_θ^∞. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is given by

$$\zeta(0)+1=\frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k))+\frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k))+$$

$$\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_2(k))+\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_1(k)),$$

where $\varphi(x)=\tau_0(xk^{-2})$, τ_0 is the unique trace on A_θ, Δ is the modular automorphism, and

$$f(u)=\frac{1}{6}u^{-1/2}-\frac{1}{3}+\mathcal{L}_1(u)-2(1+u^{1/2})\mathcal{L}_2(u)+(1+u^{1/2})^2\mathcal{L}_3(u).$$

($\mathcal{L}_m$ is the modified logarithm.)

The following theorem was proved by Alain Connes and Paula Tretkoff for conformal parameter $\tau = i$, and then for all conformal parameters by Farzad Fathizadeh and M.K.

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ be an invertible positive element. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is independent of k.

Proof.

$$\varphi(f(\Delta)(\delta_j(k))\delta_j(k)) = 0 \text{ for } j = 1, 2,$$

$$\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) = -\varphi(f(\Delta)(\delta_2(k))\delta_1(k)).$$

Therefore

$$\zeta(0) + 1 = \frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k)) + \frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k)) +$$

$$\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) + \frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_1(k))$$

$$= 0$$

. □

An argument of Moscovici: variational method. A technique of Branson-Orsted in the commutative case can be extended to the NC case, when there is a good pseudodifferential calculus and good resolvent approximation. Write, for $P=$ a NC polynomial in D and elements of $\mathcal{A}$,

$$\operatorname{Tr}\left(Pe^{-tD_{sh}^2}\right) \sim \sum_{j=0}^{\infty} a_j(P,s)\, t^{\frac{j-n-p}{2}} \quad (t \to 0)$$

Term by term differentiate w.r.t. s and observe that $\frac{d}{ds}a_p(s) = 0$. This brings you back to $h = 0$ (still you have to evaluate a zeta value using the spectrum of $\triangle'$ on A_θ).

Back to the Scalar Curvature

But: we are really interested in computing the scalar curvature as a variable function on $\mathbb{T}^2_\theta$. Gauss-Bonnet computes its total integral. Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable regular spectral triple. Consider the zeta function

$$\zeta_D(P, z) = \mathrm{Tr}\,(P|D|^{-z}), \quad P \in \Psi(\mathcal{A}, \mathcal{H}, D)$$

For the NC torus, the scalar curvature can be defined as the functional on the NC torus:

$$a \mapsto \zeta_{\triangle'}(a, 0)$$

Ongoing work: compute the scalar curvature!