Spectral Zeta Functions, Gauss-Bonnet Theorem, and Scalar Curvature for Noncommutative Tori

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Laplace spectrum; commutative background

- $(M, g) =$ closed Riemannian manifold. Laplacian on forms

\[ \triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M), \]

has pure point spectrum:

\[ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \]
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\]

- Fact: Dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of \(M\) are fully determined by the spectrum of \(\Delta\) (on all \(p\)-forms).
First examples: flat tori and round spheres

- Flat tori: $M = \mathbb{R}^m / \Gamma$, $\Gamma \subset \mathbb{R}^m$ a cocompact lattice;

$$\text{spec}(\triangle) = \{4\pi^2 \|\gamma\|^2; \gamma \in \Gamma^*\}$$

$$\varphi_\gamma(x) = e^{2\pi i \langle \gamma, x \rangle} \quad \gamma \in \Gamma^*$$
First examples: flat tori and round spheres

- **Flat tori**: $M = \mathbb{R}^m/\Gamma$, $\Gamma \subset \mathbb{R}^m$ a cocompact lattice;

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  $$\varphi_\gamma(x) = e^{2\pi i \langle \gamma, x \rangle} \quad \gamma \in \Gamma^*$$

- **Round sphere $S^n$. Eigenvalues**

  $$\bar{\lambda}_k = k(k + n - 1) \quad k = 0, 1, \cdots,$$

  with multiplicity $\binom{n+k}{k} - \binom{n+k-2}{k-2}$. 
In particular $\lambda_1(S^n) = n$ with eigenfunctions

$$\{x^1, \cdots, x^{n+1}\}$$

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Eigenspace of $\bar{\lambda}_k$: Harmonic polynomials of degree $k$.

Except for very few cases, no general formulas are known for eigenvalues.
Patterns in eigenvalues

Hard to find any pattern in eigenvalues in general, except, perhaps, that their growth is determined by the dimension of the manifold:

\[ \lambda_k \sim C k^{\frac{2}{m}} \quad k \to \infty \]
Patterns in eigenvalues

- Hard to find any pattern in eigenvalues in general, except, perhaps, that their growth is determined by the dimension of the manifold:

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- But this is far from obvious, and clues as to why such a statement should be true, and what C should be, first came from spectroscopy, and in particular attempts to find the black body radiation formula.
Method of proof: bring in the heat kernel

- Heat equation for functions: $\partial_t + \triangle = 0$
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- Heat equation for functions: $\partial_t + \Delta = 0$

- $k(t, x, y) =$ kernel of $e^{-t\Delta}$. Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}} \left( a_0(x, \Delta) + a_1(x, \Delta)t + a_2(x, \Delta)t^2 + \cdots \right)$$
Method of proof: bring in the heat kernel

- Heat equation for functions: $\partial_t + \triangle = 0$

- $k(t, x, y) = \text{kernel of } e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

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- $a_i(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.
Theorem: $a_i(x, \triangle)$ are universal polynomials in curvature tensor $R$ and its covariant derivatives:

\[
\begin{align*}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) & \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360} (2|R(x)|^2 - 2|Ric(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \ldots 
\end{align*}
\]
Heat trace asymptotics

Compute Trace\( (e^{-t\triangle}) \) in two ways:

Spectral Sum = Geometric Sum.

\[
\sum e^{-t\lambda_i} = \int_M k(t, x, x) d\text{vol}_x \sim (4\pi t)^{-\frac{m}{2}} \sum_{j=0}^{\infty} a_j t^j \quad (t \to 0).
\]

Hence

\[
a_j = \int_M a_j(x, \triangle) d\text{vol}_x,
\]

are manifestly spectral invariants:

\[
a_0 = \int_M d\text{vol}_x = \text{Vol}(M), \quad \implies \text{Weyl’s law}
\]

\[
a_1 = \frac{1}{6} \int_M S(x) d\text{vol}_x, \quad \text{total scalar curvature}
\]
Tauberian theory and $a_0 = 1$, implies Weyl’s law:

$$N(\lambda) \sim \frac{\text{Vol} (M)}{(4\pi)^{m/2} \Gamma(1 + m/2)} \lambda^{m/2} \quad \lambda \to \infty,$$

where

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

is the eigenvalue counting function.
Simplest example: flat tori

\( \Gamma \subset \mathbb{R}^m \) a cocompact lattice; \( M = \mathbb{R}^m / \Gamma \)

\[ \text{spec}(\triangle) = \left\{ 4\pi^2 \| \gamma^* \|^2; \; \gamma^* \in \Gamma^* \right\} \]

Then:

\[ K(t, x, y) = \frac{1}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2 \| x - y + \gamma \|^2 / 4t} \]

Poisson summation formula \( \Rightarrow \)

\[ \sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \| \gamma^* \|^2 t} = \frac{\text{Vol}(M)}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2 \| \gamma \|^2 / 4t} \]

And from this we obtain the asymptotic expansion of the heat trace near \( t = 0 \)

\[ \text{Tr} e^{-t \Delta} \sim \frac{\text{Vol}(M)}{(4\pi t)^{m/2}} \quad (t \to 0) \]
Application 1: heat equation proof of the Atiyah-Singer index theorem

- Dirac operator

\[ D : C^\infty(S_+) \to C^\infty(S_-) \]

McKean-Singer formula:

\[ \text{Index}(D) = \text{Tr}(e^{-tD^*D}) - \text{Tr}(e^{-tDD^*}), \quad \forall t > 0 \]
Application 1: heat equation proof of the Atiyah-Singer index theorem

- Dirac operator
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  McKean-Singer formula:
  \[
  \text{Index}(D) = \text{Tr}(e^{-tD^*D}) - \text{Tr}(e^{-tDD^*}), \quad \forall t > 0
  \]

- Heat trace asymptotics \[\implies\]
  \[
  \text{Index}(D) = \int_M a_n(x) dx,
  \]
  where \( a_n(x) = a_n^+(x) - a_n^-(x) \), \( m = 2n \), can be explicitly computed and recovers the A-S integrand (The simplest proof is due to Getzler).
Application 2: meromorphic extension of spectral zeta functions

\[ \zeta_\triangle(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \quad \text{Re}(s) > \frac{m}{2} \]

Mellin transform + asymptotic expansion:

\[ \lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1} \, dt \quad \text{Re}(s) > 0 \]

\[ \zeta_\triangle(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\text{Trace}(e^{-t\triangle}) - \text{Dim Ker } \triangle) t^{s-1} \, dt \]

\[ = \frac{1}{\Gamma(s)} \left\{ \int_0^c \cdots + \int_c^\infty \cdots \right\} \]

The second term defines an entire function, while the first term has a meromorphic extension to \( \mathbb{C} \) with simple poles within the set
Also: 0 is always a regular point.

Simplest example: For $M = S^1$ with round metric, we have

$$\zeta_{\triangle}(s) = 2\zeta(2s) \quad \text{Riemann zeta function}$$
Scalar curvature

The spectral invariants $a_i$ in the heat asymptotic expansion

$$\text{Trace}(e^{-t\triangle}) \sim (4\pi t)^{-\frac{m}{2}} \sum_{j=0}^{\infty} a_j t^j \quad (t \to 0)$$

are related to residues of spectral zeta function by

$$\text{Res}_{s=\alpha} \zeta_\triangle(s) = (4\pi)^{-\frac{m}{2}} \frac{a m^{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \quad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6} \int_M S(x) dvol_x$, we obtain a formula for scalar curvature density as follows:
Let $\zeta_f(s) := \text{Tr} \left( f \triangle^{-s} \right)$, $f \in C^\infty(M)$.

$$\text{Res} \, \zeta_f(s) \big|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2 - 1)} \int_M fS(x) \, dvol_x, \quad m \geq 3$$

$$\zeta_f(s) \big|_{s=0} = \frac{1}{4\pi} \int_M fS(x) \, dvol_x - \text{Tr}(fP) \quad m = 2$$
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$$\log \det(\triangle) = -\zeta'(0), \quad \text{Ray-Singer regularized determinant}$$
Noncommutative Geometry: Spectral Triples ($\mathcal{A}, \mathcal{H}, D$)

- $\mathcal{A} =$ involutive unital algebra, $\mathcal{H} =$ Hilbert space,

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \quad D : \mathcal{H} \to \mathcal{H}$$

$D$ has compact resolvent and all commutators $[D, \pi(a)]$ are bounded.

- An asymptotic expansion holds

$$\text{Trace } (e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

- Let $\triangle = D^2$. Spectral zeta function

$$\zeta_D(s) = \text{Tr } (|D|^{-s}) = \text{Tr } (\triangle^{-s/2}), \quad \text{Re}(s) \gg 0.$$
Curved noncommutative tori $A_\theta$

$$A_\theta = C(T^2_\theta) = \text{universal } C^*\text{-algebra generated by unitaries } U \text{ and } V$$

$$VU = e^{2\pi i \theta} UV.$$  

$$A_\theta^\infty = C^\infty(T^2_\theta) = \left\{ \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n : (a_{m,n}) \text{ Schwartz class} \right\}.$$
Differential operators on $A_\theta$

$\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$,

Infinitesimal generators of the action

$\alpha_s(U_m V^n) = e^{is \cdot (m,n)} U^m V^n \quad s \in \mathbb{R}^2$.

Analogues of $\frac{1}{i} \frac{\partial}{\partial x}, \frac{1}{i} \frac{\partial}{\partial y}$ on 2-torus.

Canonical trace $t : A_\theta \to \mathbb{C}$ on smooth elements:

$t\left( \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n \right) = a_{0,0}.$
Complex structures on $A_{\theta}$

- Let $\mathcal{H}_0 = L^2(A_{\theta}) = \text{GNS completion of } A_{\theta} \text{ w.r.t. } t$.

- Fix $\tau = \tau_1 + i\tau_2$, $\tau_2 = \Im(\tau) > 0$, and define

  $\partial := \delta_1 + \tau \delta_2$, $\partial^* := \delta_1 + \bar{\tau} \delta_2$.

- Hilbert space of $(1,0)$-forms:

  $\mathcal{H}^{(1,0)} := \text{completion of finite sums } \sum a \partial b$, $a, b \in A_{\theta}^\infty$, w.r.t.

  $\langle a \partial b, a' \partial b' \rangle := t((a' \partial b')^* a \partial b)$.

- Flat Dolbeault Laplacian: $\partial^* \partial = \delta_1^2 + 2\tau_1 \delta_1 \delta_2 + |\tau|^2 \delta_2^2$. 
Conformal perturbation of the metric

- Fix $h = h^* \in A_{\theta}^\infty$. Replace the volume form $t$ by $\varphi : A_{\theta} \to \mathbb{C}$,

  $$\varphi(a) := t(ae^{-h}), \quad a \in A_{\theta}.$$ 

- It is a twisted trace (in fact a KMS state)

  $$\varphi(ab) = \varphi(b\Delta(a)), \quad \forall a, b \in A_{\theta}.$$ 

  where

  $$\Delta(x) = e^{-h}xe^{h},$$

  is the modular automorphism of a von Neumann factor-has no commutative counterpart.

- Warning: $\triangle$ and $\Delta$ are very different operators!
Connes-Tretkoff spectral triple

- Hilbert space $\mathcal{H}_\varphi := \text{GNS}$ completion of $A_\theta$ w.r.t. $\langle \cdot, \cdot \rangle_\varphi$,

  \[ \langle a, b \rangle_\varphi := \varphi(b^* a), \quad a, b \in A_\theta \]

- View $\partial_\varphi = \partial = \delta_1 + \tau \delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$. and let

  \[ \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi \]

  \[ \mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)} \]

  \[ D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}. \]
Full perturbed Laplacian:

\[ \triangle := D^2 = \begin{pmatrix} \partial_{\phi}^* \partial_{\phi} & 0 \\ 0 & \partial_{\phi} \partial_{\phi}^* \end{pmatrix} : \mathcal{H} \to \mathcal{H}. \]

**Lemma:** \( \partial_{\phi}^* \partial_{\phi} : \mathcal{H}_{\phi} \to \mathcal{H}_{\phi} \), and \( \partial_{\phi} \partial_{\phi}^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)} \) are anti-unitarily equivalent to

\[ k \partial_{\phi}^* \partial_{\phi} : \mathcal{H}_0 \to \mathcal{H}_0, \]
\[ \partial_{\phi}^* k^2 \partial_{\phi} : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}, \]

where \( k = e^{h/2} \).
Scalar curvature for $A_\theta$

- The scalar curvature of the curved nc torus $(\mathbb{T}_\theta^2, \tau, k)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\text{Trace}(a\triangle^{-s})|_{s=0} + \text{Trace}(aP) = t(aR), \quad \forall a \in A_\theta^\infty,$$

where $P$ is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansion for the kernel of the operator $e^{-t\triangle}$, using Connes’ pseudodifferential calculus for nc tori. A good pseudo diff calculus for general nc spaces is still illusive.
Connes’ pseudodifferential calculus

- Symbols: smooth maps $\rho : \mathbb{R}^2 \to A^\infty$. \(\Psi\)DO’s: \(P_\rho : A^\infty \to A^\infty\),

\[
P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is\cdot\xi} \rho(\xi) \alpha_s(a) dsd\xi.
\]

Even for polynomial symbols these integrals are badly divergent; make sense as Fourier integral operators.

- For example:

\[
\rho(\xi_1, \xi_2) = \sum a_{ij} \xi_i^1 \xi_j^2, \quad a_{ij} \in A^\infty \quad \Rightarrow \quad P_\rho = \sum a_{ij} \delta^i_1 \delta^j_2.
\]

- Multiplication of symbol.

\[
\sigma(PQ) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1!\ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).
\]
Local expression for the scalar curvature

- Cauchy integral formula:

\[ e^{-t\triangle} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\triangle - \lambda)^{-1} d\lambda. \]

- \( B_\lambda \approx (\triangle - \lambda)^{-1} : \)

\[ \sigma(B_\lambda) \sim b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots, \]

each \( b_j(\xi, \lambda) \) is a symbol of order \(-2 - j\), and

\[ \sigma(B_\lambda(\triangle - \lambda)) \sim 1. \]

(Note: \( \lambda \) is considered of order 2.)
**Proposition**: The scalar curvature of the spectral triple attached to 
$(A_\theta, \tau, k)$ is equal to

$$\frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_{\mathbb{C}} e^{-\lambda} b_2(\xi, \lambda) \, d\lambda \, d\xi,$$

where $b_2$ is defined as above.
The computations for $k\partial^*\partial k$

- The symbol of $k\partial^*\partial k$ is equal to

$$a_2(\xi) + a_1(\xi) + a_0(\xi)$$

where

$$a_2(\xi) = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,$$

$$a_1(\xi) = 2\xi_1 k \delta_1(k) + 2|\tau|^2 \xi_2 k \delta_2(k) + 2\tau_1 \xi_1 k \delta_2(k) + 2\tau_1 \xi_2 k \delta_1(k),$$

$$a_0(\xi) = k \delta_1^2(k) + |\tau|^2 k \delta_2^2(k) + 2\tau_1 k \delta_1 \delta_2(k).$$

- The equation

$$(b_0 + b_1 + b_2 + \cdots)((a_2 + 1) + a_1 + a_0) \sim 1,$$

has a solution with each $b_j$ a symbol of order $-2 - j$. 
\[ b_0 = (a_2 + 1)^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + 1)^{-1}, \]

\[ b_1 = -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2) b_0 + \partial_2(b_0)\delta_2(a_2) b_0), \]

\[ b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1) b_0 + \partial_2(b_0)\delta_2(a_1) b_0 + \partial_1(b_1)\delta_1(a_2) b_0 + \partial_2(b_1)\delta_2(a_2) b_0 + (1/2)\partial_{11}(b_0)\delta_1^2(a_2) b_0 + (1/2)\partial_{22}(b_0)\delta_2^2(a_2) b_0 + \partial_{12}(b_0)\delta_{12}(a_2) b_0) \]

\[ = 5\xi_1^2 b_0^2 k^3 \delta_1^2(k) b_0 + 2\xi_1^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k + \text{about 800 terms}. \]
Theorem: The scalar curvature of \((A_\theta, \tau, k)\), up to an overall factor of \(-\frac{\pi}{\tau_2}\), is equal to

\[
R_1(\log \Delta)(\triangle_0(\log k)) + \\
R_2(\log \Delta_1, \log \Delta_2) \left( \delta_1(\log k)^2 + |\tau|^2 \delta_2(\log k)^2 + \tau_1 \{ \delta_1(\log k), \delta_2(\log k) \} \right) + \\
iW(\log \Delta_1, \log \Delta_2) \left( \tau_2 \left[ \delta_1(\log k), \delta_2(\log k) \right] \right)
\]
where

\[
R_1(x) = -\frac{1}{2} \frac{\sinh(x/2)}{\sinh^2(x/4)},
\]

\[
R_2(s, t) = (1 + \cosh((s + t)/2)) \times \\
- t(s + t) \cosh s + s(s + t) \cosh t - (s - t)(s + t + \sinh s + \sinh t - \sinh(s + t)) \\
\quad \frac{st(s + t) \sinh(s/2) \sinh(t/2) \sinh^2((s + t)/2)}{st \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)},
\]

\[
W(s, t) = -\frac{(-s - t + t \cosh s + s \cosh t + \sinh s + \sinh t - \sinh(s + t))}{st \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)}.
\]
The limiting case

In the commutative case, the above modular curvature reduces to a constant multiple of the formula of Gauss:

\[
\frac{1}{\tau_2} \delta_1^2 (\log k) + \frac{|\tau|^2}{\tau_2} \delta_2^2 (\log k) + 2 \frac{\tau_1}{\tau_2} \delta_1 \delta_2 (\log k).
\]
First application: Ray-Singer determinant and conformal anomaly (Connes-Moscovici)

Recall: $\log \text{Det}'(\triangle) = -\zeta'_\triangle(0)$, where $\triangle$ is the perturbed Laplacian on $\mathbb{T}_\theta^2$. One has the following *conformal variation formula*. Let $\nabla_i = \log \Delta$ which acts on the $i$-th factor of products.
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**Lemma**
(analogue of Plyakov’s formula) The log-determinant of the perturbed Laplacian $\triangle$ on $\mathbb{T}^2_\theta$ is given by

$$
\log \det'(\triangle) = \log \det'\triangle_0 + \log \varphi(1) - \frac{\pi}{12\tau_2} \varphi_0(h\triangle_0 h) -
\frac{\pi}{4\tau_2} \varphi_0 (K_2(\nabla_1)(\Box_R(h)))
$$
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\]
Second application: the Gauss-Bonnet theorem for $A_\theta$

- Heat trace asymptotic expansion relates geometry to topology, thanks to MacKean-Singer formula:

$$\sum_{p=0}^{m} (-1)^p \text{Tr} (e^{-t\Delta_p}) = \chi(M) \quad \forall t > 0$$

- This gives the spectral formulation of the Gauss-Bonnet theorem:

$$\zeta(0) + 1 = \frac{1}{12\pi} \int_{\Sigma} R \, dv = \frac{1}{6} \chi(\Sigma)$$
**Theorem** (Connes-Tretkoff; Fathizadeh-K.): Let $\theta \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, $k \in A_\theta^\infty$ be a positive invertible element. Then

$$\text{Trace}(\triangle^{-s})_{|s=0} + 2 = t(R) = 0,$$

where $\triangle$ is the Laplacian and $R$ is the scalar curvature of the spectral triple attached to $(A_\theta, \tau, k)$. 
The geometry in noncommutative geometry

- Geometry starts with metric and curvature. While there are a good number of ‘soft’ topological tools in NCG, like cyclic cohomology, K and KK-theory, and index theory, a truly noncommutative theory of curvature is still illusive. The situation is better with scalar curvature, but computations are quite tough at the moment.

- Metric aspects of NCG are informed by Spectral Geometry. Spectral invariants are the only means by which we can formulate metric ideas of NCG.

