Spectral Zeta Functions and Scalar Curvature for Noncommutative Tori

Masoud Khalkhali

Joint work with Farzad Fathizadeh

Sharif University, Tehran, Dec. 2012

- A. Connes and P. Tretkoff, *The Gauss-Bonnet Theorem for the noncommutative two torus*, Sept. 2009.
- A. Connes and H. Moscovici, *Modular curvature for noncommutative two-tori,* Oct. 2011.
- F. Fathizadeh and M. Khalkhali, *The Gauss-Bonnet Theorem for noncommutative two tori with a general conformal structure*, May 2010.
- F. Fathizadeh and M. Khalkhali, *Scalar Curvature for the Noncommutative Two Torus*, Oct. 2011.

Laplace spectrum; commutative background

- (M, g) = closed Riemannian manifold. Laplacian on forms

$$\triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

has pure point spectrum:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Fact: Dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of M are fully determined by the spectrum of Δ (on all p-forms).

Method of proof: bring in the heat kernel

- Heat equation for functions: $\partial_t + \triangle = 0$
- $k(t,x,y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t,x,x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x,\triangle) + a_1(x,\triangle)t + a_2(x,\Delta)t^2 + \cdots)$$

- $a_i(x,\triangle)$, Seeley-De Witt-Gilkey coefficients.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in curvature tensor R and its covariant derivatives:

$$
\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}
$$

Heat trace asymptotics

Compute $\mathrm{Trace}(e^{-t\triangle})$ in two ways:

$$\text{Spectral Sum} = \text{Geometric Sum}.$$

$$\sum e^{-t\lambda_i} = \int_M k(t,x,x)d\mathrm{vol}_x \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0).$$

Hence

$$a_j = \int_M a_j(x, \triangle) d\mathrm{vol}_x,$$

are manifestly spectral invariants:

$$a_0 = \int_M d\mathrm{vol}_x = \mathrm{Vol}(M), \qquad \Longrightarrow \text{Weyl's law}$$

$$a_1 = \frac{1}{6}\int_M S(x) d\mathrm{vol}_x, \qquad \text{total scalar curvature}$$

Tauberian theory and $a_0 = 1$, implies Weyl's law:

$$N(\lambda) \sim \frac{\text{Vol }(M)}{(4\pi)^{m/2}\Gamma(1+m/2)} \lambda^{m/2} \qquad \lambda \to \infty,$$

where

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

is the eigenvalue counting function.

Simplest example: flat tori

- $\Gamma \subset \mathbb{R}^m$ a cocompact lattice; $M = \mathbb{R}^m/\Gamma$

$$\mathrm{spec}(\triangle) = \{4\pi^2||\gamma^*||^2;\ \ \gamma^* \in \Gamma^*\}$$

- Then:

$$K(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2||x-y+\gamma||^2/4t}$$

- Poisson summation formula $\Longrightarrow$

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2||\gamma||^2/4t}$$

- And from this we obtain the asymptotic expansion of the heat trace near $t = 0$

$$\mathrm{Tr} e^{-t\Delta} \sim \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \qquad (t \to 0)$$

Application 1: heat equation proof of the Atiyah-Singer index theorem

- Dirac operator
$$D : C^\infty(S_+) \to C^\infty(S_-)$$
McKean-Singer formula:
$$\mathrm{Index}(D) = \mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*}), \qquad \forall t > 0$$
Heat trace asymptotics $\Longrightarrow$
$$\mathrm{Index}(D) = \int_M a_n(x)dx,$$
where $a_n(x) = a_n^+(x) - a_n^-(x), m = 2n$, can be explicitly computed and recovers the A-S integrand (The simplest proof is due to Getzler).

Application 2: meromorphic extension of spectral zeta functions

$$\zeta_{\triangle}(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

$$\begin{aligned}
\zeta_{\triangle}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle}) - \mathrm{Dim\,Ker}\,\triangle) t^{s-1}\, dt \\
&= \frac{1}{\Gamma(s)} \{ \int_0^c \cdots + \int_c^\infty \cdots \}
\end{aligned}$$

The second term defines an entire function, while the first term has a meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Simplest example: For $M = S^1$ with round metric, we have

$$\zeta_{\triangle}(s) = 2\zeta(2s) \qquad \text{Riemann zeta function}$$

Scalar curvature

The spectral invariants a_i in the heat asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_\triangle(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x) dvol_x$, we obtain a formula for scalar curvature density as follows:

Let $\zeta_f(s) := \mathrm{Tr}\,(f\triangle^{-s}),\ f \in C^\infty(M)$.

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi}\int_M fS(x)dvol_x - \mathrm{Tr}(fP) \qquad m = 2$$

$\log\det(\triangle) = -\zeta'(0),$ Ray-Singer regularized determinant

Noncommutative Geometry: Spectral Triples $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A}$= involutive unital algebra, $\mathcal{H}$ = Hilbert space,

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \qquad D : \mathcal{H} \to \mathcal{H}$$

  D has compact resolvent and all commutators $[D, \pi(a)]$ are bounded.

- An asymptotic expansion holds

$$\mathrm{Trace}\,(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

- Let $\triangle = D^2$. Spectral zeta function

$$\zeta_D(s) = \mathrm{Tr}\,(|D|^{-s}) = \mathrm{Tr}\,(\Delta^{-s/2}), \quad \mathrm{Re}(s) \gg 0.$$

Curved noncommutative tori A_θ

$A_\theta = C(\mathbb{T}^2_\theta)$ = universal C^*-algebra generated by unitaries U and V

$$VU = e^{2\pi i\theta} UV.$$

$$A_\theta^\infty = C^\infty(\mathbb{T}^2_\theta) = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n : (a_{m,n}) \text{ Schwartz class} \Big\}.$$

- Differential operators on A_θ

$$\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty,$$

  Infinitesimal generators of the action

$$\alpha_s(U^m V^n) = e^{is.(m,n)} U^m V^n \quad s \in \mathbb{R}^2.$$

  Analogues of $\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}$ on 2-torus.

- Canonical trace $\mathfrak{t} : A_\theta \to \mathbb{C}$ on smooth elements:

$$\mathfrak{t}(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = a_{0,0}.$$

Complex structures on A_θ

- Let $\mathcal{H}_0 = L^2(A_\theta)$= GNS completion of A_θ w.r.t. $\mathfrak{t}$.
- Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 = \Im(\tau) > 0$, and define
$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$
- Hilbert space of $(1,0)$-forms:
  $\mathcal{H}^{(1,0)} :=$ completion of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, w.r.t.
$$\langle a\partial b, a'\partial b'\rangle := \mathfrak{t}((a'\partial b')^* a\partial b).$$
- Flat Dolbeault Laplacian: $\partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$.

Conformal perturbation of the metric

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form $\mathfrak{t}$ by $\varphi : A_\theta \to \mathbb{C}$,
$$\varphi(a) := \mathfrak{t}(ae^{-h}), \quad a \in A_\theta.$$
- It is a twisted trace (in fact a KMS state)
$$\varphi(ab) = \varphi(b\Delta(a)), \quad \forall a, b \in A_\theta.$$
where
$$\Delta(x) = e^{-h}xe^h,$$
is the modular automorphism of a von Neumann factor-has no commutative counterpart.
- Warning: $\triangle$ and Δ are very different operators!

Connes-Tretkoff spectral triple

- Hilbert space $\mathcal{H}_\varphi :=$ *GNS* completion of A_θ w.r.t. $\langle , \rangle_\varphi$,

$$\langle a, b\rangle_\varphi := \varphi(b^* a), \quad a, b \in A_\theta$$

- View $\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$. and let

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi$$

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Full perturbed Laplacian:

$$\triangle := D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Lemma: $\partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$, and $\partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}$ are anti-unitarily equivalent to

$$k\partial^* \partial k : \mathcal{H}_0 \to \mathcal{H}_0,$$

$$\partial^* k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)},$$

where $k = e^{h/2}$.

Scalar curvature for A_θ

- The scalar curvature of the curved nc torus $(\mathbb{T}^2_\theta, \tau, k)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\text{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \text{Trace}\,(aP) = \mathfrak{t}\,(aR), \qquad \forall a \in A_\theta^\infty,$$

  where P is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansin for the kernel of the operator $e^{-t\triangle}$, using Connes' pseudodifferential calculus for nc tori. A good pseudo diff calculus for general nc spaces is still illusive.

Connes' pseudodifferential calculus

- Symbols: smooth maps $\rho : \mathbb{R}^2 \to A_\theta^\infty$. ΨDO's: $P_\rho : A_\theta^\infty \to A_\theta^\infty$,

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi.$$

  Even for polynomial symbols these integrals are badly divergent; make sense as Fourier integral operators.

- For example:

$$\rho(\xi_1, \xi_2) = \sum a_{ij} \xi_1^i \xi_2^j, \quad a_{ij} \in A_\theta^\infty \quad \Rightarrow \quad P_\rho = \sum a_{ij} \delta_1^i \delta_2^j.$$

- Multiplication of symbol.

$$\sigma(PQ) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1! \ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).$$

Local expression for the scalar curvature

- Cauchy integral formula:

$$e^{-t\triangle} = \frac{1}{2\pi i}\int_C e^{-t\lambda}(\triangle - \lambda)^{-1}\, d\lambda.$$

- $B_\lambda \approx (\triangle - \lambda)^{-1}$:

$$\sigma(B_\lambda) \sim b_0(\xi,\lambda) + b_1(\xi,\lambda) + b_2(\xi,\lambda) + \cdots ,$$

each $b_j(\xi,\lambda)$ is a symbol of order $-2-j$, and

$$\sigma(B_\lambda(\triangle - \lambda)) \sim 1.$$

(Note: λ is considered of order 2.)

Proposition: The scalar curvature of the spectral triple attached to (A_θ, τ, k) is equal to

$$\frac{1}{2\pi i}\int_{\mathbb{R}^2}\int_C e^{-\lambda} b_2(\xi,\lambda)\, d\lambda\, d\xi,$$

where b_2 is defined as above.

The computations for $k\partial^*\partial k$

- The symbol of $k\partial^*\partial k$ is equal to

$$a_2(\xi) + a_1(\xi) + a_0(\xi)$$

where

$$a_2(\xi) = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,$$
$$a_1(\xi) = 2\xi_1 k \delta_1(k) + 2|\tau|^2 \xi_2 k \delta_2(k) + 2\tau_1 \xi_1 k \delta_2(k) + 2\tau_1 \xi_2 k \delta_1(k),$$
$$a_0(\xi) = k\delta_1^2(k) + |\tau|^2 k \delta_2^2(k) + 2\tau_1 k \delta_1 \delta_2(k).$$

- The equation

$$(b_0 + b_1 + b_2 + \cdots)((a_2 + 1) + a_1 + a_0) \sim 1,$$

has a solution with each b_j a symbol of order $-2 - j$.

$$b_0 = (a_2+1)^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + 1)^{-1},$$

$$b_1 = -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0),$$

$$\begin{aligned} b_2 &= -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1)b_0 + \partial_2(b_0)\delta_2(a_1)b_0 + \\ &\quad \partial_1(b_1)\delta_1(a_2)b_0 + \partial_2(b_1)\delta_2(a_2)b_0 + (1/2)\partial_{11}(b_0)\delta_1^2(a_2)b_0 + \\ &\quad (1/2)\partial_{22}(b_0)\delta_2^2(a_2)b_0 + \partial_{12}(b_0)\delta_{12}(a_2)b_0) \end{aligned}$$

$$\begin{aligned} &= 5\xi_1^2 b_0^2 k^3 \delta_1^2(k) b_0 + 2\xi_1^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k \\ &\quad + \text{about 800 terms.} \end{aligned}$$

To perform the $\mathbb{R}^2$ integration and simplify, need a rearrangement lemma (Connes)

The computation of $\int_0^\infty \bullet \, rdr$ of these terms is achieved by:

For all $\rho_j \in A_\theta^\infty$ and $m_j > 0$ one has

$$\int_0^\infty (k^2u+1)^{-m_0}\rho_1(k^2u+1)^{-m_1}\cdots\rho_\ell(k^2u+1)^{-m_\ell}u^{\sum m_j-2}du$$

$$= k^{-2(\sum m_j-1)}F_{m_0,m_1,\cdots,m_\ell}(\Delta_{(1)},\Delta_{(2)},\cdots,\Delta_{(\ell)})(\rho_1\rho_2\cdots\rho_\ell),$$

where

$$F_{m_0,m_1,\cdots,m_\ell}(u_1,u_2,\cdots,u_\ell) = \int_0^\infty (u+1)^{-m}\prod_1^\ell (u\prod_1^j u_h+1)^{-m_j}u^{\sum m_j-2}$$

and $\Delta_{(i)}$ signifies that Δ acts on the i-th factor.

Final formula for the scalar curvature (Connes-Moscovici, Fathizadeh-K, Oct. 2011)

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\big(\triangle_0(\log k)\big)+$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2+|\tau|^2\delta_2(\log k)^2+\tau_1\big\{\delta_1(\log k), \delta_2(\log k)\big\}\Big)+$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2\big[\delta_1(\log k), \delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1 + \cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s-t+t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

The limiting case

In the commutative case, the above modular curvature reduces to a constant multiple of the formula of Gauss:

$$\frac{1}{\tau_2}\delta_1^2(\log k) + \frac{|\tau|^2}{\tau_2}\delta_2^2(\log k) + 2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$

First application: Ray-Singer determinant and conformal anomaly (Connes-Moscovici)

Recall: $\log \mathrm{Det}'(\triangle) = -\zeta'_{\triangle}(0)$, where $\triangle$ is the perturbed Laplacian on $\mathbb{T}^2_\theta$. One has the following *conformal variation formula*. Let $\nabla_i = \log \Delta$ which acts on the i-th factor of products.

Lemma

The log-determinant of the perturbed Laplacian $\triangle$ on $\mathbb{T}^2_\theta$ is given by

$$\log Det'(\triangle) = \log Det' \triangle_0 + \log \varphi(1) - \frac{\pi}{12\tau_2}\varphi_0(h\triangle_0 h) -$$

$$\frac{\pi}{4\tau_2}\varphi_0\left(K_2(\nabla_1)(\square_{\Re}(h))\right),$$

Analogue of Osgood-Phillips-Sarnak functional on the space of selfadjoint elements of A_θ^∞:

$$F(h) := -\log \mathrm{Det}'(\triangle) + \log \varphi(1) \;=\; -\log \mathrm{Det}'\left(e^{\frac{h}{2}} \triangle_0 e^{\frac{h}{2}}\right) + \log \varphi_0(e^{-h}).$$

Since $\quad \triangle_{h+c} = e^c \triangle_h$ for any $c \in \mathbb{R}$, one has

$$\zeta_{\triangle_{h+c}}(z) \;=\; e^{-cz}\, \zeta_{\triangle_h}(z).$$

Therefore

$$\begin{aligned} F(h+c) &= \zeta'_{\triangle_{h+c}}(0) \,+\, \log \varphi_0(e^{-h-c}) \\ &= -c\, \zeta_{\triangle_h}(0) \,+\, \zeta'_{\triangle_h}(0) \,+ \log \varphi_0(e^{-c}) \,+\, \log \varphi_0(e^{-h}) \\ &= F(h). \end{aligned}$$

Theorem

The functional $F(h)$ has the expression

$$F(h) = -(2\log 2\pi + \log(|\eta(\tau)|^4)) + \frac{\pi}{4\tau_2}\varphi_0\left((K_2 - \frac{1}{3})(\nabla_1)(\Box_{\Re}(h))\right).$$

One has $F(h) \geq F(0)$ for all h and equality holds if and only if h is a scalar.

Second application: the Gauss-Bonnet theorem for A_θ

- Heat trace asymptotic expansion relates geometry to topology, thanks to MacKean-Singer formula:

$$\sum_{p=0}^{m}(-1)^p \mathrm{Tr}\left(e^{-t\Delta_p}\right) = \chi(M) \qquad \forall t > 0$$

- This gives the spectral formulation of the Gauss-Bonnet theorem:

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R\, dv = \frac{1}{6}\chi(\Sigma)$$

Theorem (Connes-Tretkoff; Fathizadeh-K.): Let $\theta \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, $k \in A_\theta^\infty$ be a positive invertible element. Then

$$\mathrm{Trace}(\triangle^{-s})_{|_{s=0}} + 2 = \mathfrak{t}(R) = 0,$$

where $\triangle$ is the Laplacian and R is the scalar curvature of the spectral triple attached to (A_θ, τ, k).

The geometry in noncommutative geometry

- Geometry starts with metric and curvature. While there are a good number of 'soft' topological tools in NCG, like cyclic cohomology, K and KK-theory, and index theory, a truly noncommutative theory of curvature is still illusive. The situation is better with scalar curvature, but computations are quite tough at the moment.
- Metric aspects of NCG are informed by Spectral Geometry. Spectral invariants are the only means by which we can formulate metric ideas of NCG.