Mathematics 9302A - Riemann Surfaces
UWO, Math Department
Summer 2019

- **Instructor:** Masoud Khalkhali, Professor of Mathematics, UWO
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  Office hours: TBA, or by appointment.

- **Lectures**: MW 1-3 PM, MC 108.

- **Grading**: Based on a final presentation (%50), Homework (%30) and active class participation (%20). This is a small class so I expect students to attend all the lectures.

- **Historical Introduction**: *Riemann surfaces* were first defined in Riemann's 1851 PhD dissertation [7] where he laid down the geometric foundations of the theory of functions of one complex variable. The theory was advanced much further in Riemann's 1857 monumental paper on the theory of *abelian functions* [8]. It took mathematicians many years to fully grasp, develop, and extend Riemann's ideas to its modern status. Today, the theory of Riemann surfaces occupies a central place in modern mathematics and mathematical physics, specially in areas such as geometric analysis, potential theory and PDE's, number theory, algebraic geometry, string theory, and conformal field theory.

  One should however not get the impression that Riemann surface theory started in Riemann's thesis! Riemann was on the shoulders of giants like Abel and Jacobi, not to mention Cauchy's work in complex analysis, and the works of Gauss, Legendre, and Euler on *elliptic integrals and elliptic functions*. In fact as the names of some of the most important theorems and concepts in the subject would suggest, (e.g. *Abel's addition theorem, the Abel-Jacobi map, Jacobian of a Riemann surface*, etc.), many deep results, at least in their germinal forms, were obtained by Abel and Jacobi more than twenty years prior to Riemann's thesis. It is fair to say that Riemann surfaces were invented (or discovered!) by Riemann in order to put the whole theory of *algebraic functions in one variable*, as well as the theory of abelian functions, on a firm geometric and analytic basis and to forge ahead equipped with this new geometric insight.

This subject has somehow two distinct flavors: algebraic geometric and complex manifold theoretic. One of Riemann's results that we shall prove in this course states that any compact Riemann surface is algebraic, i.e. is the set of solutions of a homogeneous polynomial equation. Thus the two approaches are equivalent as long as we work over $\mathbb{C}$. With the appearance of Weyl's 1913 classic [9], Riemann's approach and his results found its definitive form. For more on the history of the subject and its development one can check [6, 2] and references therein. Remmert's article carefully follows the evolution of the subject from Riemann to present times from the point of view of complex manifold theory, while Dieudonné treats it as part of algebraic geometry.

Textbook: O. Forster, *Lectures on Riemann surfaces*. This is a very clear and lucid presentation of some of the main topics that we shall cover in this course.

Resources: The literature in Riemann surface theory is quite vast. Here are some additional references you can benefit from. This list is by no means comprehensive.

• *Riemann Surfaces*, by Simon Donaldson, Oxford University Press. Among all modern expositions of the theory, this one is particularly close to Riemann's original analytic approach based on potential theory, PDE's and ideas of mathematical physics. It is also remarkable in its breath and scope and the clarity of exposition while keeping the required prerequisites to a bare minimum.

• J.-B. Bost,*Introduction to compact Riemann surfaces, Jacobians, and abelian varieties*, in From number theory to physics (Les Houches, 1989), Springer, Berlin.1992, pp. 64–211. This is a particularly inspiring text full of historical background and references to original works (check Abel's amazing 2-page proof of his addition theorem, or his discovery of what later came to be known as the *genus* of a curve).

• H. M. Farkas, I. Kra, *Riemann Surfaces.*

• J. Jost, *Compact Riemann Surfaces.*

• F. Kirwan, *Complex Algebraic Curves.* This one is more on the algebraic geometry side and at a more elementary level. Very well written and suitable for an advanced undergraduate course.

• H. McKean, V. Moll, *Elliptic curves: function theory, geometry, arithmetic.* A good place to start learning about connections between Riemann surfaces and arithmetic. Covers Kronecker-Weber and Mordel-Weil theorems. Very close to the style of originals.

Course Outine: Here is a tentative list of topics I plan to cover.

• Riemann surfaces and holomorphic maps between them, examples: algebraic curves, Riemann surfaces from analytic continuation, differential equations, conformal structures. Problem of moduli.

• Covering spaces and monodromy, Riemann's existence theorem.

• Differential forms, de Rham and Dolbeault cohomology, Laplace operators, harmonic forms, The Dirichlet principle and Hodge theory à la Riemann.

• Elliptic integrals and functions, doubly periodic function, Weierstrass $\wp$-function, The field of meromorphic functions, Theta functions, classification of Riemann surfaces of genus one, the modular curve.

• The Riemann-Hurwitz formula, the degree-genus formula. Field of meromorphic functions, birational equivalence, connections with algebraic number theory. Hyperbolic surfaces, Gauss-Bonnet theorem.

• The main theorem for compact Riemann surfaces and its consequences: The Riemann-Roch formula, the uniformization theorem, automorphisms of Riemann surfaces, Weierstrass points.

• Divisors, line bundles, sheaf cohomology, Jacobi's inversion problem, Abel-Jacobi map, Jacobian of Riemann surfaces. Abelian varieties and abelian functions.

• Moduli spaces of Riemann surfaces, Beltrami differentials, compactification of moduli spaces.

• (Time permitting) Belyi's theorem, Grothendieck's dessins dénfant.

- **Conflict exams**: If you have a conflict with one of the exam times, please consult the Faculty of Science policy on missed course work. Based on that, if you think your situation qualifies you to take the conflict exam, please contact me as soon as possible, no later than a week before the exam in question.
- **Medical accommodations**: If you are unable to meet a course requirement due to illness or other serious circumstances, you must provide valid medical or other supporting documentation to the Dean's Office as soon as possible and contact me immediately. It is your responsibility to make alternative arrangements with me once the accommodation has been approved. In the event of a missed final exam, a "Recommendation of Special Examination" form must be obtained from the Dean's Office. For further information, please consult the University policy on medical accommodation.

- **Missed homework**: Late homework will not be accepted. Homeworks can always be submitted in advance. For extended absences or medical emergencies, these are handled the same way as for exams. In that case, a homework grade could be dropped; there will be no make-up homework.

- **Academic integrity**: Working on homework with your peers is allowed, in fact encouraged. However, each student must write their own solutions. Handing in suspiciously similar solutions will be considered an instance of cheating. Scholastic offences are taken seriously and will not be tolerated. For more information, please consult the University policy on scholastic discipline.

- **Accessibility**: Please consult Services for Students with Disabilities (SSD) regarding accessibility services on campus. Please contact me if you require material in an alternate format or other accommodations to make this course more accessible to you.

References

[1] J.-B. Bost, , Introduction to compact Riemann surfaces, Jacobians, and abelian varieties , in From number theory to physics (Les Houches, 1989), Springer, Berlin.1992, pp. 64–211.

[2] J. Dieudonné, History of Algebraic geometry.

[3] S. Donaldson, Riemann Surfaces Oxford Graduate Texts in Mathematics 22, 2011.

[4] H. M. Farkas, I. Kra, , Riemann Surfaces (2nd ed.), Berlin, New York: Springer-Verlag.

[5] J. Jost, Compact Riemann Surfaces, Berlin, New York: Springer-Verlag.

[6] R. Remmert, From Riemann Surfaces to Complex Spaces.

[7] B. Riemann, Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse. Inauguraldissertation, Göttingen 1851; Werke, pp. 3-48.

[8] B. Riemann, Theorie der Abelschen Functionen, J. Reine Angew. Math., 54 (1857), pp. 115–155; Werke, pp. 88-142.

[9] H. Weyl, Die Idee der Riemannschen Fläche. Teubner, Leipzig, 1913. Annotated reedition, 1997. The concept of a Riemann surface (3rd ed.), New York: Dover Publications.