Riemann Surfaces
Assignment 1

1. (Orientation) Let \( B(V) \) denote the set of all \textit{ordered basis} of a finite dimensional real vector space \( V \). Two ordered bases are called \textit{equivalent} if the map that sends one to the other has positive determinant.

   a) Show that this is an equivalence relation on \( B(V) \) and that there are only \textit{two} equivalence classes. Each equivalence class is called an \textit{orientation} for \( V \). Let \( f : V_1 \to V_2 \) be an invertible linear map between oriented vector spaces. Define what it means to say \( f \) is \textit{orientation preserving}.

   b) Let \( W \) be a finite dimensional complex vector spaces and \( V = W_{\mathbb{R}} \) its underlying real vector space. Show that \( V \) has a canonical orientation. If \( f : W_1 \to W_2 \) is a \( \mathbb{C} \)-linear invertible map, show that the induced map between real vector spaces is orientation preserving. How all this is related to the fact that a Riemann surface has a canonical orientation and an (invertible) holomorphic map is orientation preserving?

2. Let \( X \) be a compact connected Riemann surface and \( F : X \to \mathbb{C} \) be a holomorphic function. Show that \( F \) is constant. (Hint: use maximum principle from complex analysis.)

3. A \textit{meromorphic function} on a Riemann surface \( X \) is, by definition, a holomorphic map \( F : X \to \mathbb{C}P^1 \) which is not identically equal to \( \infty \).

   a) Define \textit{zeros} and \textit{poles} and their \textit{orders} for a meromorphic function. (You have to show that your definitions are independent of the choice of holomorphic coordinates). Why the \textit{residue} of a meromorphic function at a pole is not well-defined?

   b) Let \( \frac{f(z)}{g(z)} \) be a rational function. Show that it defines a meromorphic function on \( \mathbb{C}P^1 \). Find its zeros and poles and their orders in terms of linear factorizations of \( f \) and \( g \). What is the \textit{degree} of this map? Let \( n_i \)
denote the order of zeros and $p_j$ the orders of poles of a meromorphic function on $\mathbb{C}P^1$. Show that

$$\sum n_i = \sum p_j.$$  

c) Let $K(X)$ denote the set of meromorphic functions on $X$. Show that $K(X)$ is naturally a field.

d) Show that any meromorphic function on $\mathbb{C}P^1$ is a rational function and hence $K(\mathbb{C}P^1) = \mathbb{C}(z)$ is the field of rational functions in one variable.

e) Show that the poles and zeros of a meromorphic function on $\mathbb{C}P^1$ can be placed anywhere you wish, provided they are the same in number.

f) Let $p_1, p_2, \ldots, p_n$ be a collection of points on $\mathbb{C}P^1$, repetitions permitted, and let $L$ be the space of meromorphic functions with poles of orders at most $d_i$ at $p_i$. Show that $L$ is a complex vector space of dimension $\sum d_i + 1$.

4. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_{2g+2}$ be real numbers.

a) Sketch a graph of real points of the curve

$$y^2 = \prod_{i=1}^{2g+2} (x - \alpha_i)$$

b) By using homogenization, show that by adding two points, one obtains a compact Riemann surface.

5. Prove Euler’s formula for homogeneous functions we used in class.

6. Show that the set of points $(x, y)$ is $\mathbb{C}^2$ where $y^2 = \sin(x)$ is naturally a Riemann surface.
7. a) Use Liouville’s theorem to show that $\text{Aut}(\mathbb{C})$ consists of maps $z \to az + b$ for $a \neq 0$.

b) Show that the automorphisms of the Riemann sphere are given by Möbius maps $\text{PSL}(2, \mathbb{C})$.

c) Use the Schwartz Lemma to identify the stabilizer of $0$ is $\text{Aut}(D)$ and hence identify $\text{Aut}(D)$ and $\text{Aut}(H)$.

8. Let $X(R_1, R_2)$ denote the open annular region between concentric circles of radius $R_1$ and $R_2$ in the plane. Show that $X(R_1, R_2)$ is (biholomorphically) equivalent to $X(R'_1, R'_2)$ iff

$$\frac{R_1}{R_2} = \frac{R'_1}{R'_2}.$$ 

Conclude that there are uncountably many inequivalent Riemann surfaces with the topology of an annular region (= cylinder).