# Scalar Curvature, Gauss-Bonnet Theorem and Einstein-Hilbert Action for Noncommutative Tori

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## What is curvature?

Classical geometry: $R^i_{jkl}$, $R_{ij}$, $R$.

Einstein-Hilbert action: $\int_M R\, dvol$.

Einstein field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$

Chern-Weil theory: $\mathrm{Tr}(e^{\Omega})$ $\Omega^i_j = R_{ijkl} dx^k \wedge dx^l$.

## Curvature in NCG

Connection-Curvature formalism of Connes in 1981 (NC Chern-Weil theory):

$$\nabla : E \to E \otimes_A \Omega^1 A, \qquad \nabla \in \mathsf{End}_{\mathbb{C}}(E \otimes_A \Omega A)$$

$$\nabla^2 \in \mathsf{End}_{\Omega A}(E \otimes_A \Omega A) = \mathsf{End}_A(E) \otimes_A \Omega A.$$

Any cyclic cocycle $\varphi : A^{\otimes(2n+1)} \to \mathbb{C}$ defines a closed graded trace $\int_\varphi : \Omega A \to \mathbb{C}$. Can define $\int_\varphi \mathsf{Tr}(e^{\Omega})$, etc. But won't discuss it here.

In particular in his 1981 paper Connes shows how to define the curvature of vector bundles over NC tori using this idea.

How to define the scalar curvature of a spectral triple $(A, H, D)$? This is also answered by Connes since late 1980's and is based on ideas of spectral geometry. But computing it in concrete examples is only acheived in the last few years!

A spectral triple is a NC Riemannian manifold. It is tempting to think that one might be able to define a Levi-Civita type connection for a spectral triple and then define the curvature of this connection. For many reasons this does not work in NCG in general.

## Spectral geometry

- $(M, g)$ = closed Riemannian manifold. Laplacian on forms

$$\triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

has pure point spectrum:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Fact: Dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of $M$ are fully determined by the spectrum of $\Delta$ (on all $p$-forms).

## Heat trace asymptotics

- Heat equation for functions: $\partial_t + \triangle = 0$
- $k(t, x, y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, \triangle)+a_1(x, \triangle)t+a_2(x, \Delta)t^2+\cdots)$$

- $a_i(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in the curvature tensor $R = R^1_{jkl}$ and its covariant derivatives:

$$
\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathsf{Ric}(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}
$$

## Spectral Triples

Noncommutative geometric spaces are described by spectral triples:

$$(\mathcal{A}, \mathcal{H}, D),$$

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \qquad (*\text{-representation}),$$

$$D = D^* : Dom(D) \subset \mathcal{H} \to \mathcal{H},$$

$$D\,\pi(a) - \pi(a)\,D \in \mathcal{L}(\mathcal{H}).$$

**Examples.**

$$\big(C^\infty(M), L^2(M, S), D = \text{Dirac operator}\big).$$

$$\Big(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), \frac{1}{i}\,\frac{\partial}{\partial x}\Big).$$

## Noncommutative Local Invariants

The local geometric invariants such as scalar curvature of $(A, \mathcal{H}, D)$ are detected by the high frequency behavior of the spectrum of $D$ and the action of $A$ via heat kernel asymptotic expansions of the form

$$\text{Trace}\left(a\, e^{-tD^2}\right) \sim_{t \searrow 0} \sum_{j=0}^{\infty} a_j(a, D^2)\, t^{(-n+j)/2}, \qquad a \in A.$$

## Noncommutative 2-Torus $A_\theta = C(\mathbb{T}^2_\theta)$

It is the universal $C^*$-algebra generated by $U$ and $V$ s.t.

$$U^* = U^{-1},$$

$$V^* = V^{-1},$$

$$VU = e^{2\pi i\theta} UV,$$

where $\theta \in \mathbb{R}$ is fixed.

The geometry of the Kronecker foliation $dy = \theta dx$ on the ordinary torus $\mathbb{R}^2/\mathbb{Z}^2$ is closely related to the structure of this algebra.

A representation of $A_\theta$:

$$U\xi(x) = e^{2\pi i x}\xi(x), \qquad V\xi(x) = \xi(x+\theta), \qquad \xi \in L^2(\mathbb{R}).$$

## Action of $\mathbb{T}^2 = (\frac{\mathbb{R}}{2\pi\mathbb{Z}})^2$ on $A_\theta$ and Smooth Elements

- $$\alpha_s : A_\theta \to A_\theta, \quad s \in \mathbb{R}^2,$$
  $$\alpha_s(U^m V^n) = e^{is.(m,n)} U^m V^n, \qquad m, n \in \mathbb{Z}.$$

- $$A_\theta^\infty := \{a \in A_\theta; \qquad s \mapsto \alpha_s(a) \text{ is smooth from } \mathbb{R}^2 \text{ to } A_\theta\}$$
  $$= \Big\{ \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n \in A_\theta; \quad (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \Big\}.$$

- $$\delta_j = \frac{\partial}{\partial s_j} \Big|_{s=0} \alpha_s \ : A_\theta^\infty \to A_\theta^\infty.$$

## The Derivations $\delta_1, \delta_2$ and the Volume Form

- $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$ are defined by:

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V,$$

$$\delta_i(a\,b) = \delta_i(a)\,b + a\,\delta_i(b), \qquad a, b \in A_\theta^\infty.$$

- Tracial state $\varphi_0 : A_\theta \to \mathbb{C}$ (analog of integration):

$$\varphi_0\,(1) = 1, \qquad \varphi_0\,(U^m V^n) = 0 \quad \text{if} \quad (m,n) \neq (0,0).$$

## Conformal Structure on $A_\theta$ (Connes)

The Dolbeault operators associated with $\tau \in \mathbb{C}, \Im(\tau) > 0$ are

$$\partial = \delta_1 + \bar{\tau}\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)},$$
$$\bar{\partial} = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(0,1)}.$$

The conformal structure represented by $\tau$ is encoded in

$$\psi(a,b,c) = -\varphi_0\big(a\,\partial(b)\,\bar{\partial}(c)\big), \qquad a,b,c \in A_\theta^\infty,$$

which is a positive Hochschild cocycle.

## Conformal Perturbation (Connes-Tretkoff)

Let $h = h^* \in A_\theta^\infty$ and replace the trace $\varphi_0$ by

$$\varphi : A_\theta \to \mathbb{C},$$

$$\varphi(a) := \varphi_0(a\, e^{-h}), \qquad a \in A_\theta.$$

$\varphi$ is a KMS state with the modular group

$$\sigma_t(a) = e^{ith}\, a\, e^{-ith}, \qquad a \in A_\theta,$$

and the modular automorphism

$$\Delta(a) := \sigma_i(a) = e^{-h}\, a\, e^{h}, \qquad a \in A_\theta.$$

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big), \qquad a, b \in A_\theta.$$

## A Spectral Triple $(A_\theta^\infty, \mathcal{H}, D)$

$$\mathcal{H} := \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

$$D := \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

$$\partial_\varphi := \partial = \delta_1 + \bar{\tau}\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

## Anti-Unitary Equivalence of the Laplacians

$$D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}.$$

**Lemma:** Let

$$k = e^{h/2}.$$

We have

$$\begin{aligned} \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi \qquad &\sim \qquad k\bar{\partial}\partial k : \mathcal{H}_0 \to \mathcal{H}_0, \\ \partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)} \qquad &\sim \qquad \bar{\partial} k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}. \end{aligned}$$

(The Tomita anti-unitary map $J$ is used.)

## Conformal Geometry of $\mathbb{T}^2_\theta$ with $\tau = i$ (Cohen-Connes, late 80's)

Let

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \qquad \text{be the eigenvalues of } \partial_\varphi^* \partial_\varphi,$$

and

$$\zeta(s) = \sum \lambda_j^{-s}, \qquad \Re(s) > 1.$$

Then

$$\zeta(0) + 1 =$$

$$\varphi\big(f(\Delta)(\delta_1(e^{h/2}))\,\delta_1(e^{h/2})\big) + \varphi\big(f(\Delta)(\delta_2(e^{h/2}))\,\delta_2(e^{h/2})\big),$$

where

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1+u^{1/2})\mathcal{L}_2(u) + (1+u^{1/2})^2\mathcal{L}_3(u),$$

$$\mathcal{L}_m(u) = (-1)^m (u-1)^{-(m+1)}\Big(\log u - \sum_{j=1}^{m} (-1)^{j+1} \frac{(u-1)^j}{j}\Big).$$

## The Gauss-Bonnet theorem for $\mathbb{T}^2_\theta$

**Theorem.** (Connes-Tretkoff; Fathizadeh-Kh.) For any $\theta \in \mathbb{R}$, complex parameter $\tau \in \mathbb{C} \setminus \mathbb{R}$ and Weyl conformal factor $e^h, h = h^* \in A_\theta^\infty$, we have

$$\zeta(0) + 1 = 0.$$

## Final Part of the Proof

$$\zeta(0)+1=$$

$$\frac{2\pi}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_1(\frac{h}{2}))\,\delta_1(\frac{h}{2})\Big)+\frac{2\pi|\tau|^2}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_2(\frac{h}{2}))\,\delta_2(\frac{h}{2})\Big)$$

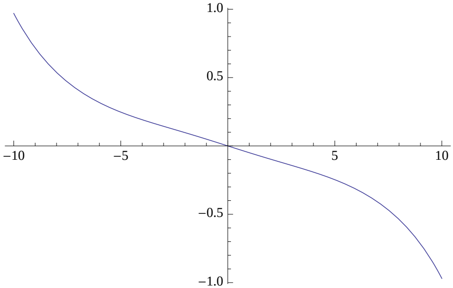
$$+\frac{2\pi\Re(\tau)}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_1(\frac{h}{2}))\,\delta_2(\frac{h}{2})\Big)+\frac{2\pi\Re(\tau)}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_2(\frac{h}{2}))\,\delta_1(\frac{h}{2})\Big),$$

where

$$K(x)=-\frac{\left(3x-3\sinh\left(\frac{x}{2}\right)-3\sinh(x)+\sinh\left(\frac{3x}{2}\right)\right)\mathsf{csch}^2\left(\frac{x}{2}\right)}{3x^2}$$

is an odd entire function, and $\nabla=\log\Delta$.

$$K(x) = -\frac{x}{20} + \frac{x^3}{2240} - \frac{23x^5}{806400} + O\left(x^6\right).$$



## Scalar Curvature for $(A_\theta^\infty, \mathcal{H}, D)$

It is the unique element $R \in A_\theta^\infty$ such that

$$\zeta_a(0) = \varphi_0(a\,R), \qquad a \in A_\theta^\infty,$$

where

$$\zeta_a(s) := \mathrm{Trace}(a\,|D|^{-2s}), \qquad \mathrm{Re}(s) \gg 0.$$

Equivalently, consider small-time heat kernel expansions:

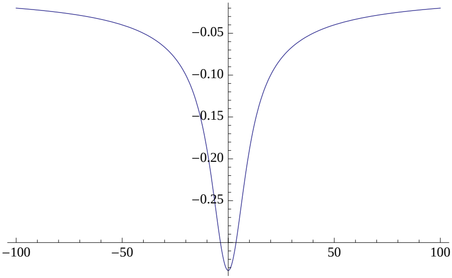
$$\mathrm{Trace}(a\,e^{-tD^2}) \sim \sum_{n \geq 0} B_n(a, D^2)\,t^{\frac{n-2}{2}}, \qquad a \in A_\theta^\infty.$$

## Final Formula for the Scalar Curvature of $\mathbb{T}^2_\theta$

**Theorem.** (Connes-Moscovici; Fathizadeh-Kh.) Up to an overall factor of $\frac{-\pi}{\Im(\tau)}$, $R$ is equal to

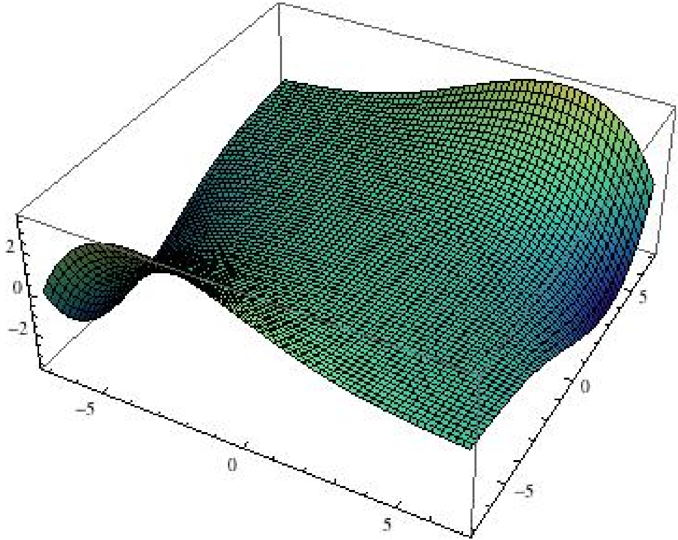
$$R_1(\nabla)\big(\delta_1^2(\frac{h}{2}) + 2\,\tau_1\,\delta_1\delta_2(\frac{h}{2}) + |\tau|^2\,\delta_2^2(\frac{h}{2})\big)$$
$$+R_2(\nabla,\nabla)\Big(\delta_1(\frac{h}{2})^2 + |\tau|^2\,\delta_2(\frac{h}{2})^2 + \Re(\tau)\,\big\{\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})\big\}\Big)$$
$$+i\,W(\nabla,\nabla)\Big(\Im(\tau)\,[\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})]\Big).$$

$$R_1(x) = \frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)}.$$



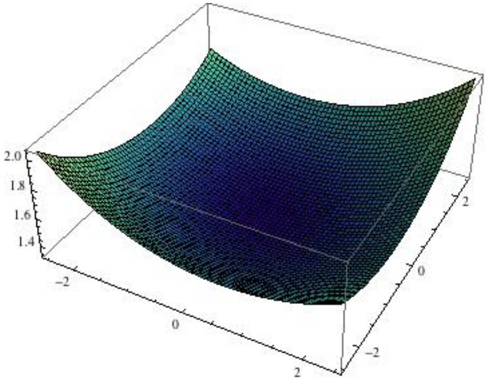
$R_2(s,t) =$

$$-\frac{(1+\cosh((s+t)/2))(-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t)}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$



$$W(s,t) =$$

$$\frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$



## Derivation of the Asymptotic Expansion

Approximate $e^{-tD^2}$ by pseudodifferential operators:

$$e^{-tD^2} = \frac{1}{2\pi i} \int_C e^{-t\lambda} \, (D^2 - \lambda)^{-1} \, d\lambda,$$

$$B_\lambda \, (D^2 - \lambda) \approx 1,$$

$$\sigma(B_\lambda) = b_0 + b_1 + b_2 + \cdots \, .$$

## Connes' pseudodifferential calculus (1980)

- Symbols $\rho : \mathbb{R}^2 \to A_\theta^\infty \Rightarrow P_\rho : A_\theta^\infty \to A_\theta^\infty$

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \, \rho(\xi) \, \alpha_s(a) \, ds \, d\xi, \qquad a \in A_\theta^\infty.$$

- Differential operators:

$$\rho(\xi_1, \xi_2) = \sum a_{ij} \, \xi_1^i \, \xi_2^j, \quad a_{ij} \in A_\theta^\infty \quad \Rightarrow \quad P_\rho = \sum a_{ij} \, \delta_1^i \, \delta_2^j.$$

- $\Psi$DO's on $A_\theta^\infty$ form an algebra:

$$\sigma(P\,Q) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1! \ell_2!} \, \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \, \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).$$

## Rearrangement Lemma

For any $m = (m_0, m_1, \ldots, m_\ell) \in \mathbb{Z}_{>0}^{\ell+1}$ and $\rho_1, \ldots, \rho_\ell \in A_\theta^\infty$

$$\int_0^\infty u^{|m|-2}(e^h u + 1)^{-m_0} \prod_1^\ell \rho_j \, (e^h u + 1)^{-m_j} \, du$$

$$= e^{-(|m|-1)h} \, F_m(\Delta, \ldots, \Delta)\Big( \prod_1^\ell \rho_j \Big),$$

where

$$F_m(u_1, \ldots, u_\ell) = \int_0^\infty \frac{x^{|m|-2}}{(x+1)^{m_0}} \prod_1^\ell \Big( x \prod_1^j u_k + 1 \Big)^{-m_j} \, dx.$$

# Examples of $F_m$

$$F_{(3,4)}(u) = \frac{60u^3\log(u) + (u-1)(u(u(3(u-9)u-47)+13)-2)}{6(u-1)^6u^3}$$

$$F_{(2,2,1)}(u,v) =$$

$$\frac{(v-1)\Big((u-1)(uv-1)(u(u(v-1)+v)-1)-u^2(v-1)(2uv+u-3)\log(uv)\Big)+(u(2v-3)+1)(uv}{(u-1)^3u^2(v-1)^2(uv-1)^2}$$

## Identities Relating $\delta_i(e^h)$ and $\delta_i(h)$

$$e^{-h}\,\delta_i(e^h) = g_1(\Delta)\big(\delta_i(h)\big),$$

$$e^{-h}\,\delta_i^2(e^h) = g_1(\Delta)\big(\delta_i^2(h)\big) + 2\,g_2(\Delta_{(1)},\Delta_{(2)})\big(\delta_i(h)\,\delta_i(h)\big),$$

where

$$g_1(u) = \frac{u-1}{\log u},$$

$$g_2(u,v) = \frac{u(v-1)\log(u) - (u-1)\log(v)}{\log(u)\log(v)(\log(u)+\log(v))}.$$

## Noncommutative 4-Torus $\mathbb{T}^4_\theta$

$C(\mathbb{T}^4_\theta)$ is the universal $C^*$-algebra generated by 4 unitaries

$$U_1, U_2, U_3, U_4,$$

satisfying

$$U_k U_\ell = e^{2\pi i \theta_{k\ell}} U_\ell U_k,$$

for a skew symmetric matrix

$$\theta = (\theta_{k\ell}) \in M_4(\mathbb{R}).$$

## Action of $\mathbb{T}^4 = (\mathbb{R}/2\pi\mathbb{Z})^4$ on $C(\mathbb{T}^4_\theta)$

$$\mathbb{R}^4 \ni s \mapsto \alpha_s \in \mathsf{Aut}\Big(C(\mathbb{T}^4_\theta)\Big),$$

$$\alpha_s(U^m) := e^{is\cdot m}\, U^m, \quad U^m := U_1^{m_1}\, U_2^{m_2}\, U_3^{m_3}\, U_4^{m_4}, \quad m_j \in \mathbb{Z}.$$

$$\delta_j = \frac{\partial}{\partial s_j}\Big|_{s=0}\, \alpha_s \ : C^\infty(\mathbb{T}^4_\theta) \to C^\infty(\mathbb{T}^4_\theta),$$

$$\begin{aligned} \delta_j(U_k) &:= U_k \qquad \text{if} \qquad k = j, \\ &:= 0 \qquad \text{if} \qquad k \neq j. \end{aligned}$$

## Complex Structure on $\mathbb{T}^4_\theta$

$$\partial = \partial_1 \oplus \partial_2, \qquad \bar{\partial} = \bar{\partial}_1 \oplus \bar{\partial}_2,$$
$$\partial_1 = \frac{1}{2}(\delta_1 - i\delta_3), \qquad \partial_2 = \frac{1}{2}(\delta_2 - i\delta_4),$$
$$\bar{\partial}_1 = \frac{1}{2}(\delta_1 + i\delta_3), \qquad \bar{\partial}_2 = \frac{1}{2}(\delta_2 + i\delta_4).$$

## Volume Form on $\mathbb{T}^4_\theta$

$$\varphi_0 : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi_0(1) := 1,$$

$$\varphi_0(U_1^{m_1}\, U_2^{m_2}\, U_3^{m_3}\, U_4^{m_4}) := 0, \qquad (m_1, m_2, m_3, m_4) \neq (0,0,0,0).$$

$$\varphi_0(a\, b) = \varphi_0(b\, a), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

$$\varphi_0(a^*\, a) > 0, \qquad a \neq 0.$$

## Conformal Perturbation (Connes-Tretkoff)

Let $h = h^* \in C^\infty(\mathbb{T}^4_\theta)$ and replace the trace $\varphi_0$ by

$$\varphi : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi(a) := \varphi_0(a\, e^{-2h}), \qquad a \in C(\mathbb{T}^4_\theta).$$

$\varphi$ is a KMS state with the modular group

$$\sigma_t(a) = e^{2ith}\, a\, e^{-2ith}, \qquad a \in C(\mathbb{T}^4_\theta),$$

and the modular automorphism

$$\Delta(a) := \sigma_i(a) = e^{-2h}\, a\, e^{2h}, \qquad a \in C(\mathbb{T}^4_\theta).$$

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

## Perturbed Laplacian on $\mathbb{T}^4_\theta$

$$d = \partial \oplus \bar{\partial} : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)},$$

$$\triangle_\varphi := d^* d.$$

**Remark.** If $h = 0$ then $\varphi = \varphi_0$ and

$$\triangle_{\varphi_0} = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = \partial^* \partial$$

(the underlying manifold is Kähler).

## Scalar Curvature for $\mathbb{T}^4_\theta$

It is the unique element $R \in C^\infty(\mathbb{T}^4_\theta)$ such that

$$\mathsf{Res}_{s=1}\, \zeta_a(s) = \varphi_0(a\, R), \qquad a \in C^\infty(\mathbb{T}^4_\theta),$$

where

$$\zeta_a(s) := \mathsf{Trace}(a\, \triangle_\varphi^{-s}), \qquad \Re(s) \gg 0.$$

## Explicit Formula for $\triangle_\varphi$

**Lemma.** (Fathizadeh-Kh.) Up to an anti-unitary equivalence $\triangle_\varphi$ is given by

$$e^h\bar{\partial}_1 e^{-h}\partial_1 e^h + e^h\,\partial_1 e^{-h}\bar{\partial}_1\, e^h + e^h\bar{\partial}_2 e^{-h}\,\partial_2 e^h + e^h\partial_2 e^{-h}\bar{\partial}_2 e^h,$$

where $\partial_1, \partial_2$ are analogues of the Dolbeault operators.

## Connes' Pseudodifferential Calculus (1980)

A smooth map $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$ is a symbol of order $m \in \mathbb{Z}$, if for any $i, j \in \mathbb{Z}^4_{\geq 0}$, there exists a constant $c$ such that

$$||\partial^j \delta^i \big(\rho(\xi)\big)|| \leq c(1 + |\xi|)^{m-|j|},$$

and if there exists a smooth map $k : \mathbb{R}^4 \setminus \{0\} \to C^\infty(\mathbb{T}^4_\theta)$ such that

$$\lim_{\lambda \to \infty} \lambda^{-m} \rho(\lambda \xi) = k(\xi), \qquad \xi \in \mathbb{R}^4 \setminus \{0\}.$$

- Given a symbol $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$, the corresponding $\psi$DO is:

$$P_\rho(a) = (2\pi)^{-4} \int \int e^{-is.\xi}\, \rho(\xi)\, \alpha_s(a)\, ds\, d\xi, \qquad a \in C^\infty(\mathbb{T}^4_\theta).$$

- Differential operators:

$$\rho(\xi) = \sum a_\ell\, \xi^\ell, \quad a_\ell \in C^\infty(\mathbb{T}^4_\theta) \quad \Rightarrow \quad P_\rho = \sum a_\ell\, \delta^\ell.$$

- $\Psi$DO's on $\mathbb{T}^4_\theta$ form an algebra:

$$\sigma(P\,Q) \sim \sum_{\ell \in \mathbb{Z}^4_{\geq 0}} \frac{1}{\ell!}\, \partial_\xi^\ell \rho(\xi)\, \delta^\ell(\rho'(\xi)).$$

• A symbol $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$ of order $m$ is elliptic if $\rho(\xi)$ is invertible for any $\xi \neq 0$, and if there exists a constant $c$ such that

$$||\rho(\xi)^{-1}|| \leq c(1 + |\xi|)^{-m},$$

when $|\xi|$ is sufficiently large.

• Example of an elliptic operator:

$$\triangle_\varphi = e^h \bar{\partial}_1 e^{-h} \partial_1 e^h + e^h \, \partial_1 e^{-h} \bar{\partial}_1 \, e^h + e^h \bar{\partial}_2 e^{-h} \, \partial_2 e^h + e^h \partial_2 e^{-h} \bar{\partial}_2 e^h.$$

## Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^4$

**Theorem.** (Fathizadeh-Kh.) We have

$$R = e^{-h}\, k(\nabla)\Big(\sum_{i=1}^{4} \delta_i^2(h)\Big) + e^{-h}\, H(\nabla,\nabla)\Big(\sum_{i=1}^{4} \delta_i(h)^2\Big),$$

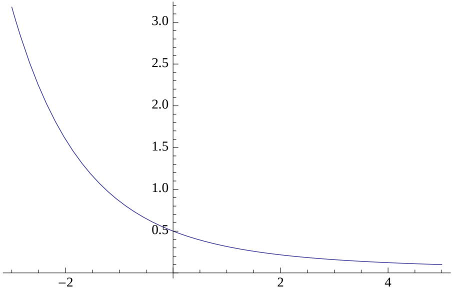
where

$$\nabla(a) = [-h, a], \qquad a \in C(\mathbb{T}_\theta^4),$$

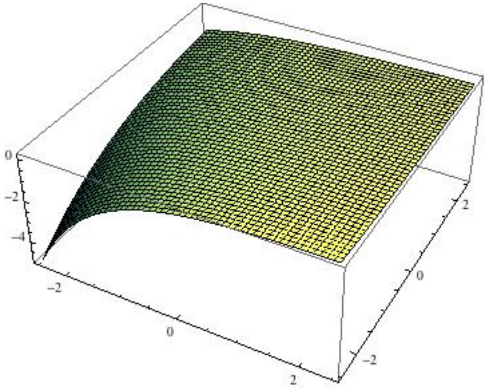
$$k(s) = \frac{1-e^{-s}}{2s},$$

$$H(s,t) = -\frac{e^{-s-t}\left(\left(-e^{s}-3\right)s\left(e^{t}-1\right)+\left(e^{s}-1\right)\left(3e^{t}+1\right)t\right)}{4\,s\,t\,(s+t)}.$$

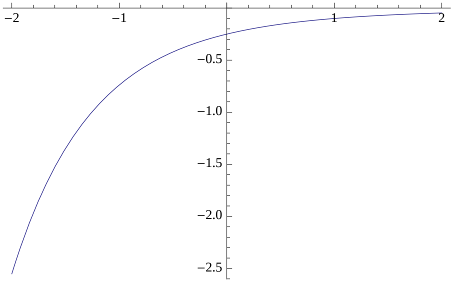
$$k(s) = \frac{1}{2} - \frac{s}{4} + \frac{s^2}{12} - \frac{s^3}{48} + \frac{s^4}{240} - \frac{s^5}{1440} + O\left(s^6\right).$$



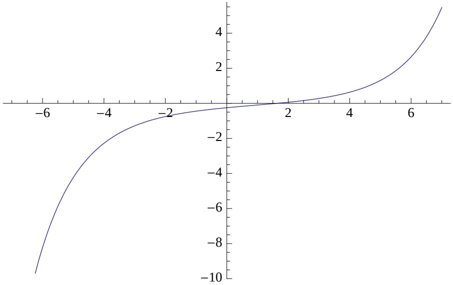
$$
\begin{aligned}
H(s,t) \quad = \quad & \left(-\frac{1}{4}+\frac{t}{24}+O\left(t^{3}\right)\right)+s\left(\frac{5}{24}-\frac{t}{16}+\frac{t^{2}}{80}+O\left(t^{3}\right)\right) \\
& +s^{2}\left(-\frac{1}{12}+\frac{7t}{240}-\frac{t^{2}}{144}+O\left(t^{3}\right)\right)+O\left(s^{3}\right).
\end{aligned}
$$



$$
\begin{aligned}
H(s,s) &= -\frac{e^{-2s}\left(e^{s}-1\right)^{2}}{4s^{2}} \\
&= -\frac{1}{4}+\frac{s}{4}-\frac{7s^{2}}{48}+\frac{s^{3}}{16}-\frac{31s^{4}}{1440}+\frac{s^{5}}{160}+O\left(s^{6}\right).
\end{aligned}
$$



$$
\begin{aligned}
G(s) \quad := \quad H(s,-s) \quad &= \quad \frac{-4s-3e^{-s}+e^{s}+2}{4s^2} \\
&= \quad -\frac{1}{4}+\frac{s}{6}-\frac{s^2}{48}+\frac{s^3}{120}-\frac{s^4}{1440}+\frac{s^5}{5040}+O\left(s^6\right).
\end{aligned}
$$



## Einstein-Hilbert Action for $\mathbb{T}^4_\theta$

**Theorem**. (Fathizadeh-Kh.) We have the local expression (up to a factor of $\pi^2$)

$$\begin{aligned} \varphi_0(R) &= \frac{1}{2}\sum_{i=1}^{4}\varphi_0\Big(e^{-h}\delta_i^2(h)\Big) \\ &\quad + \sum_{i=1}^{4}\varphi_0\Big(G(\nabla)(e^{-h}\delta_i(h))\,\delta_i(h)\Big). \end{aligned}$$

## Extremum of the Einstein-Hilbert Action

**Theorem.** (Fathizadeh-Kh.) For any Weyl factor $e^{-h} \in C^\infty(\mathbb{T}^4_\theta)$

$$\varphi_0(R) \leq 0,$$

and the equality happens if and only if $h$ is a constant.

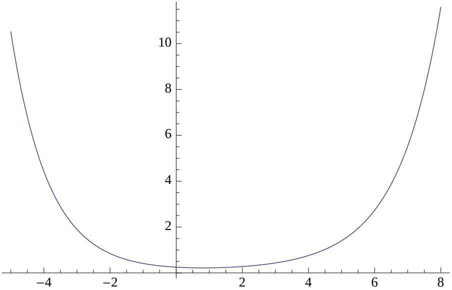
**Proof.**

$$\varphi_0(R) = \sum_{i=1}^{4} \varphi_0\big(e^{-h} T(\nabla)(\delta_i(h))\, \delta_i(h)\big),$$

where

$$T(s) = \frac{1}{2}\frac{e^{-s}-1}{-s} + G(s) = \frac{-2s + e^s - e^{-s}(2s+3) + 2}{4s^2}.$$

$$T(s) = \frac{1}{4} - \frac{s}{12} + \frac{s^2}{16} - \frac{s^3}{80} + \frac{s^4}{288} - \frac{s^5}{2016} + O\left(s^6\right).$$



## Analogue of Weyl's Law for $\mathbb{T}^4_\theta$

**Theorem.** (Fathizadeh-Kh.) For the eigenvalue counting function

$$N(\lambda) = \#\{\lambda_j \leq \lambda\}$$

of the Laplacian $\triangle_\varphi$ on $\mathbb{T}^4_\theta$, we have

$$N(\lambda) \sim \frac{\pi^2 \varphi_0(e^{-2h})}{2} \lambda^2 \qquad (\lambda \to \infty).$$

**Corollary.**

$$\lambda_j \sim \frac{\sqrt{2}}{\pi \varphi_0(e^{-2h})^{1/2}} \, j^{1/2} \qquad (j \to \infty),$$

$$\mathsf{Tr}_\omega\Big((1 + \triangle_\varphi)^{-2}\Big) = \frac{\pi^2}{2} \varphi_0(e^{-2h}).$$

## Karamata's Tauberian theorem

Let $\mu$ be a positive measure on $\mathbb{R}_+$ such that

$$\int_0^\infty e^{-t\lambda} d\mu(\lambda) < \infty, \qquad \forall t > 0,$$

and

$$\lim_{t\to 0^+} t^\alpha \int_0^\infty e^{-t\lambda} d\mu(\lambda) = C$$

for some positive constants $\alpha$ and $C$. Then for any continuous function $f$ on $[0,1]$ we have

$$\lim_{t\to 0^+} t^\alpha \int_0^\infty f(e^{-t\lambda}) e^{-t\lambda} d\mu(\lambda) = \frac{C}{\Gamma(\alpha)} \int_0^\infty f(e^{-t}) t^{\alpha-1} e^{-t}\, dt.$$

## The Dixmier trace $\mathsf{Tr}_\omega : \mathcal{L}^{1,\infty}(\mathcal{H}) \to \mathbb{C}$

For any $T \in \mathcal{K}(\mathcal{H})$, let

$$\mu_1(T) \geq \mu_2(T) \geq \cdots \geq 0$$

be the sequence of eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$.

- $$\mathcal{L}^{1,\infty}(\mathcal{H}) := \Big\{T \in \mathcal{K}(\mathcal{H}); \qquad \sum_{n=1}^{N} \mu_n(T) = O\left(\mathsf{log} N\right)\Big\}.$$
- $$\mathsf{Tr}_\omega(T) := \lim_\omega \Big(\frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T)\Big), \qquad 0 \leq T \in \mathcal{L}^{1,\infty}(\mathcal{H}).$$

## Noncommutative Residue (Wodzicki)

Let $P$ be a classical $\psi$DO acting on smooth sections of a vector bundle $E$ over a closed smooth manifold $M$ of dimension $n$.

- **Definition:**

$$\mathrm{Res}(P) = (2\pi)^{-n} \int_{S^*M} \mathrm{tr}(\rho_{-n}(x,\xi))\, dx\, d\xi,$$

where $S^*M \subset T^*M$ is the unit cosphere bundle on $M$ and $\rho_{-n}$ is the component of order $-n$ of the complete symbol of $P$.

- **Theorem:** Res is the unique trace on $\Psi(M, E)$.

## A Noncommutative Residue for $\mathbb{T}^4_\theta$

Classical symbols: $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$

$$\rho(\xi) \sim \sum_{i=0}^{\infty} \rho_{m-i}(\xi) \qquad (\xi \to \infty),$$

$$\rho_{m-i}(t\,\xi) = t^{m-i}\,\rho_{m-i}(\xi), \qquad t > 0, \quad \xi \in \mathbb{R}^4.$$

**Proposition.** The linear functional

$$\mathrm{Res}(P_\rho) := \int_{\mathbb{S}^3} \varphi_0\big(\rho_{-4}(\xi)\big)\, d\xi$$

is the unique trace on classical pseudodifferential operators on $\mathbb{T}^4_\theta$.

## Analogue of Connes' Trace Theorem for $\mathbb{T}_\theta^4$

**Theorem.** (Fathizadeh-Kh.) For any classical symbol $\rho$ of order $-4$ on $\mathbb{T}_\theta^4$, we have

$$P_\rho \in \mathcal{L}^{1,\infty}(\mathcal{H}_0),$$

and

$$\mathrm{Tr}_\omega(P_\rho) = \frac{1}{4}\mathsf{Res}(P_\rho).$$

**Remark.** Weyl's law is a special case of this theorem: let

$$\rho(\xi) = \frac{1}{(1+|\xi|^2)^2}.$$