Computing the Modular Curvature for the Noncommutative Two Torus

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Recall: Gauss-Bonnet theorem for the NC torus

- A. Connes and P. Tretkoff, The Gauss-Bonnet Theorem for the noncommutative two torus (September 2009, and Sept. 1991).
- F. Fathizadeh and M. Khalkhali, The Gauss-Bonnet Theorem for noncommutative two tori with a general conformal structure (May 2010).

GB for NCT

- $A_\theta =$ universal C^*-algebra generated by unitaries U and V satisfying $VU = e^{2\pi i\theta} UV$,

- $\mathfrak{t} : A_\theta \to \mathbb{C}$, the standard trace,

- $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$; derivations uniquely defined by:

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V.$$

- The Hilbert space

$$\mathcal{H}_0 = L^2(A_\theta, \mathfrak{t}),$$

completion of A_θ w.r.t. inner product

$$\langle a, b \rangle = \mathfrak{t}(b^* a).$$

- The derivations

$$\delta_1, \delta_2 : \mathcal{H}_0 \to \mathcal{H}_0$$

are formally selfadjoint unbounded operators (analogues of $\frac{1}{i}\frac{d}{dx}, \frac{1}{i}\frac{d}{dy}$).

Complex structure

- $\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0,$

$$\partial = \delta_1 + \tau\delta_2, \qquad \partial^* = \delta_1 + \bar{\tau}\delta_2.$$

- Hilbert space of $(1,0)$-forms, $\mathcal{H}^{(1,0)}$, completion of the linear span of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, with

$$\langle a\partial b, a'\partial b'\rangle := \mathfrak{t}(a'^* a\partial b(\partial b')^*).$$

Laplacian

$$\partial = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$$

Unbounded operator; adjoint given by

$$\partial^* = \delta_1 + \bar{\tau}\delta_2.$$

- Laplacian

$$\partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

Conformal perturbation of the metric

- Vary the metric within the conformal class by $h = h^* \in A_\theta^\infty$: Define a new state $\varphi : A_\theta \to \mathbb{C}$ by

$$\varphi(a) = \mathfrak{t}(ae^{-h}), \qquad a \in A_\theta.$$

It is a KMS state with modular group

$$\sigma_t(x) = e^{ith} x e^{-ith}.$$

and modular automorphism

$$\Delta(x) = e^{-h} x e^{h}.$$

The perturbed Laplacian

- $\mathcal{H}_\varphi =$ completion of A_θ w.r.t. $\langle , \rangle_\varphi$,

$$\langle a, b\rangle_\varphi = \varphi(b^* a), \qquad a, b \in A_\theta.$$

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

- The new Laplacian:

$$\partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi.$$

- Lemma: (Connes-Tretkoff) $\partial_\varphi^* \partial_\varphi$ is anti-unitarily equivalent to the positive unbounded operator $k\partial^*\partial k$ acting on $\mathcal{H}_0$, where $k = e^{h/2}$.

Spectral zeta function

Let $\triangle' = \partial_\varphi^* \partial_\varphi$.

$$\zeta(s) = \sum \lambda_i^{-s} = \text{Trace}\,(\triangle'^{-s}), \qquad \text{Re}(s) > 1.$$

Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt$$

gives us

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Trace}^+(e^{-t\triangle'}) t^{s-1} dt,$$

where

$$\mathrm{Trace}^{+}(e^{-t\Delta'}) = \mathrm{Trace}\,(e^{-t\Delta'}) - \mathrm{Dim\ Ker}(\Delta').$$

Asymptotic expansion of the trace of the heat kernel:

$$\mathrm{Trace}\,(e^{-t\Delta'}) \sim \sum_{n=0}^{\infty} a_n t^{\frac{n-1}{2}} \qquad (t \to 0).$$

$\zeta(s)$ has a holomorphic extension to $\mathbb{C} \setminus 1$ with a simple pole at $s = 1$. In particular it is holomorphic at $s = 0$.

The Gauss-Bonnet theorem for NC torus

- Gauss-Bonnet for NC torus (Connes-Tretkoff; Fathizadeh-K.): For all k, θ, τ, the value $\zeta(0)$ of the zeta function ζ of the operator $\partial_\varphi^* \partial_\varphi \sim k \partial^* \partial k$ is given by

$$\zeta(0) + 1 = 0.$$

Remark: A simpler proof, based on variational techniques, was later found by Henri Moscovici.

The Connes-Tretkoff spectral triple

$$(A_\theta, \mathcal{H}, D), \qquad \mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

It is a regular spectral triple w.r.t left A_θ-action, but a twisted spectral triple w.r.t. right unitary action $a \mapsto J_\varphi a^* J_\varphi$: the following operator is bounded:

$$Da^{op} - (k^{-1}ak)^{op} D.$$

Full perturbed Laplacian:

$$\triangle := D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix}.$$

$$\partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi,$$

is anti-unitarily equivalent to

$$k \partial^* \partial k : \mathcal{H}_0 \to \mathcal{H}_0,$$

In a similar manner:

$$\partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}$$

is anti-unitarily equivalent to

$$\partial^* k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}.$$

Modular Curvature for NC Torus

- The modular curvature of the spectral triple attached to $(\mathbb{T}_\theta, \tau, k)$ is the unique element $R \in A_\theta^\infty$ satisfying the equation

$$\zeta_a(0) = \mathfrak{t}(aR), \qquad \forall a \in A_\theta^\infty$$

where

$$\zeta_a(s) := \text{Trace}\,(a|D|^{-s}) \qquad \text{Re}(s) >> 0.$$

- We find a formula for R using Connes' pseudodifferential calculus (1980). Symbols:

$$\rho : \mathbb{R}^2 \to A_\theta^\infty.$$

Local expression for scalar curvature

Cauchy integral formula:

$$e^{-t\triangle} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\triangle - \lambda)^{-1} \, d\lambda$$

one can approximate the inverse of the operator $(\triangle - \lambda)$ by a pseudodifferential operator B_λ whose symbol has an expansion of the form

$$b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots$$

where $b_j(\xi, \lambda)$ is a symbol of order $-2 - j$, and

$$\sigma(B_\lambda(\triangle - \lambda)) \sim 1.$$

Note: λ is considered of order 2.

Prop: The scalar curvature of the spectral triple attached to $(\mathbb{T}_\theta^2, \tau, k)$ is equal to

$$\frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_C e^{-\lambda} b_2(\xi, \lambda) \, d\lambda \, d\xi,$$

where b_2 is defined as above.

By a homogeneity argument, one can set $\lambda = -1$ and multiply the final answer by -1 as in Connes-Tretkoff. Hence, in the sequel we will write $b_j(\xi)$ for $b_j(\xi, -1)$.

The computations for $k\partial^*\partial k$

- The symbol of $k\partial^*\partial k$ is equal to

$$a_2(\xi) + a_1(\xi) + a_0(\xi)$$

where

$$a_2(\xi) = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,$$
$$a_1(\xi) = 2\xi_1 k \delta_1(k) + 2|\tau|^2 \xi_2 k \delta_2(k) + 2\tau_1 \xi_1 k \delta_2(k) + 2\tau_1 \xi_2 k \delta_1(k),$$
$$a_0(\xi) = k\delta_1^2(k) + |\tau|^2 k \delta_2^2(k) + 2\tau_1 k \delta_1 \delta_2(k).$$

- The equation

$$(b_0 + b_1 + b_2 + \cdots)\,\sigma(k\partial^*\partial k + 1) = (b_0 + b_1 + b_2 + \cdots)((a_2 + 1) + a_1 + a_0) \sim 1,$$

has a solution where each b_j can be chosen to be a symbol of order $-2-j$.

$$b_0 = (a_2+1)^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + 1)^{-1},$$

$$b_1 = -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0),$$

$$\begin{aligned} b_2 = & -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1)b_0 + \partial_2(b_0)\delta_2(a_1)b_0 + \\ & \partial_1(b_1)\delta_1(a_2)b_0 + \partial_2(b_1)\delta_2(a_2)b_0 + (1/2)\partial_{11}(b_0)\delta_1^2(a_2)b_0 + \\ & (1/2)\partial_{22}(b_0)\delta_2^2(a_2)b_0 + \partial_{12}(b_0)\delta_{12}(a_2)b_0) \end{aligned}$$

$$\begin{aligned} = & \ 5\xi_1^2 b_0^2 k^3 \delta_1^2(k) b_0 + 2\xi_1^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k \\ & + \text{about 800 terms.} \end{aligned}$$

To integrate b_2 over the ξ-plane, pass to the coordinates

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta, \qquad \xi_2 = \frac{r}{\tau_2}\sin\theta,$$

After the integration with respect to θ, up to a factor of $\frac{r}{\tau_2}$ which is the Jacobian of the change of variables, one gets

$$4|\tau|^2\pi r^6 b_0^2 k^2\delta_2(k) b_0^2 k^3\delta_2(k) b_0 k + 4\tau_1\pi r^6 b_0^2 k^2\delta_2(k) b_0^2 k^3\delta_1(k) b_0 k$$

$$+\ldots \quad \text{(more than 80 terms)}$$

Terms with two b_0^i factors

These are the following terms

$$-4\pi r^4 b_0^3 k^4 \delta_1^2(k) b_0 k - 4|\tau|^2 \pi r^4 b_0^3 k^4 \delta_2^2(k) b_0 k + ...(23 \text{ terms})$$

where

$$b_0 = (r^2 k^2 + 1)^{-1}.$$

The computation of $\int_0^\infty \bullet\, r dr$ of these terms is achieved by

Lemma (Connes and Tretkoff): For any $\rho \in A_\theta^\infty$ and every non-negative integer m, one has

$$\int_0^\infty \frac{k^{2m+2}u^m}{(k^2u+1)^{m+1}}\rho\frac{1}{(k^2u+1)}du = \mathcal{D}_m(\rho),$$

where $\mathcal{D}_m = \mathcal{L}_m(\Delta)$, Δ is the modular automorphism and $\mathcal{L}_m$ is the modified logarithm:

$$\begin{aligned}\mathcal{L}_m(u) &= \int_0^\infty \frac{x^m}{(x+1)^{m+1}}\frac{1}{(xu+1)}dx \\ &= (-1)^m(u-1)^{-(m+1)}\Big(\log u - \sum_{j=1}^{m}(-1)^{j+1}\frac{(u-1)^j}{j}\Big).\end{aligned}$$

Up to an overall factor of π, $\int_0^\infty \bullet\, r dr$ of the terms with two positive powers of b_0 is equal to

$$\begin{aligned} & f_1(\Delta)(k^{-1}\delta_1^2(k)) + f_2(\Delta)(k^{-2}\delta_1(k)^2) \\ + \quad & |\tau|^2 f_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 f_2(\Delta)(k^{-2}\delta_2(k)^2) \\ + \quad & \tau_1 f_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\ + \quad & \tau_1 f_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)), \end{aligned}$$

where

$$\begin{aligned} f_1(u) \; &:= \; -2\mathcal{L}_2(u)u^{1/2} - 2\mathcal{L}_2(u) + \mathcal{L}_1(u)u^{1/2} + 3\mathcal{L}_1(u) - \mathcal{L}_0(u) \\ &= \; -\frac{u^{1/2}(2 - 2u + (1+u)\log u)}{(-1+u^{1/2})^3(1+u^{1/2})^2}, \end{aligned}$$

and

$$f_2(u) := -4\mathcal{L}_2(u) + 4\mathcal{L}_1(u) = 2\frac{-1+u^2-2u\log u}{(-1+u)^3}.$$

Terms with three b_0^i

These terms are the following:

$$4|\tau|^2\pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0 k + 4\tau_1 \pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0 k$$

$$+\cdots \quad \text{(62 terms)}$$

For computing $\int_0^\infty \bullet\, r dr$ of these terms we will use the following lemma in which two variable modified logarithm functions appear:

Lemma: (Connes) For any $\rho, \rho' \in A_\theta^\infty$ and positive integers m, m', we have

$$\int_0^\infty \frac{1}{(k^2u+1)^m} \rho \frac{k^{2(m+m')} u^{m+m'-1}}{(k^2u+1)^{m'}} \rho' \frac{1}{k^2u+1}\, du = \mathcal{D}_{m,m'}(\Delta_{(1)}, \Delta_{(2)})(\rho\rho').$$

The function $\mathcal{D}_{m,m'}$ is defined by

$$\mathcal{D}_{m,m'}(u,v) = \int_0^\infty \frac{1}{(xu^{-1}+1)^m} \frac{x^{m+m'-1}}{(x+1)^{m'}} \frac{1}{xv+1}\, dx,$$

and $\Delta_{(i)}$ signifies the action of Δ on the i-th factor of the product.

After the integrations, up to an overall factor of π, we find the following expression

$$
\begin{aligned}
& f_1(\Delta)(k^{-1}\delta_1^2(k)) + f_2(\Delta)(k^{-2}\delta_1(k)^2) \\
+ \;& F(\Delta_{(1)},\Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k))) \\
+ \;& |\tau|^2 f_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 f_2(\Delta)(k^{-2}\delta_2(k)^2) \\
+ \;& |\tau|^2 F(\Delta_{(1)},\Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \;& \tau_1 f_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\
+ \;& \tau_1 F(\Delta_{(1)},\Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \;& \tau_1 f_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)) \\
+ \;& \tau_1 F(\Delta_{(1)},\Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k))),
\end{aligned}
$$

where we have

$$f_1(u) = -\frac{u^{1/2}(2-2u+(1+u)\log u)}{(-1+u^{1/2})^3(1+u^{1/2})^2},$$

$$f_2(u) = 2\frac{-1+u^2-2u\log u}{(-1+u)^3},$$

$F(u,v) =$

$$\begin{aligned}
&2\mathcal{D}_{2,2}(u,v)u^{-1}v^{1/2} + 2\mathcal{D}_{2,2}(u,v)u^{-1} + 2\mathcal{D}_{2,2}(u,v)u^{-3/2}v^{1/2} \\
&+2\mathcal{D}_{2,2}(u,v)u^{-3/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2} \\
&+4\mathcal{D}_{3,1}(u,v)u^{-5/2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-5/2} - 2\mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} \\
&-2\mathcal{D}_{1,2}(u,v)u^{-1/2} - 4\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} - 6\mathcal{D}_{2,1}(u,v)u^{-1} \\
&-6\mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - 8\mathcal{D}_{2,1}(u,v)u^{-3/2} + 2\mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2} \\
&+4\mathcal{D}_{1,1}(u,v)u^{-1/2}
\end{aligned}$$

$$=$$

$$
\begin{aligned}
&(2u(-(((-1+uv)(1+\sqrt{u}(-1-\sqrt{v}-(-2+\sqrt{u}+u)v+uv^{3/2})))/\\
&((-1+\sqrt{u})(-1+\sqrt{v}))) + (\sqrt{u}\sqrt{v}(-1-\sqrt{u}+u+u(-2-\sqrt{u}+2u)\\
&\sqrt{v}+u(-1+\sqrt{u}+u)v+u^{5/2}v^{3/2})\log u)/((-1+\sqrt{u})^2(1+\sqrt{u}))\\
&+(\sqrt{v}(1-\\
&\sqrt{u}\sqrt{v}(-1-\sqrt{v}+v+uv(-1+\sqrt{v}+v)+\sqrt{u}(-2+\sqrt{v}+2v)))\log v)\\
&((-1+\sqrt{v})^2(1+\sqrt{v}))))/(-1+uv)^3.
\end{aligned}
$$

Computations for $\partial^* k^2 \partial$

The symbol of $\partial^* k^2 \partial$ is equal to $c_2(\xi) + c_1(\xi)$ where

$$c_2(\xi) = \xi_1^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + |\tau|^2 \xi_2^2 k^2,$$

$$c_1(\xi) = (\delta_1(k^2) + \overline{\tau}\delta_2(k^2))\xi_1 + (\tau\delta_1(k^2) + |\tau|^2\delta_2(k^2))\xi_2.$$

After similar computations, the second component of the scalar curvature is:

$$
\begin{aligned}
& g_1(\Delta)(k^{-1}\delta_1^2(k)) + g_2(\Delta)(k^{-2}\delta_1(k)^2) \\
+ \quad & G(\Delta_{(1)},\Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k))) \\
+ \quad & |\tau|^2 g_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 g_2(\Delta)(k^{-2}\delta_2(k)^2) \\
+ \quad & |\tau|^2 G(\Delta_{(1)},\Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \quad & \tau_1 g_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\
+ \quad & \tau_1 G(\Delta_{(1)},\Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \quad & \tau_1 g_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)) \\
+ \quad & \tau_1 G(\Delta_{(1)},\Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k))) \\
+ \quad & i\tau_2 L(\Delta_{(1)},\Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k))) \\
- \quad & i\tau_2 L(\Delta_{(1)},\Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))
\end{aligned}
$$

$$g_1(u) = \frac{-1+u^2-2u\log u}{(-1+u^{1/2})^3(1+u^{1/2})^2},$$

$$g_2(u) = 2\frac{-1+u^2-2u\log u}{(-1+u)^3},$$

$G(u,v) =$

$$\begin{aligned}
&2\mathcal{D}_{2,2}(u,v)u^{-1}v^{1/2} + 2\mathcal{D}_{2,2}(u,v)u^{-1} + 2\mathcal{D}_{2,2}(u,v)u^{-3/2}v^{1/2} \\
&+2\mathcal{D}_{2,2}(u,v)u^{-3/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2} \\
&+4\mathcal{D}_{3,1}(u,v)u^{-5/2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-5/2} - 4\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
&-4\mathcal{D}_{2,1}(u,v)u^{-1} - 4\mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - 4\mathcal{D}_{2,1}(u,v)u^{-3/2} \\
&-\mathcal{D}_{1,2}(u,v)v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2} \\
&-\mathcal{D}_{1,2}(u,v) - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
&-\mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-1} - \mathcal{D}_{2,1}(u,v)u^{-1}
\end{aligned}$$

$$
\begin{aligned}
&-\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2}-\mathcal{D}_{2,1}(u,v)u^{-3/2}-\mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2}\\
&+\mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2}+\mathcal{D}_{1,1}(u,v)v^{1/2}+\mathcal{D}_{1,1}(u,v)u^{-1/2}\\
&+\mathcal{D}_{1,1}(u,v)
\end{aligned}
$$

$$
\begin{aligned}
= \;& -(\sqrt{u}(u(-1+v)^2(-1+uv(-4+u(4+v)))\log(1/u)+(-1+u)\\
&((1+u(-2+v))(-1+v)(-1+uv)(1+uv)+(-1+u)v\\
&(-1+u(-4+v(4+uv)))\log v)))/((-1+\sqrt{u})^2(1+\sqrt{u})(-1+\sqrt{v})^2\\
&(1+\sqrt{v})(-1+uv)^3),
\end{aligned}
$$

$$
\begin{aligned}
L(u,v) \;:=\; & -\mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} - \mathcal{D}_{1,2}(u,v)v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2} \\
& -\mathcal{D}_{1,2}(u,v) - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
& -\mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-1} + \mathcal{D}_{2,1}(u,v)u^{-1} \\
& +\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} + \mathcal{D}_{2,1}(u,v)u^{-3/2} + \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} \\
& +\mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2} + \mathcal{D}_{1,1}(u,v)v^{1/2} + \mathcal{D}_{1,1}(u,v)u^{-1/2} \\
& +\mathcal{D}_{1,1}(u,v)
\end{aligned}
$$

$$
\begin{aligned}
= \; & (\sqrt{u}(u(-1+v)^2 \log(1/u) + (-1+u)((-1+v)(-1+uv) + (v-uv) \\
& \log v)))/((-1+\sqrt{u})^2(1+\sqrt{u})(-1+\sqrt{v})^2(1+\sqrt{v})(-1+uv)).
\end{aligned}
$$

Modular Curvature in Terms of $\log(k)$

To express the curvature in terms of $\log k$, we need identities that relate $k^{-1}\delta_i\delta_j(k)$ and $k^{-2}\delta_i(k)^2$, for $i, j = 1, 2$, to $\log k$:

Lemma: For $i, j = 1, 2$, we have

$$k^{-2}\delta_i(k)\delta_j(k) = 4\frac{\Delta - \Delta^{1/2}}{\log \Delta}(\delta_i(\log k))\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_j(\log k)).$$

Also we have

$$\begin{aligned} k^{-1}\delta_i\delta_j(k) &= 2\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_i\delta_j(\log k)) + g(\Delta_{(1)}, \Delta_{(2)})(\delta_j(\log k)\delta_i(\log k)) \\ &\quad + g(\Delta_{(1)}, \Delta_{(2)})(\delta_i(\log k)\delta_j(\log k)), \end{aligned}$$

where

$$g(u,v) := 4\frac{(\sqrt{uv}-1)\log u - (\sqrt{u}-1)\log(uv)}{\log v \log u \log(uv)},$$

and $\Delta_{(i)}$ signifies the action of Δ on the i-th factor of the product.

Using the above lemma, the first half of the modular curvature, up to an overall factor of $\frac{-\pi}{\tau_2}$, is expressed as:

$$K(\log\Delta)\big(\delta_1^2(\log k)+|\tau|^2\delta_2^2(\log k)+2\tau_1\delta_1\delta_2(\log k)\big)+$$
$$H(\log\Delta_{(1)},\log\Delta_{(2)})\big(\delta_1(\log k)\delta_1(\log k)+|\tau|^2\delta_2(\log k)\delta_2(\log k)$$
$$+\tau_1\delta_1(\log k)\delta_2(\log k)+\tau_1\delta_2(\log k)\delta_1(\log k)\big),$$

where

$$K(x) := -\frac{2e^{x/2}(2 + e^x(-2 + x) + x)}{(-1 + e^x)^2 x},$$

and

$H(s, t) :=$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$

$$-\frac{(s-t)(s+t+\sinh s + \sinh t - \sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}.$$

Final Formula for the Modular Curvature

Theorem: The modular curvature of (T_θ^2, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$
\begin{aligned}
& R_1(\log\Delta)\big(\delta_1^2(\log k) + |\tau|^2\delta_2^2(\log k) + 2\tau_1\delta_1\delta_2(k)\big) \\
+ \quad & R_2(\log\Delta_{(1)}, \log\Delta_{(2)})\Big(\delta_1(\log k)\delta_1(\log k) + |\tau|^2\delta_2(\log k)\delta_2(\log k) \\
& +\tau_1\big(\delta_1(\log k)\delta_2(\log k) + \delta_2(\log k)\delta_1(\log k)\big)\Big) \\
+ \quad & iW(\log\Delta_{(1)}, \log\Delta_{(2)})\Big(\tau_2\big(\delta_1(\log k)\delta_2(\log k) - \delta_2(\log k)\delta_1(\log k)\big)\Big),
\end{aligned}
$$

where

$$R_1(x) := K(x) + S(x) = \frac{2\coth(x/4)}{x} - \frac{1}{2\sinh^2(x/4)} = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) := H(s,t) + T(s,t) = (1+\cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$

and

$$W(s,t) = -\frac{(-s-t+t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

Theorem: The graded modular curvature, up to an overall factor of $\frac{-\pi}{\tau_2}$, is given by

$$
\begin{aligned}
& R_1^{\gamma}(\log\Delta)\big(\delta_1^2(\log k) + |\tau|^2\delta_2^2(\log k) + 2\tau_1\delta_1\delta_2(k)\big) \\
+ \quad & R_2^{\gamma}(\log\Delta_{(1)}, \log\Delta_{(2)})\Big(\delta_1(\log k)\delta_1(\log k) + |\tau|^2\delta_2(\log k)\delta_2(\log k) + \\
& \qquad \tau_1\big(\delta_1(\log k)\delta_2(\log k) + \delta_2(\log k)\delta_1(\log k)\big)\Big) \\
- \quad & iW(\log\Delta_{(1)}, \log\Delta_{(2)})\Big(\tau_2\big(\delta_1(\log k)\delta_2(\log k) - \delta_2(\log k)\delta_1(\log k)\big)\Big),
\end{aligned}
$$

where

$$R_1^\gamma(x) := K(x) - S(x) = -\frac{x + 2\sinh(x/2)}{x + x\cosh(x/2)} = -\frac{\frac{1}{2} + \frac{\sinh(x/2)}{x}}{\cosh^2(x/4)},$$

$$R_2^\gamma(s,t) := H(s,t) - T(s,t) = (1 - \cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$

and

$$W(s,t) = -\frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

The above local expressions for the modular curvature and the functions involved match precisely with Connes-Moscovici's result. (The two experiments gave exactly the same result!)

The limiting case

$$\lim_{x\to 0} R_1(x) = \frac{1}{3}, \quad \lim_{x\to 0} R_1^{\gamma}(x) = -1,$$

$$\lim_{s,t\to 0} R_2(s,t) = \lim_{s,t\to 0} R_2^{\gamma}(s,t) = 0,$$

$$\lim_{s,t\to 0} W(s,t) = \frac{2}{3}.$$

So, in the commutative case, the above modular curvatures reduce to constant multiples of

$$\frac{1}{\tau_2}\delta_1^2(\log k) + \frac{|\tau|^2}{\tau_2}\delta_2^2(\log k) + 2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$