Scalar curvature, Connes' trace theorem, and Einstein-Hilbert action for noncommutative 4-tori

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(joint work with Farzad Fathizadeh)

In my Oberwolfach talk I gave a report on our joint (with Farzad Fathizadeh) new paper *Scalar Curvature for Noncommutative Four-Tori* [9]. In this paper we study the curved geometry of noncommutative 4-tori $\mathbb{T}^4_\theta$. We use a Weyl conformal factor to perturb the standard volume form and obtain the Laplacian that encodes the local geometric information. Connes' pseudodifferential calculus is then used to explicitly compute the terms in the small time heat kernel expansion of the perturbed Laplacian which correspond to the volume and scalar curvature of $\mathbb{T}^4_\theta$. We establish the analogue of Weyl's law, define a noncommutative residue, prove the analogue of Connes' trace theorem, and find explicit formulas for the local functions that describe the scalar curvature of $\mathbb{T}^4_\theta$. We also study the analogue of the Einstein-Hilbert action for these spaces and show that metrics with constant scalar curvature are critical for this action.

Spectral geometry has played an important role in the development of metric aspects of noncommutative geometry [2, 3]. After the seminal paper [5], in which the analogue of the Gauss-Bonnet theorem is proved for noncommutative two tori $\mathbb{T}^2_\theta$ with complex parameter $\tau = i$, there has been much progress in understanding the local differential geometry of these noncommutative spaces [6, 4, 7, 8, 13]. In these works, the flat geometry of $\mathbb{T}^2_\theta$ is conformally perturbed by means of a Weyl factor given by a positive invertible element in $C^\infty(\mathbb{T}^2_\theta)$ (see also [1] for a preliminary version). Connes' pseudodifferential calculus for C^*-dynamical systems is employed crucially to apply heat kernel techniques to geometric operators on $\mathbb{T}^2_\theta$ to derive small time heat kernel expansions that encode local geometric information such as scalar curvature. A purely noncommutative feature is the appearance of the modular automorphism of the Tomita-Takesaki theory for the KMS state implementing the conformal perturbation of the metric in the computations and in the final formula for the curvature [4, 7]. Among other results, in [9] we show that modular automorphism appears also in the final formula for the scalar curvature of the noncommutative 4-tori.

We consider the noncommutative 4-torus $\mathbb{T}^4_\theta$ with the simplest structure of a noncommutative abelian variety. We perturb the standard volume form on this space conformally and analyse the corresponding perturbed Laplacian. Using Connes' pseudodifferential calculus for $\mathbb{T}^4_\theta$ we derive the small time heat kernel expansion for the perturbed Laplacian. This enables us to prove the analogue of Weyl's law for $\mathbb{T}^4_\theta$ by studying the asymptotic distribution of the eigenvalues of the perturbed Laplacian on this space. We define a noncommutative residue on the algebra of classical pseudodifferential operators on $\mathbb{T}^4_\theta$, and show that it gives the unique continuous trace on this algebra. We also prove the analogue of Connes' trace theorem for $\mathbb{T}^4_\theta$ by showing that this noncommutative residue and the Dixmier trace coincide on pseudodifferential operators of order -4. We have performed the computation of the scalar curvature for $\mathbb{T}^4_\theta$, and found explicit formulas for the local functions that describe the curvature in terms of the modular automorphism of the conformally perturbed volume form and derivatives of the logarithm of the Weyl factor. Then, by integrating this curvature, we define and find an explicit formula for the analogue of the Einstein-Hilbert action for $\mathbb{T}^4_\theta$. Finally, we show that the extremum of this action occurs at metrics with constant scalar curvature. We record here some of the main results of our paper [9] that were presented in my talk in Oberwolfach.

Theorem 1. *(Noncommutative Weyl's law) The eigenvalue counting function N of the Laplacian $\triangle_\varphi$ on $\mathbb{T}^4_\theta$ satisfies*

$$N(\lambda) \sim \frac{\pi^2 \varphi_0(e^{-2h})}{2}\lambda^2 \qquad (\lambda \to \infty). \tag{1}$$

Theorem 2. *(Noncommutative Connes' trace theorem) Let ρ be a classical pseudodifferential symbol of order -4 on $\mathbb{T}^4_\theta$. Then P_ρ is a measurable operator in $\mathcal{L}^{1,\infty}(\mathcal{H}_0)$, and under the assumption that all nonzero entries of θ are irrational, we have*

$$⨍ P_\rho = \frac{1}{4}\,\mathrm{res}(P_\rho).$$

Following [3, 4, 7] we define the scalar curvature of $\mathbb{T}^4_\theta$ equipped with the perturbed Laplacian $\triangle_\varphi$ as follows.

Definition 1. *The scalar curvature of the noncommutative 4-torus equipped with the perturbed volume form is the unique element $R \in C^\infty(\mathbb{T}^4_\theta)$ such that*

$$\mathrm{res}_{s=1}\mathrm{Trace}(a\triangle_\varphi^{-s}) = \varphi_0(aR),$$

for any $a \in C^\infty(\mathbb{T}^4_\theta)$.

Theorem 3. *The scalar curvature R of $\mathbb{T}^4_\theta$, up to a factor of π^2, is equal to*

$$e^{-h}K(\nabla)\Big(\sum_{i=1}^{4}\delta_i^2(h)\Big) + e^{-h}H(\nabla_{(1)},\nabla_{(2)})\Big(\sum_{i=1}^{4}\delta_i(h)^2\Big), \tag{2}$$

where

$$K(s) = \frac{1-e^{-s}}{2s},$$
$$H(s,t) = -\frac{e^{-s-t}\left((-e^s-3)\,s\,(e^t-1)+(e^s-1)\,(3e^t+1)\,t\right)}{4st(s+t)}.$$

A natural analogue of the Einstein-Hilbert action for $\mathbb{T}^4_\theta$ is $\varphi_0(R)$, where R is the scalar curvature given by (2). In the following theorem we find an explicit formula for this action.

Theorem 4. *A local expression for the Einstein-Hilbert action for $\mathbb{T}^4_\theta$, up to a factor of π^2, is given by*

$$\varphi_0(R) = \frac{1}{2}\varphi_0\Big(\sum_{i=1}^{4} e^{-h}\delta_i^2(h)\Big) + \varphi_0\Big(\sum_{i=1}^{4} G(\nabla)(e^{-h}\delta_i(h))\delta_i(h)\Big), \tag{3}$$

where $G(s) = \frac{-4s-3e^{-s}+e^{s}+2}{4s^2}$.

The Einstein-Hilbert action $\varphi_0(R)$ attains its maximum if and only if the Weyl factor e^{-h} is a constant. This is done by combining the two terms in the explicit formula (3) for $\varphi_0(R)$, and observing that it can be expressed by a non-negative function. We note that the function G in (3), is neither bounded below nor bounded above.

Theorem 5. *The maximum of the Einstein-Hilbert action is equal to* 0*, and it is attained if and only if the Weyl factor is a constant. That is, for any Weyl factor* $e^{-h}, h = h^* \in C^\infty(\mathbb{T}^4_\theta)$*, we have*

$$\varphi_0(R) \leq 0,$$

and the equality happens if and only if h is a constant.

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