Curvature of the determinant line bundle in noncommutative geometry

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Holomorphic determinants

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- Zeta regularized determinant: For $D \in \mathcal{A}$, let $\Delta = D^* D$.

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- Spectral zeta function:

$$\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \qquad Re(s) \gg 0$$

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- But $D \rightsquigarrow \sqrt{\det \Delta}$ is not holomorphic!
- Quillen's approach: based on determinant line bundle.

The determinant line

- Let $\lambda =$ top exterior power functor. Given $T : V \to W$, let

$$\lambda T : \lambda V \to \lambda W,$$

$$\det T := \lambda T \in (\lambda V)^* \otimes \lambda W \qquad \leftarrow \text{ determinant line}$$

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- Goal: globalize this and construct a line bundle Det $\to$ Fred over Fredholm operators s.t.

$$\mathrm{Det}_T \simeq \lambda(ker T)^* \otimes \lambda(coker T)$$

The determinant line bundle

- Space of Fredholm operators:

$$F = \mathrm{Fred}(H_0, H_1) = \{T : H_0 \to H_1; \;\; T \text{ is Fredholm}\}$$

$$K_0(X) = [X, F]$$

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- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.

$$(DET)_T = \lambda(KerT)^* \otimes \lambda(KerT^*)$$

2) There map $\sigma : F_0 \to DET$

$$\sigma(T) = \begin{cases} 1 & T \quad invertible \\ 0 & otherwise \end{cases}$$

is a holomorphic section of DET over F_0.

Sketch proof

- Open cover: $\text{Fred} = \bigcup U_F, \quad dim F < \infty,$

$$U_F = \{T \in \text{Fred};\ Im(T) + F = H\}$$

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- Open cover: Fred $= \bigcup U_F, \quad dimF < \infty,$

$$U_F = \{T \in \text{Fred};\ Im(T) + F = H\}$$

- Over U_F define

$$\text{Det}_T = \lambda(T^{-1}F)^* \otimes \lambda(F)$$

- Fact: These glue together nicely to define a lie bundle over Fred.

$$0 \to Ker(T) \to T^{-1}F \to F \to coker(T) \to 0$$

shows that

$$\text{Det}_T \simeq \lambda(kerT)^* \otimes \lambda(cokerT)$$

From det section to det function

- Use elliptic theory to pull back DET to a holomorphic line bundle $\mathcal{L} \to \mathcal{A}$ with
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- If $\mathcal{L}$ admits a canonical global section s, then
$$\sigma(D) = det(D)s$$
defines a holomorphic determinant. s is defined once we have a canonical flat connection.

Families of Cauchy-Riemann operators

- $E \to M$ smooth vector bundle over a compact Riemann surface.

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- Let $\mathcal{A} =$ space of $\bar{\partial}$-connections $D : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$ on E.

$$D = \bar{\partial} + A, \qquad A \in \Omega^{0,1}(End(E))$$

  It is an affine space over $\mathcal{B} = \Omega^{0,1}(\mathrm{End}(E))$.

$$\mathcal{A}/\mathcal{G} \simeq \{\text{holomorphic structures on } E\}$$

Quillen's metric

- D is an elliptic 1st order PDE and defines a Fredholm operator

$$D : L^2(E) \to W^1(\Omega^{0,1}(E)), \quad D \in \mathrm{Fred}(H_0, H_1)$$

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- This defines a map $f : \mathcal{A} \to Fred(H_0, H_1)$. Pull back DET along f

$$\mathcal{L} := f^*(DET)$$

- $\mathcal{L}$ is a holomorphic line bundle over $\mathcal{A}$.

Quillen's metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Pick an o. n. basis for $ker(D)$ and $ker(D^*)$. Get a basis v for $\mathcal{L}_D \simeq \lambda(kerD)^* \otimes \lambda(kerD^*)$. Let

$$||v||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta.$$

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

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- A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from

$$\bar{\partial}\partial \log ||s||^2,$$

where s is any local holomorphic frame.

The curvature of $\mathcal{L}$

- A Hermitian metric on $\mathcal{B} = \Omega^{0,1}(EndE)$

$$||B||^2 = \frac{i}{2\pi} \int_M Tr_E(B^* B)$$

where $B = \alpha(z) d\bar{z}, B^* = \alpha(z)^* dz$.

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- Kaehler form of $\mathcal{A}$. Fix $D_0 \in \mathcal{A}$, let $q(D) = ||D - D_0||^2$, and

$$\omega = \partial\bar{\partial}q.$$

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- Theorem (Quillen): The curvature of the determinant line bundle is the symplectic form ω.

A holomorphic determinant

- Modify the metric to get a flat connection:

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- Get a flat holomorphic global section of norm 1. This gives a holomorphic determinant function

$$det(D, D_0) : \mathcal{A} \to \mathbb{C}$$

It satisfies

$$|det(D, D_0)|^2 = e^{||D-D_0||^2} det_\zeta(D^* D)$$

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- Cauchy-Riemann operators on $\mathcal{A}_\theta$:

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with $\alpha \in \mathcal{A}_\theta$.

Connes' Pseudodifferential Calculus

- Symbols of order m: smooth maps $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ with

$$||\delta^{(i_1,i_2)}\partial^{(j_1,j_2)}\sigma(\xi)|| \leq c(1+|\xi|)^{m-j_1-j_2},$$

and there exists a smooth map $k : \mathbb{R}^2 \to A_\theta^\infty$ such that

$$\lim_{\lambda\to\infty} \lambda^{-m}\sigma(\lambda\xi_1, \lambda\xi_2) = k(\xi_1, \xi_2).$$

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- To a symbol σ of order m, one associates an operator

$$P_\sigma(a) = \int\int e^{-is\cdot\xi}\sigma(\xi)\alpha_s(a)\, ds\, d\xi.$$

  The operator P_σ is said to be a pseudodifferential operator of order m.

Classical symbols

- Product formula:

$$\sigma(PQ) \sim \sum_{\ell=(\ell_1,\ell_2)\geq 0} \frac{1}{\ell!} \partial^\ell(\sigma(\xi))\delta^\ell(\sigma'(\xi)).$$

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- Classical symbol of order $\alpha \in \mathbb{C}$: f or any N and each $0 \leq j \leq N$ there exist $\sigma_{\alpha-j} : \mathbb{R}^2\backslash\{0\} \to \mathcal{A}_\theta$ positive homogeneous of degree $\alpha - j$, and a symbol $\sigma^N \in \mathcal{S}^{\Re(\alpha)-N-1}(\mathcal{A}_\theta)$, such that

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.$$

We denote the set of classical symbols of order α by $\mathcal{S}^\alpha_{cl}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi^\alpha_{cl}(\mathcal{A}_\theta)$.

Noncommutative residue

- The Wodzicki residue of a classical pseudodifferential operator P_σ is defined as
$$\mathrm{Res}(P_\sigma) = \varphi_0\left(\mathrm{res}(P_\sigma)\right),$$
where $\mathrm{res}(P_\sigma) := \int_{|\xi|=1} \sigma_{-2}(\xi)d\xi$.

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- t is evident from its definition that Wodzicki residue vanishes on differential operators and on non-integer order classical pseudodifferential operators.

A cutoff integral

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- For $\mathrm{ord}(P) \geq -2$ the integral is divergent, but, assuming P is classical, one has an asymptotic expansion as $R \to \infty$

$$\int_{B(R)} \sigma(\xi)d\xi \sim \sum_{j=0,\alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma)R^{\alpha-j+2} + \beta(\sigma)\log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi)d\xi$.

The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi) d\xi$.

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- The canonical trace of a classical pseudo $P \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ of non-integral order α is defined as

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- Theorem: The functional TR is the analytic continuation of the ordinary trace on trace-class pseudodifferential operators.

- Let $A \in \Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$ be of order $\alpha \in \mathbb{Z}$ and let Q be a positive elliptic classical pseudodifferential operator of positive order q. We have

$$\mathrm{Res}_{z=0}\mathrm{TR}(AQ^{-z}) = \frac{1}{q}\mathrm{Res}(A).$$

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- Proof: For the holomorphic family $\sigma(z) = \sigma(AQ^{-z})$, $z = 0$ is a pole for the map $z \mapsto ⨍ \sigma(z)(\xi)d\xi$ whose residue is given by

$$\mathrm{Res}_{z=0}\left(z \mapsto ⨍ \sigma(z)(\xi)d\xi\right) = -\frac{1}{\alpha'(0)}\int_{|\xi|=1}\sigma_{-2}(0)d\xi$$

$$= -\frac{1}{\alpha'(0)}\mathrm{res}(A).$$

Taking trace on both sides gives the result.

- Prop: We have $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ for any $A, B \in \Psi_{cl}(\mathcal{A}_\theta)$, provided that $ord(A) + ord(B) \notin \mathbb{Z}$.

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- z-derivatives of a classical holomorphic family of symbols is not classical anymore. So we introduce log-polyhomogeneous symbols which include the z-derivatives of the symbols of the holomorphic family $\sigma(AQ^{-z})$.

Logarithmic symbols

- Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

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- Example: $\log Q$ where $Q \in \Psi^q_{cl}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$.

$$\log Q := Q \left. \frac{d}{dz} \right|_{z=0} Q^{z-1} = Q \left. \frac{d}{dz} \right|_{z=0} \frac{i}{2\pi} \int_C \lambda^{z-1} (Q - \lambda)^{-1} d\lambda.$$

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- Wodzicki residue:

$$\mathrm{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.$$

Variations of LogDet and the curvature form

- Recall: for our canonical section

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}$$

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- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$, and compute

$$\bar{\partial}\partial \log \|\sigma\|^2 = \delta_{\bar{w}}\delta_w \zeta'_\Delta(0) = \delta_{\bar{w}}\delta_w \frac{d}{dz}\mathrm{TR}(\Delta^{-z})|_{z=0}.$$

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- The first variational formula is given by

$$\delta_w \zeta(z) = \delta_w \mathrm{TR}(\Delta^{-z}) = \mathrm{TR}(\delta_w \Delta^{-z}) = -z\mathrm{TR}(\delta_w \Delta \Delta^{-z-1}).$$

- Using the properties of TR, we have

$$\delta_w \zeta'(0) = \left. \frac{d}{dz} \delta_w \zeta(z) \right|_{z=0} =$$

$$-\varphi_0 \left(\fint \sigma(\delta_w \Delta \Delta^{-1}) - \frac{1}{2} \mathrm{res}\left(\delta_w \Delta \Delta^{-1} \log \Delta\right) \right).$$

The curvature of the determinant line bundle

- For a holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ is given by :

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{2}\varphi_0\left(\delta_w D\delta_{\bar{w}}\mathrm{res}(\log\Delta\, D^{-1})\right).$$

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- Using the symbol calculus and properties of the canonical trace we prove:
  Theorem (A. Fathi, A. Ghorbanpour, MK.) The curvature of the determinant line bundle for the noncommutative two torus is given by

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right).$$