From Cyclic Cohomology to Hopf Cyclic Cohomology in Five Lectures

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NCGOA07

May 2007, Vanderbilt University, USA

Abstract

These are slides of five lectures I gave during the 5th annual noncommutative geometry conference at Vanderbilt university. The theme of this year's meeting was "Index theory, Hopf Cyclic Cohomology and Noncommutative Geometry".

Hopf cyclic cohomology and its corresponding noncommutative Chern-Weil theory was discovered by Connes and Moscovici in 1998 in the course of their breakthrough analysis of transverse index theory on foliated spaces. The subject has since gone through a rapid development and currently is the subject of intense study. Ideally, in conjunction with Hopf cyclic cohomology, I should have also spoken about the local index formula of Connes-

Moscovici as well but this obviously requires more time and space than allowed in a conference. The lectures started with cyclic cohomology as was discovered by Connes as a receptacle for a noncommutative Chern character map from K-homology and leads to Hopf cyclic cohomology. My emphasis was on examples, main ideas and general patterns and applications rather than a dry formal presentation.

References include:

1. Alain Connes, Noncommutative differential geometry, Publ. Math. IHES no. 62 (1985), 41-144.

2. Alain Connes, Noncommutative geometry, Academic Press (1994).

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mology and the transverse index theorem, Comm. Math. Phys. 198 (1998), no. 1, 199–246.

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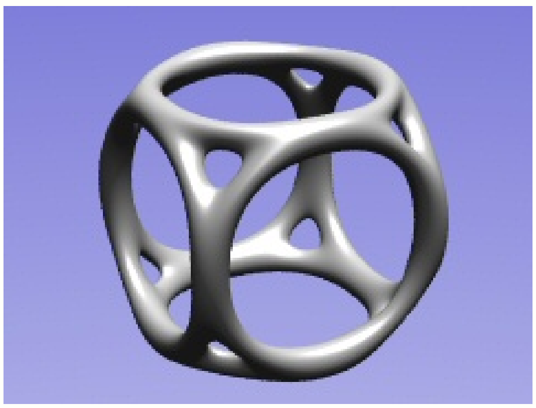
What is Noncommutative Geometry?

..........In mathematics each object (or subject) can be looked at in two different ways:

Geometric or *Algebraic*



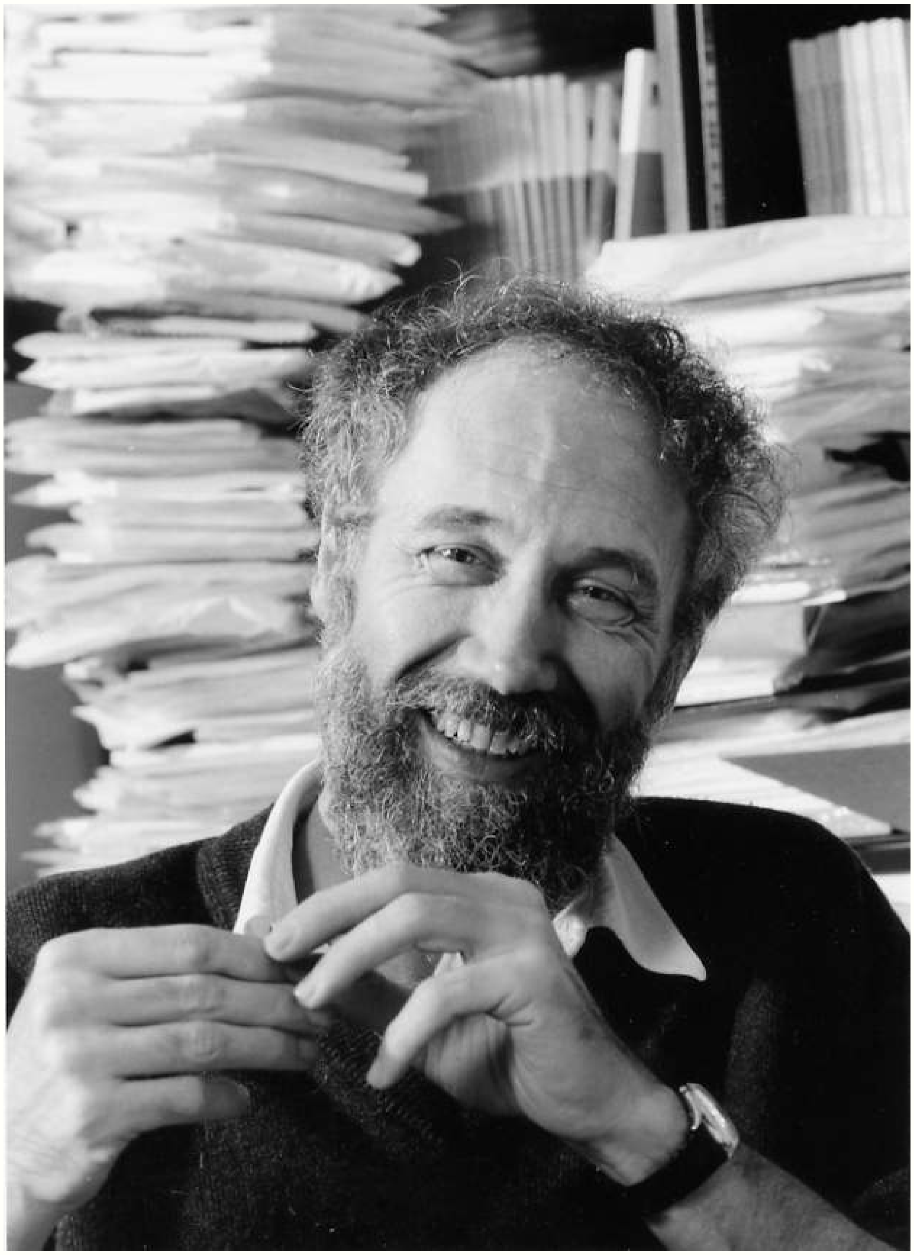
Geometry

$= C^\infty(M)$

Algebra

Classical Geometry $=$ Commutative Algebra

Fundamental duality theorems like Gelfand-Naimark, Serre-Swan, or Hilbert's Nullstellensatz, among others, suggest an equivalence or duality between spaces and commutative algebras.



But Alain Connes, in his Noncommutative Geometry, has taught us that there is a far more fascinating world of noncommutative spaces that have a rich geometry and topology waiting to be explored. Noncommutative Geometry (NCG) builds on this idea of duality between algebra and geometry and vastly extends it by treating special classes of noncommutative algebras as the algebra of coordinates of a noncommutative space.

A paradigm to bear in mind throughout noncommutative geometry is the classical inclusions

smooth ⊂ continuous ⊂ measurable

which reflects in inclusions of algebras

$$C^\infty(M) \subset C(M) \subset L^\infty(M)$$

This permeates throughout NCG: in the NC world one studies a noncommutative space from a

- measure theoretic point of view (von Neumann algebras)
- continuous topological point of view (C*- algebras)
- differentiable point of view (smooth algebras)
- algebraic geometric point of view (abstract associative algebras, abelian or triangulated categories)

Needless to say the subject is already quite mature with many deep theorems and applications. In these lectures we touch only some selected aspects of noncommutative geometry related to cyclic cohomology.

Cyclic Cocycles

In these lectures we shall pursue cyclic cohomology as the right NC analogue of de Rham homology, closely following Alain Connes' 1981 paper, Non commutative Differential Geometry, publication Math IHES (published in 1985) and his 1994 book, Noncommutative Geometry.

We start by giving many examples of cyclic cocycles. Classes of cyclic cocycles modulo exact cyclic cochains define cyclic cohomology.

A motivating example: Let $M =$ closed (i.e. compact without boundary), smooth, oriented, n-manifold and

$$\mathcal{A} = C^\infty(M)$$

the algebra of smooth complex valued functions on M. For $f^0, \cdots, f^n \in \mathcal{A}$, let

$$\varphi(f^0, \cdots, f^n) = \int_M f^0 df^1 \wedge \cdots \wedge df^n.$$

The $(n+1)$-linear cochain

$$\varphi : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n+1} \to \mathbb{C}$$

has the properties:

- φ is continuous in the Fréchet space topology of $\mathcal{A}$.

Topology of $\mathcal{A}$:

$$f_n \to f$$

if for each partial derivative ∂^α,

$$\partial^\alpha f_n \to \partial^\alpha f$$

uniformly (in a coordinate system).

• φ is a Hochschild cocycle, i.e.

$$b\varphi = 0$$

where

$$\begin{aligned} & (b\varphi)(f^0, \cdots, f^{n+1}) = \\ & \sum_{i=0}^{n} (-1)^i \int_M f^0 df^1 \cdots d(f^i f^{i+1}) \cdots df^{n+1} \\ & +(-1)^{n+1} \int_M f^{n+1} f^0 df^1 \cdots df^n \\ = \; & 0, \end{aligned}$$

where we only used the Leibnitz rule for the de Rham differential d and the graded commutativity of the algebra $(\Omega M, d)$ of differential forms on M.

- φ is cyclic:

$$\varphi(f^n, f^0, \cdots, f^{n-1}) = (-1)^n \varphi(f^0, \cdots, f^n)$$

This is more interesting. In fact since

$$\int_M (f^n df^0 \cdots df^{n-1} - (-1)^n f^0 df^1 \cdots df^n)$$

$$= \int_M d(f^n f^0 df^1 \cdots df^{n-1}).$$

we see that the cyclic property of φ follows from Stokes' formula

$$\int_M d\omega = 0,$$

which is valid for any $(n-1)$-form ω on a closed manifold M.

Definition: Let $\mathcal{A}$ be any algebra, commutative or not. A cyclic n-cocycle on $\mathcal{A}$ is an $(n+1)$-linear functional

$$\varphi : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathbb{C}$$

satisfying the two conditions

$$b\varphi = 0, \quad \lambda\varphi = \varphi$$

where

$$(\lambda\varphi)(a^0, \cdots, a^n) = (-1)^n \varphi(a^n, a^0, \cdots, a^{n-1}).$$

Examples:

1) $n = 0$. A cyclic 0-cocycle on $\mathcal{A}$ is a linear map

$$\varphi : \mathcal{A} \to \mathbb{C}$$

with

$$\varphi(ab) = \varphi(ba)$$

for all a and b, i.e. φ is a trace.

2) A cyclic 1-cocycle is a bilinear map

$$\varphi : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$$

with

$$(b\varphi)(a, b, c) = \varphi(ab, c) - \varphi(a, bc) + \varphi(ca, b) = 0$$

$$\varphi(b, a) = -\varphi(a, b)$$

for all a, b, c.

Example (winding number through cyclic 1-cocycles). Let $\mathcal{A} = C^\infty(S^1)$.

$$\varphi(f^0, f^1) = \frac{1}{2\pi i} \int_{S^1} f^0 df^1$$

is a cyclic 1-cocycle on $\mathcal{A}$. Clearly if f is invertible then

$$\varphi(f^{-1}, f) = \frac{1}{2\pi i} \int_{S^1} f^{-1} df = W(f, 0)$$

is the winding number of f.

Thus: cyclic cocycles can carry non-trivial topological information.

Reinterpret: The formula $< [\varphi], [f] >:= \varphi(f^{-1}, f)$ defines a pairing

$$HC^1(\mathcal{A}) \times K_1(\mathcal{A}) \longrightarrow \mathbb{C}$$

Remark: We shall see that this example can be generalized, in the odd case, to a pairing

$$HC^{2n+1}(\mathcal{A}) \times K_1(\mathcal{A}) \longrightarrow \mathbb{C}$$

for any $\mathcal{A}$, commutative or not. And in the even case:

$$HC^{2n}(\mathcal{A}) \times K_0(\mathcal{A}) \longrightarrow \mathbb{C}$$

(Connes-Chern characters for K-theory)

A remarkable property of cyclic cocycles is that, unlike de Rham cocycles which make sense only over smooth manifolds or commutative algebras, they can be defined over any noncommutative algebra and the resulting cohomology theory is the right generalization of de Rham homology of currents on a smooth manifold.

Before developing cyclic cohomology any further we give one more example.

3. Let $V \subset M$ be a closed p-dimensional oriented submanifold. Then

$$\varphi(f^0, \cdots, f^p) = \int_V f^0 df^1 \cdots df^p$$

defines a cyclic p-cocycle on $\mathcal{A}$. We can replace V by any *closed current* on M and obtain more cyclic cocycles.

Recall: A p-dimensional *current* C on M is a continuous linear functional

$$C : \Omega^p M \to \mathbb{C}$$

on the space of p-forms on M. We write $\langle C, \omega \rangle$ instead of $C(\omega)$. For example a zero dimensional current on M is just a distribution on M. The differential of a current is defined by

$$\langle dC, \omega \rangle = \langle C, d\omega \rangle$$

The complex of currents on M:

$$C^0(M) \xleftarrow{d} C^1(M) \xleftarrow{d} C^2(M) \xleftarrow{d} \cdots$$

Its homology is the *de Rham homology* of M.

Let C be a p-dimensional current on M. The $(p+1)$-linear functional

$$\varphi_C(f^0, \cdots, f^p) = \langle C\,,\, f^0 df^1 \cdots df^p \rangle$$

is a Hochschild cocycle on $\mathcal{A}$. If C is closed then φ_C is a cyclic p-cocycle on $\mathcal{A}$. So we have a map

$$\{\text{closed de Rham } p\text{-currents on } M\} \longrightarrow$$

$$\{\text{cyclic } p\text{-cocycles on } C^\infty(M)\}$$

Dually, as we shall see later, there is a map

$$\{\text{cyclic } p\text{-cycles on } C^\infty(M)\} \longrightarrow$$

$$\{\text{closed differential forms on } M\}$$

defined by

$$f^0 \otimes f^1 \otimes \cdots f^n \mapsto f^0 df^1 df^2 \cdots df^n$$

Remark: using these maps, we shall find an exact relation between de Rham homology of currents $H_*^{dR}(M)$ (resp. de Rham cohomology) and cyclic cohomology of $\mathcal{A} = C^\infty(M)$ (resp. cyclic homology of $\mathcal{A}$) (Connes' theorem).

From Quantized Calculus to Cyclic Cocycles

Where cyclic cocycles come from?

Answer: From quantizations of differential and integral calculus, i.e. from noncommutative analogues of $(\Omega M,\, d,\, \int)$:

Definition: An n-dimensional cycle is a triple $(\Omega,\, d,\, \int)$ where $(\Omega,\, d)$ is a differential graded algebra and

$$\int : \Omega^n \to \mathbb{C}$$

is a closed graded trace.

Thus:

$$\Omega = \bigoplus_{n \geq 0} \Omega^n$$

is a graded algebra,

$$d : \Omega \to \Omega$$

is a graded derivation of degree 1 and

$$\int d\omega = 0, \qquad \int [\omega_1, \omega_2] = 0$$

($\omega \in \Omega^{n-1}$ and $[\omega_1, \omega_2] = \omega_1\omega_2 - (-1)^{|\omega_1||\omega_2|}\omega_2\omega_1$.)

Definition: An n-dimensional cycle over an algebra $\mathcal{A}$ is a quadruple

$$(\Omega,\, d,\, \int,\, \rho)$$

where $(\Omega,\, d,\, \int)$ is an n-cycle and $\rho : \mathcal{A} \to \Omega^0$ is an algebra map.

Given an n-cycle $(\Omega, d, \int, \rho)$ over $\mathcal{A}$ its character

$$\varphi : \mathcal{A}^{\otimes(n+1)} \to \mathbb{C}$$

is defined by

$$\varphi(a^0, \cdots, a^n) = \int \rho(a^0) d(\rho a^1) \cdots d(\rho a^n).$$

Claim: φ is a cyclic n-cocycle on $\mathcal{A}$.

Conversely: Any cyclic cocycle on $\mathcal{A}$ is obtained in this way!

Proof: Introduce the DGA of noncommutative differential forms $(\Omega\mathcal{A}, d)$ on $\mathcal{A}$: let $\Omega^0\mathcal{A} = \mathcal{A}$ and $\Omega^n\mathcal{A}=$ linear span of expressions

$$(a^0 + \lambda I)da^1 da^2 \cdots da^n.$$

Alternatively: $(\Omega\mathcal{A} = \oplus_n \Omega^n\mathcal{A}, d)$ is the universal DGA with a non-unital map $\mathcal{A} \to \Omega^0\mathcal{A}$.

d is uniquely defined by

$$d(a^0 + \lambda I)da^1 \cdots da^n = da^0 da^1 \cdots da^n,$$

and the product is defined using the Leibniz rule $d(ab) = adb + da.b$.

Example:

$$(a^0da^1da^2)(b^0db^1) = a^0da^1d(a^2b^0)db^1$$
$$-a^0d(a^1a^2)db^0db^1 + a^0a^1da^2db^0db^1.$$

Define a closed graded n-trace on $(\Omega\mathcal{A}, d)$ by

$$\int (a^0 + \lambda I)da^1 \cdots da^n = \varphi(a^0, a^1, \cdots, a^n).$$

Its character is clearly φ.

Example: Let $\mathcal{A} = \mathbb{C}$ and

$$\Omega = \mathbb{C}[e, de], \quad e^2 = e.$$

Define $\int : \Omega^2 \to \mathbb{C}$ by

$$\int edede = 1$$

It is a closed graded 2-dimensional cycle on A. The corresponding cyclic cocycle on $\mathbb{C}$:

$$\sigma(1,1,1) = 1.$$

Example: Let

$$\Omega^p = \Omega^p(M) \otimes M_k(\mathbb{C})$$

denote matrix valued differential p-forms on M, and define $\int : \Omega^n \to \mathbb{C}$ by

$$\int \omega = \int_M \mathrm{tr}(w).$$

This defines an n-cycle over the algebra

$$C^\infty(M, M_k(\mathbb{C}))$$

of matrix valued functions on M. This example can be generalized:

Example: (Tensor product of cycles)

Let $(\Omega_i, d, \int, \rho)$ be cycles over $\mathcal{A}_i$ for $i = 1, 2$ of dimensions m and n, respectively. Then we have a cycle

$$(\Omega_1 \otimes \Omega_2,\, d,\, \int,\, \rho)$$

over $\mathcal{A}_1 \otimes \mathcal{A}_2$ of dimension $m+n$ defined as a tensor product of DGA's with a closed graded trace

$$\int (\omega_1 \otimes \omega_2) = (\int \omega_1)(\int \omega_2)$$

Tensoring with the 2-dimensioal cycle σ on $\mathbb{C}$ yields a map

$$S : \{\text{cyclic n-cocycles on A}\} \longrightarrow$$

$$\{\text{cyclic n+2-cocycles on A}\}$$

and, as we shall see later, a map

$$S : HC^n(\mathcal{A}) \longrightarrow HC^{n+2}(\mathcal{A}), \quad n = 0, 1, \cdots$$

A necessary digression: Operator Ideals

A major source of quantized calculi and cyclic cocycles are Fredholm Modules and Spectral Triples. These are cycles for (noncommutative) K-homology and Connes' Chern character associates a cyclic cocycle to them:

$$\{\text{p-summable Fredholm modules on A}\} \rightsquigarrow$$

$$\{\text{cyclic cocycles on A}\}$$

To understand this circle of ideas properly some doses of operator ideals, operator trace, and the Dixmier trace is in order.

Let $\mathcal{H}$ = complex (separable) Hilbert space,

$\mathcal{L}(H)$= algebra of bounded linear operators T : $\mathcal{H} \to \mathcal{H}$.

$$||T|| := \sup\{||TX||, ||X|| \leq 1\} < \infty$$

For any T, its adjoint T^* is defined by

$$< TX, Y > = < X, T^*Y > .$$

$\mathcal{L}(H)$ is a C^*-algebra in which every other C^*-algebra can be embedded.

(2-sided) ideals of $\mathcal{H}$:

- $\mathcal{F}(\mathcal{H})$= finite rank operators
- $\mathcal{K}(\mathcal{H})$ = compact operators

$T : \mathcal{H} \to \mathcal{H}$ is called compact if

$$\overline{T(\text{unit ball})} \quad \text{is compact}$$

Facts: $\mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}$ = unique proper closed ideal of $\mathcal{L}(H)$ ($\mathcal{H}$ is separable). For any proper ideal $\mathcal{I}$,

$$\mathcal{F}(\mathcal{H}) \subset \mathcal{I} \subset \mathcal{K}(\mathcal{H})$$

Uncountably many $\mathcal{I}$ (Calkin's classification). They are classified through their singular numbers.

Example: The Schatten ideals $\mathcal{L}^p(\mathcal{H})$, $1 \le p < \infty$

$$T \in \mathcal{L}^p(\mathcal{H}) \quad \text{if} \quad \sum_{n=1}^{\infty} \mu_n(T)^p < \infty$$

$\mu_n(T)$= n-th eigenvalue of $|T| = (T^*T)^{\frac{1}{2}}$

$$\mu_1 \ge \mu_2 \ge \cdots \longrightarrow 0$$

- $\mathcal{L}^p(\mathcal{H})$ is a 2-sided ideal
- $\mathcal{L}^p(\mathcal{H}) \subset \mathcal{L}^q(\mathcal{H})$ if $p \le p$
- For $T \in \mathcal{L}^1(\mathcal{H})$ = trace class operators

$$Tr(T) := \sum_{n=1}^{\infty} < Te_n, e_n >$$

independent of the o.n. basis $\{e_n\}$.

Example: Let

$$T(e_n) = \lambda_n e_n, \quad n = 1, 2, \cdots$$

be a diagonal operator. Then

T is bounded iff $|\lambda_n| \leq M$ for some M and all n

T is compact iff $\lambda_n \to 0$ as $n \to \infty$

$T \in \mathcal{L}^p(\mathcal{H})$ iff $(\lambda_n) \in \ell^p$.

Example: Let

$$(Tf)(x) = \int K(x, y) f(y) d\mu$$

be an integral operator with $K \in L^2(X \times X)$ ($\mathcal{H} = L^2(X, \mu)$). Then $T \in \mathcal{L}^2(\mathcal{H})$ is a Hilbert-Schmidt operator with $||T||_2 = ||K||_2$.

It is harder to get $T \in \mathcal{L}^1(\mathcal{H})$. Certainly K being continuous suffices. Then

$$Tr(T) = \int K(x,x)$$

The Dixmier Trace

Operator trace Tr is the unique (up to scale) positive normal trace on $\mathcal{L}(\mathcal{H})$. Normal= completely additive:

$$Tr(\sum_i T_i) = \sum_i Tr(T_i)$$

for positive $T_i's$. Non-normal positive traces exist on $\mathcal{L}(\mathcal{H})$ (Dixmier, 1966)

Idea of construction: amenability of the ax+b group plus some extra work with singular values.

Connes' approach: Let

$$\mathcal{L}^{1,\infty}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(LogN)\},$$

where

$$\sigma_N(T) = \sum_i^N \mu_i(T).$$

We have strict inclusions

$$\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}^{1,\infty}(\mathcal{H}) \subset \bigcap_{p>1} \mathcal{L}^p(\mathcal{H}).$$

e.g. for T with

$$T(e_n) = \frac{1}{n} e_n,$$

$T \in \mathcal{L}^{1,\infty}(\mathcal{H})$ but $T \notin \mathcal{L}^1(\mathcal{H})$.

Roughly speaking, the Dixmier trace

$$Tr_\omega(T)$$

picks up the logarithmic divergencies of $Tr(T)$. e.g. in 'good cases' when the limit on RHS exists

$$Tr_\omega(T) = \mathrm{Lim} \frac{\sigma_N(T)}{\mathrm{Log} N} \quad \text{as} \quad N \to \infty. \tag{1}$$

In general for $T \in \mathcal{L}^{1,\infty}(\mathcal{H})$, let

$$f_T(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda \sigma_u(T) \frac{du}{u}.$$

f_T is asymptotically additive in T:

$$\operatorname*{Lim}_{\lambda \to \infty} (f_{T_1+T_2}(\lambda) - f_{T_1}(\lambda) - f_{T_2}(\lambda)) = 0.$$

Thus $T \mapsto [f_T]$ sends positive compact operators in $\mathcal{L}^{1,\infty}(\mathcal{H})$ additively to $C_b[1,\infty)/C_0[1,\infty)$:

$$\mathcal{L}^{1,\infty}(\mathcal{H})^+ \longrightarrow C_b[1,\infty)/C_0[1,\infty).$$

Let

$$\omega : C_b[1,\infty)/C_0[1,\infty) \to \mathbb{C}$$

be a state (i.e. positive, linear map, with $\omega(1) = 1$). Then

$$Tr_\omega(T) := \omega([f_T])$$

is a positive, unitary invariant, additive and homogeneous functional on $\mathcal{L}^{1,\infty}(\mathcal{H})^+$ and hence has a unique extension to a positive trace

$$\mathrm{Tr}_\omega : \mathcal{L}^{1,\infty}(\mathcal{H}) \longrightarrow \mathbb{C},$$

called the Dixmier trace.

$\mathrm{Tr}_\omega(T)$ depends on the state ω, but for geometrically defined operators the definition is actually independent of the choice of ω. This happens when the limit in (**??**) exists. In particular, for $T \in \mathcal{L}^1(\mathcal{H})$, a trace class operator, we have

$$\mathrm{Tr}_\omega(T) = 0.$$

Equivalently:

$$\mathrm{Tr}_\omega(T_1) = \mathrm{Tr}_\omega(T_2)$$

if

$$T_1 - T_2 \in \mathcal{L}^1(\mathcal{H})$$

is a trace class operator, i.e. an infinitesimal of first order. This is a cohomological property of Tr_ω which makes it very flexible for computations!

From P-summable Fredholm Modules to Quantized Calculus

The proper context for noncommutative index theory and K-homology is the following notion of an "elliptic operator on a noncommutative space" :

Definition: A p-summable Fredholm module over an algebra $\mathcal{A}$ is a pair $(\mathcal{H}, F)$ where

1. $\mathcal{H}$ is a Hilbert space,
2. $\mathcal{H}$ is a left $\mathcal{A}$-module, i.e. there is an algebra map

$$\mathcal{A} \to \mathcal{L}(\mathcal{H})$$

3. there is an $F \in \mathcal{L}(\mathcal{H})$ with $F^2 = I$ and

$$[F, a] = Fa - aF \in \mathcal{L}^p(\mathcal{H}). \tag{2}$$

for all $a \in \mathcal{A}$.

$(H,\ F)$ is called even if

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

is $\mathbb{Z}_2$-graded, F is odd, and $\mathcal{A}$ acts by even operators.

If instead of (**??**), we just have

$$[F,\ a] \in \mathcal{K}(H), \tag{3}$$

we say we have a Fredholm module over $\mathcal{A}$.

The difference between p-summable Fredholm modules and Fredholm modules is like the difference between p-times differentiable functions and continuous functions. The p-summability condition (**??**) singles out 'smooth subalgebras' $\mathcal{A}$ of a C^*-algebra A.

This principle is easily corroborated in the commutative case, using the fundamental commutation formula

$$[\frac{d}{dx}, f] = f'$$

which relates boundedness of LHS to the regularity of f.

That is why if $\mathcal{A}$ is a C^*-algebra, the natural condition to consider is condition (**??**) instead of (**??**). In general any Fredholm module over a C^*-algebra A defines a series of subalgebras of ‘smooth functions’ in A, via (**??**), which are closed under holomorphic functional calculus and have the same K-theory as A.

Example: Let

$$D : C^\infty(E^+) \to C^\infty(E^-)$$

be an elliptic differential operator on a closed manifold M. Assume D is injective and let

$$F = \frac{D}{|D|} : L^2(E^+) \longrightarrow L^2(E^-)$$

be the phase of D. Let $\mathcal{A} = C^\infty(M)$ act by multiplication on sections of E^+ and E^-. It can be shown that

$$(L^2(E^+) \oplus L^2(E^-), F)$$

is an even p-summable Fredholm module over $\mathcal{A}$ for any $p > n =$ dimension M.

Proposition: let (H, F) be a 1-summable even Fredholm module over $\mathcal{A}$. Its character $\mathrm{Ch}\,(H, F) : \mathcal{A} \to$

$\mathbb{C}$ defined by

$$\mathrm{Ch}\,(H, F)(a) = \frac{1}{2}\mathrm{Tr}\,(\varepsilon F[F,\, a])$$

is a trace on $\mathcal{A}$. (prove it as a good exercise!)

This construction is an example of Connes' Chern character from K-homology $\longrightarrow$ cyclic cohomology. It can be extended to higher dimensions.

The cycle $(\Omega,\, d,\, \int)$ associated to a Fredholm module:

Let (H, F) = even p-summable Fredholm module on $\mathcal{A}$ and $n \geq p$ an integer. Then for all $a \in \mathcal{A}$,

$$da := [F, a] \in \mathcal{L}^n(\mathcal{H}).$$

Let

$$\Omega^q = \text{linear span of operators}$$

$$(a^0 + \lambda I) da^1 da^2 \cdots da^q \in \mathcal{L}^{\frac{n}{q}}(\mathcal{H})$$

and

$$\Omega = \bigoplus_{i=0}^{n} \Omega^i$$

$$d\omega = [F, \omega] = F\omega - (-1)^{|\omega|}\omega F$$

Check: $d^2 = 0$ and

$$d(\omega_1 \omega_2) = d(\omega_1)\omega_2 + (-1)^{|\omega_1|}\omega_1 d(\omega_2)$$

Define: $\int : \Omega^n \to \mathbb{C}$ by

$$\int \omega = \text{Trace}(\varepsilon\omega)$$

where ε is the grading operator.

Check: $\int$ is a closed graded trace.

Using the general construction

n-cycles over A $\longrightarrow$ cyclic n-cocycles over A for each $n \geq p$ we obtain a cyclic n-cocycle φ on $\mathcal{A}$:

$$\varphi(a^0, \cdots, a^n) = \int a^0 da^1 \cdots da^n$$

$$= \text{Trace}\,(\varepsilon a^0 [F, a^1][F, a^2] \cdots [F, a^n])$$

For symmetry reasons, if n=odd, we have $\varphi = 0$ and only even cocycles survive.

A mystery: The 1-summable case we had before suggests that something more should be true! In fact for each

$$n \geq p - 1$$

one can define a cyclic cocycle on $\mathcal{A}$ as follows:

For any T such that $[F,T] \in \mathcal{L}^1(\mathcal{H})$ let

$$\mathrm{Tr}_s(T) = \mathrm{Trace}(\varepsilon F[F,T]).$$

1) $\mathrm{Tr}_s(T) = 0$ if T is odd

2) $\mathrm{Tr}_s(T) = \mathrm{Trace}(\varepsilon T)$ if $T \in \mathcal{L}^1(\mathcal{H})$

3) $[F,\Omega^n] \subset \mathcal{L}^1(\mathcal{H})$ and

$$\mathrm{Tr}_s : \Omega^n \to \mathbb{C}$$

is a closed graded trace on $\mathcal{A}$.

The corresponding cyclic n-cocycle is given by

$$\varphi(a^0,\cdots,a^n) = \mathrm{Tr}_s(a^0 da^1 \cdots da^n)$$

$$= \mathrm{Trace}(\varepsilon F[F,a^0][F,a^1][F,a^2]\cdots[F,a^n]).$$

Example: Let $A = C^\infty(V)$, $V = \mathbb{C}/\Gamma$ a two dimensional torus. Let

$$\bar{\partial} : H^+ = \{\xi \in L^2(V), \bar{\partial}\xi \in L^2(V)\} \to H^- = L^2(V).$$

Let $C^\infty(V)$ act on H^+ and H^- by multiplication operators and let

$$F = \begin{pmatrix} 0 & (\bar{\partial} + \varepsilon)^{-1} \\ (\bar{\partial} + \varepsilon) & 0 \end{pmatrix}$$

where the fixed parameter ε is not in the spectrum of $\bar{\partial}$. Then

1. (H, F) is p-summable for any $p > 2$
2. Its character is given by

$$\mathrm{Tr}_s(f^0[F, f^1][F, f^2]) = \frac{-1}{2\pi i} \int f^0 df^1 \wedge df^2.$$

Refinements

There are two issues:

A) Bounded versus unbounded operators: F or D ?

B) Non-local versus local trace in cyclic cocycles.

Fredholm modules only capture the topology of a NC space. Its metric aspects are rather hidden and can best be expressed using unbounded operators. Then we are dealing with an spectral triple.

In practice a differential operator D is an unbounded operator between L^2-sections of vector bundles. To obtain a bounded F with $F^2 = 1$ from D, assuming D is injective, the recipe is

$$F = \frac{D}{|D|}$$

Definition: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of

1. A unital involutive algebra $\mathcal{A}$
2. An action of $\mathcal{A}$ by bounded operators on $\mathcal{H}$
3. A self-adjoint (unbounded, in general) operator D on $\mathcal{H}$ with compact resolvent:

$$(D + i)^{-1} \in \mathcal{K}(\mathcal{H})$$

A spectral triple is even if $\mathcal{H}$ is $\mathbb{Z}_2$-graded, D is odd and A acts by even operators. It is called finitely summable if the singular values μ_n of the resolvent of D satisfy

$$\mu_n = \mathrm{O}(n^{-\alpha})$$

for some $\alpha > 0$.

To address the second issue Connes advanced the idea of using the Dixmier trace Tr_ω instead of the usual operator trace Tr. (More on this later).

Cyclic Cohomology 101

$\mathcal{A}$= any algebra, commutative or not.

$$C^n(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}^{\otimes(n+1)}, \mathbb{C})$$

Thus: n-cochains = $(n+1)$-linear functionals

$$\varphi(a_0, a_1, \cdots, a_n)$$

on $\mathcal{A}$.

Define

b, $b' : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$, and $\lambda : C^n(\mathcal{A}) \to C^n(\mathcal{A})$,

by

$$(b\varphi)(a_0, \cdots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \cdots, a_n)$$

$$(b'\varphi)(a_0, \cdots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1})$$

$$\lambda\varphi(a_0, \cdots, a_n) = (-1)^n \varphi(a_n, a_0, \cdots, a_{n-1})$$

By a direct computation one checks that:

$$b^2 = 0, \quad (1-\lambda)b = b'(1-\lambda), \quad b'^2 = 0. \qquad (4)$$

Hochschild complex of $\mathcal{A}$:

$$C^0(\mathcal{A}) \xrightarrow{b} C^1(\mathcal{A}) \xrightarrow{b} C^2(\mathcal{A}) \xrightarrow{b} \cdots \qquad (5)$$

Its cohomology is the Hochschild cohomology of $\mathcal{A}$ (with coefficients in $\mathcal{A}^*$-more about this later!) shall be denoted as

$$HH^n(\mathcal{A}), \quad n = 0, 1, 2, \cdots$$

A cochain $\varphi \in C^n$ is called cyclic if

$$(1 - \lambda)\varphi = 0,$$

or equivalently

$$\varphi(a_n, a_0, \cdots, a_{n-1}) = (-1)^n \varphi(a_0, \cdots, a_n).$$

Using (**??**) we obtain the very interesting fact:

The space of cyclic cochains is invariant under b, i.e. for all n,

$$b\, C_\lambda^n(\mathcal{A}) \subset C_\lambda^{n+1}(\mathcal{A}).$$

We therefore have a subcomplex of the Hochschild complex, called the *Connes complex* of $\mathcal{A}$:

$$C_\lambda^0(\mathcal{A}) \xrightarrow{b} C_\lambda^1(\mathcal{A}) \xrightarrow{b} C_\lambda^2(\mathcal{A}) \xrightarrow{b} \cdots \qquad (6)$$

The cohomology of this complex is called the *cyclic cohomology* of $\mathcal{A}$ and will be denoted by

$$HC^n(\mathcal{A}), \quad n = 0, 1, 2, \cdots$$

A cocycle for cyclic cohomology is called a *cyclic* *cocycle*. It satisfies the two conditions:

$$(1-\lambda)\varphi = 0, \quad \text{and} \quad b\varphi = 0.$$

Examples:

1. For any $\mathcal{A}$

$$HC^0(\mathcal{A}) = HH^0(\mathcal{A}) \;=\; \text{space of traces on}\, \mathcal{A}$$

In particular if $\mathcal{A}$ is commutative then

$$HC^0(\mathcal{A}) \simeq \mathcal{A}^*$$

2. $\mathcal{A} = \mathbb{C}$.

$$C^n(\mathbb{C}) = \mathbb{C}$$

$$b|C^{2n} = 0, \quad b|C^{2n+1} = id$$

$$C_\lambda^{2n}(\mathbb{C}) = \mathbb{C}, \quad C_\lambda^{2n+1}(\mathbb{C}) = 0$$

$$\text{Hochschild}: \quad \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{id} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{id} \cdots$$

$$\text{Connes}: \quad \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

We obtain:

$$HH^0(\mathbb{C}) = \mathbb{C}, \quad HH^n(\mathbb{C}) = 0, \quad n \geq 1$$

$$HC^{2n}(\mathbb{C}) = \mathbb{C}, \quad HC^{2n+1}(\mathbb{C}) = 0$$

3. Let $\mathcal{A} = C^\infty(M)$. By a result Connes, its continuous Hochschild cohomology is

$$HH^n_{cont}(C^\infty(M)) \simeq C^n(M)$$

and its continuous cyclic cohomology is

$$HC^n_{cont}(C^\infty(M)) \simeq Z^n(M) \oplus H^{dr}_{n-2}(M) \oplus H^{dr}_{n-4}(M) \oplus \cdots$$

Remark: later we shall see analogues of these results for Hochschild and cyclic homology. Note that computing the algebraic Hochschild and cyclic (co)homology of $C^\infty(M)$ is totally hopeless! More on this later.....

3. Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation and $\tau : \mathcal{A} \to \mathbb{C}$ an invariant trace, i.e.

$$\tau(\delta(a)) = 0$$

for all $a \in \mathcal{A}$. Then one checks that

$$\varphi(a_0, a_1) = \tau(a_0 \delta(a_1)) \qquad (7)$$

is a cyclic 1-cocycle on $\mathcal{A}$.

Remark: compare with

$$\varphi(f^0, f^1) = \int_{S^1} f^0 df^1$$

we had earlier.

This example can be generalized. Let δ_1 and δ_2 be a pair of *commuting* derivations which leave a trace τ invariant. Then

$$\varphi(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))) \tag{8}$$

is a cyclic 2-cocycle on $\mathcal{A}$.

Here is a concrete example with $\mathcal{A} = \mathcal{A}_\theta$ a smooth noncommutative torus. Let δ_1, $\delta_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ be the unique derivations defined by

$$\delta_1(U) = U, \quad \delta_1(V) = 0; \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

They commute with each other and preserve the standard trace τ on $\mathcal{A}_\theta$. The resulting cyclic 1-cocycles $\varphi_1(a_0, a_1) = \tau(a_0\delta_1(a_1))$ and $\varphi_1'(a_0, a_1) = \tau(a_0\delta_2(a_1))$ form a basis for the periodic cyclic cohomology $HP^1(\mathcal{A}_\theta)$. Similarly, the corresponding cocycle (**??**) together with τ form a basis for $HP^0(\mathcal{A}_\theta)$.

Connes' Long exact Sequence

Consider the short exact sequence of complexes

$$0 \to C_\lambda \to C \to C/C_\lambda \to 0$$

Its associated long exact sequence is

$$\cdots \longrightarrow HC^n(\mathcal{A}) \longrightarrow HH^n(\mathcal{A}) \longrightarrow H^n(C/C_\lambda) \qquad (9)$$

$$\longrightarrow HC^{n+1}(\mathcal{A}) \longrightarrow \cdots$$

We need to identify the cohomology groups

$$H^n(C/C_\lambda) = ?$$

Consider the short exact sequence

$$0 \longrightarrow C/C_\lambda \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} C_\lambda \longrightarrow 0, \qquad (10)$$

where

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n : C^n \longrightarrow C^n.$$

The relations

$$N(1-\lambda) = (1-\lambda)N = 0, \qquad bN = Nb'$$

can be verified and they show that $1-\lambda$ and N are morphisms of complexes.

Exercise Show that (**??**) is exact (the interesting part is to show that $\operatorname{Ker} N \subset \operatorname{Im}(1-\lambda)$).

Assume $\mathcal{A}$ is unital. The middle complex (C, b') in (**??**) is exact with contracting homotopy $s : C^n \to C^{n-1}$

$$(s\varphi)(a_0, \cdots, a_{n-1}) = (-1)^n \varphi(a_0, \cdots, a_{n-1}, 1).$$

Thus:

$$H^n(C/C_\lambda) \simeq HC^{n-1}(\mathcal{A}).$$

Using this in (**??**), we obtain Connes' long exact sequence relating Hochschild and cyclic cohomology:

$$\cdots \longrightarrow HC^n(\mathcal{A}) \xrightarrow{I} HH^n(\mathcal{A}) \xrightarrow{B} HC^{n-1}(\mathcal{A}) \qquad (11)$$

$$\xrightarrow{S} HC^{n+1}(\mathcal{A}) \longrightarrow \cdots$$

The operators B and S can be made more explicit by finding the connecting homomorphisms in the above long exact sequences. Remarkably, there is a formula for *Connes' boundary operator*

$$B : HH^n(\mathcal{A}) \longrightarrow HC^{n-1}(\mathcal{A})$$

on the level of cochains:

$$B = Ns(1 - \lambda) = NB_0,$$

with $B_0 : C^n \to C^{n-1}$ defined by:

$$B_0\varphi(a_0, \cdots, a_{n-1}) = \varphi(1, a_0, \cdots, a_{n-1})$$

$$-(-1)^n \varphi(a_0, \cdots, a_{n-1}, 1).$$

The *periodicity operator*

$$S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$$

is related to Bott periodicity and it is used to define the *periodic cyclic cohomology* of $\mathcal{A}$ as a direct limit:

$$HP^i(\mathcal{A}) = \varinjlim HC^{2n+i}(\mathcal{A}), \qquad i = 0, 1$$

Morita Invariance

A typical application of Connes' long exact sequence (**??**) is to extract information about cyclic cohomology from Hochschild cohomology. For example, assume $f : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism such that

$$f^* : HH^n(\mathcal{B}) \to HH^n(\mathcal{A})$$

is an isomorphism for all $n \geq 0$. Then, using the five lemma and the IBS sequence, we conclude that

$$f^* : HC^n(\mathcal{B}) \to HC^n(\mathcal{A})$$

is an isomorphism for all n. In particular from Morita invariance of Hochschild cohomology one obtains the Morita invariance of cyclic cohomology.

The *generalized trace map*

$$Tr : C^n(\mathcal{A}) \longrightarrow C^n(M_k(\mathcal{A})),$$

is defined as

$$(Tr\varphi)(a_0 \otimes m_0, \cdots, a_n \otimes m_n) \; = \; tr(m_0 \cdots m_n)\varphi(a_0, \cdots, a_n)$$

where $m_i \in M_k(\mathbb{C}), a_i \in \mathcal{A}$ and tr is the standard trace of matrices.

It is easy to see that:

1. Tr is a chain map.
2. Let $i : \mathcal{A} \to M_k(\mathcal{A})$ be the map

$$a \mapsto a \otimes E_{11}$$

and define a map

$$i^* : C^n(M_k(\mathcal{A})) \longrightarrow C^n(\mathcal{A}).$$

Then

$$i^* \circ Tr = id.$$

$i^* \circ Tr$ however is not equal to id on the nose. There is instead a homotopy between $Tr \circ i^*$ and id.

We need to know, also when defining the Connes-Chern character, that inner automorphisms act by identity on Hochschild and cyclic cohomology and inner derivations act by zero. Let $u \in \mathcal{A}$ be invertible, and let $a \in \mathcal{A}$. They induce the chain maps

$$\Theta_u, \quad L_a : C^n(\mathcal{A}) \to C^n(\mathcal{A})$$

by

$$(\Theta_u\varphi)(a_0,\cdots,a_n)=\varphi(ua_0u^{-1},\cdots,ua_nu^{-1}),$$

$$(L_a\varphi)(a_0,\cdots,a_n)=$$

$$\sum_{i=0}^{n}\varphi(a_0,\cdots,[a,a_i],\cdots a_n).$$

Θ induces the identity map on Hochschild homology and L_a induces the zero map.

The maps (cf. e.g. Loday), $h_i : A^{\otimes n+1}\to A^{\otimes n+2}$, $i=0,\cdots,n$

$$h_i(a_0\otimes\cdots\otimes a_n)=(a_0u^{-1}\otimes ua_1u^{-1},\cdots,u\otimes a_{i+1}\cdots\otimes a_n)$$

define a homotopy

$$h=\sum_{i=0}^{n}(-1)^i h_i^*$$

between id and Θ_u.

For L_a the maps $h_i : A^{\otimes n+1} \to A^{\otimes n+2}$, $i = 0, \cdots, n$,

$$h_i(a_0 \otimes \cdots \otimes a_n) = (a_0 \otimes \cdots a_i \otimes a \cdots \otimes a_n),$$

define a homotopy between L_a and 0.

Remark: Let $\mathcal{A}$ and $\mathcal{B}$ be unital Morita equivalent algebras. Let X be an equivalence $\mathcal{A} - \mathcal{B}$ bimodule and Y its inverse bimodule. Let M be an $\mathcal{A} - \mathcal{A}$ bimodule and $N = Y \otimes_{\mathcal{A}} M \otimes_{\mathcal{A}} X$ the corresponding $\mathcal{B}$-bimodule. Morita invariance of Hochschild cohomology (with coefficients) states that there is a natural isomorphism

$$H^n(\mathcal{A}, M) \simeq H^n(\mathcal{B}, N),$$

for all $n \geq 0$. In the above we sketch a proof of this for $\mathcal{B} = M_k(\mathcal{A})$ and $M = \mathcal{A}^*$.

Connes' Spectral Sequence

The cyclic complex and the long exact sequence, as useful as they are, are not powerful enough for computations. A much deeper relation between Hochschild and cyclic cohomology is encoded in Connes' spectral sequence. This spectral sequence resembles in many ways the Hodge to de Rham spectral sequence for complex manifolds.

$\mathcal{A}$ = unital algebra. We have

$$b^2 = 0, \quad bB + Bb = 0, \quad B^2 = 0.$$

The middle relation follows from $b's + sb' = 1$, $(1-\lambda)b = b'(1-\lambda)$ and $Nb' = bN$.

Connes' (b, B)- bicomplex of $\mathcal{A}$:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
C^2(\mathcal{A}) & \xrightarrow{B} & C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) \\
b\uparrow & & b\uparrow & & \\
C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) & & \\
b\uparrow & & & & \\
C^0(\mathcal{A}) & & & &
\end{array}
$$

Theorem: (Connes, 1981) The map

$$\varphi \mapsto (0, \cdots, 0, \varphi)$$

is a quasi-isomorphism of complexes

$$(C_\lambda(\mathcal{A}), b) \to (Tot\mathcal{B}(\mathcal{A}), b + B).$$

This is a consequence of the vanishing of the E^2 term of the second spectral sequence (filtration by columns) of $\mathcal{B}(A)$. To prove this consider the short exact sequence of b-complexes

$$0 \longrightarrow \mathrm{Im}\, B \longrightarrow \mathrm{Ker}\, B \longrightarrow \mathrm{Ker}\, B/\mathrm{Im}\, B \longrightarrow 0$$

By a hard lemma of Connes (NCDG, Lemma 41), the induced map

$$H_b(\mathrm{Im} B) \longrightarrow H_b(\mathrm{Ker}\, B)$$

is an isomorphism. It follows that $H_b(\mathrm{Ker}\, B/\mathrm{Im}\, B)$ vanish. To take care of the first column one appeals to the fact that

$$\mathrm{Im}\, B \simeq Ker(1-\lambda)$$

is the space of cyclic cochains.

Topological Algebras

For applications of cyclic (co)homology to noncommutative geometry, it is crucial to consider topological algebras, topological resolutions, and continuous chains and cochains on them. For example while the algebraic Hochschild and cyclic groups of the algebra of smooth functions on a manifold are not known, their topological counterparts are computed by Connes as we recall below.

There is no difficulty in defining *continuous analogues* of Hochschild and cyclic cohomology groups for Banach algebras. One simply replaces bimodules by Banach bimodules (where the left and right module actions are bounded operators) and cochains by continuous cochains. Since the multiplication of a Banach algebra is a bounded map, all operators including the Hochschild boundary and the cyclic operator extend to this continuous setting.

The resulting Hochschild and cyclic theory for Banach and C^*-algebras, however, are hardly useful and tend to vanish in many interesting examples.

This is hardly surprising since the definition of any Hochschild and cyclic cocycle of dimension bigger than zero involves differentiating the elements of the algebra in one way or another. This is in sharp contrast with topological K-theory where the right setting is the setting of Banach or C^*-algebras.

Exercise: Let X be a compact Hausdorff space. Show that any derivation $\delta : C(X) \longrightarrow C(X)$ is identically zero. (hint: first show that if $f = g^2$ and $g(x) = 0$ for some $x \in X$, then $\delta(f)(x) = 0$.)

Compare this with the fact that: there exists a one-one correspondence between derivations

$$\delta : C^\infty(M) \to C^\infty(M)$$

and smooth vector fields on M given, in local co-ordinated, by

$$\delta = \sum X^i \frac{\partial}{\partial x_i}$$

Remark: By results of Connes and Haggerup, a C^*-algebra is *amenable* if and only if it is *nuclear*. Amenability refers to the property that for all $n \geq 1$,

$$H^n_{cont}(A, M^*) = 0,$$

for any Banach dual bimodule M^*. In particular, by using Connes' long exact sequence, we find that, for any nuclear C^*-algebra A,

$$HC^{2n}_{cont}(A) = A^*, \quad \text{and} \quad HC^{2n+1}_{cont}(A) = 0,$$

for all $n \geq 0$.

The right class of topological algebras for Hochschild and cyclic cohomology turns out to be the class of *locally convex algebras:*

Definition: An algebra $\mathcal{A}$ equipped with a locally convex topology is called a locally convex algebra if its multiplication map

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

is jointly continuous.

Examples of locally convex algebras:

$$\mathcal{A} = C^\infty(M), \quad A = \mathcal{A}_\theta$$

smooth functions on a closed manifold and smooth noncommutative tori. The topology of $C^\infty(M)$ is defined by the sequence of seminorms

$$\|f\|_n = \sup |\partial^\alpha f|; \quad |\alpha| \leq n,$$

where the supremum is over a fixed, finite, coordinate cover for M.

The sequence of norms

$$p_k(a) = \text{Sup}\,\{(1 + |n| + |m|)^k |a_{mn}|\}$$

defines a locally convex topology on the smooth noncommutative torus $\mathcal{A}_\theta$. The multiplication of $\mathcal{A}_\theta$ is continuous in this topology.

Given locally convex topological vector spaces V_1 and V_2, their *projective tensor product* is a locally convex space $V_1 \hat{\otimes} V_2$ together with a universal jointly continuous bilinear map

$$V_1 \otimes V_2 \to V_1 \hat{\otimes} V_2$$

It follows from the universal property that for any locally convex space W, we have a natural isomorphism

$$B(V_1 \times V_2, W) \simeq L(V_1 \hat{\otimes} V_2, W)$$

between continuous bilinear maps and continuous linear maps .

One of the nice properties of the projective tensor product is that for smooth compact manifolds M and N, the natural map

$$C^{\infty}(M)\hat{\otimes}C^{\infty}(N) \to C^{\infty}(M \times N)$$

is an isomorphism. This is crucial for computations.

A *topological left $\mathcal{A}$-module* is a locally convex topological vector space $\mathcal{M}$ endowed with a continuous left $\mathcal{A}$-module action $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$. A *topological free* left $\mathcal{A}$-module is a module of the type $\mathcal{M} = \mathcal{A}\hat{\otimes}V$ where V is a locally convex space. A *projective* module is a module which is a direct summand in a free module.

Given a locally convex algebra $\mathcal{A}$, let

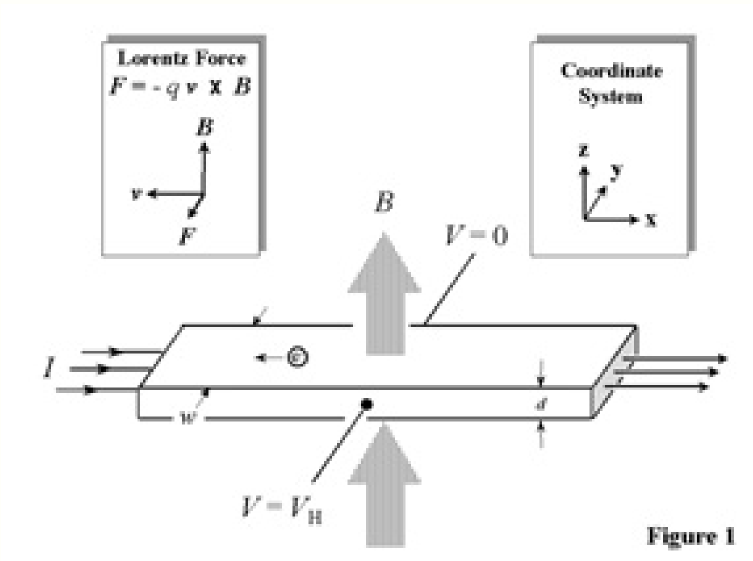
$$C^n_{\text{cont}}(\mathcal{A}) = \text{Hom}_{\text{cont}}(\mathcal{A}^{\widehat{\otimes} n}, \mathbb{C})$$

be the space of continuous $(n+1)$-linear functionals on A and let $C^n_{\text{cont},\,\lambda}(\mathcal{A})$ denote the space of continuous cyclic cochains on $\mathcal{A}$.

All of algebraic definitions and results extend to this topological setting. In particular one defines topological Hochschild and cyclic cohomology groups of a locally convex algebra. The right class of topological projective resolutions are those resolutions that admit a continuous linear splitting. This extra condition is needed when one wants to prove comparison theorems for resolutions. We won't go into details here since this is very well explained in Connes' original article (NCDG).

Cyclic Cohomology in Physics

1. The quantum Hall effect:



In low temperatures and under strong magnetic fields the Hall conductivity σ_H takes on only quantized values

$$\sigma_H = \nu \frac{e^2}{h} \quad \nu \in \mathbb{Z}$$

(there is a rational q.h.e as well which is more subtle and we won't consider here. Classically σ_H can take on any real values.)

How to understand the integrality of a quantity?

Classical examples from differential topology: Let

$$\alpha = \int_M \omega$$

e.g. M= a 2-dimensional Riemannian manifold and

$$\alpha = \frac{1}{2\pi}\int_M K = \text{total curvature}$$

Gauss-Bonnet says this is an integer. One way to prove integrality results is to use the index theorem and look for a an elliptic operator D and show that

$$\alpha = \int_M \omega = \text{index}(D)$$

which is a priori an integer.

Cyclic cohomology and Connes-Chern character allows one to extend this method to noncommutative manifolds and that is exactly what is needed in quantum Hall effect.

Let $\varphi \in Z_\lambda^{2n}(A)$ be a cyclic $2n$-cocycle on an algebra A, $E \in A$ a projection (it can be a projection in $M_q(A)$ as well) and let

$$\alpha = \varphi(E, E, \cdots, E) = \int_\varphi EdEdE \cdots dE$$

Assume there is a p-summable Fredholm module (H, F) on A with

$$Ch(H, F) = \varphi$$

Then Connes' index formula shows that

$$\varphi(E, E, \cdots, E) = \text{index}\,(F_e^+)$$

from which the integrality of α of course follows.

The noncommutative manifold A for the quantum

Hall effect is the Brilouin Zone which is Morita equivalent to a noncommutative torus A_θ.

Kubo's formula expresses σ_H by

$$\sigma_H = \varphi(E, E, E)$$

where φ is a cyclic 2-cocycle on A and $E = E_\mu$ is the spectral projection of the Hamiltonian H corresponding to energies smaller than the Fermi level μ. For the construction of the $(2, \infty)$-summable Fredholm module we refer to Connes' 1994 book or Bellissard's articles cited there.

2. Noncommutative Yang-Mills theory:

Classical setting of Yang-Mills theory: A fixed back ground manifold M, representing the spacetime, a principal G-bundle over M, and a vector bundle E over M (associated to a representation of G).

matter fields: (fermions) sections of E

Yang-Mills fields: (aka vector potential; gauge field): (bosons) connections $A = A_\mu dx^\mu$ on E, carriers of force.

Thus, locally, A is a matrix valued (or Lie algebra valued, in general) 1-form on M. Yang-Mills action:

$$YM(A) = \int_M ||F||^2$$

where

$$F = dA + A^2$$

is the curvature (aka: force field).

The set up for noncommutative Yang-Mills theory: Let $\mathcal{A}$ be an $*$-algebra, τ a Hochschild 4-cocycle on $\mathcal{A}$ and

$$\int_\tau : \Omega^4\mathcal{A} \to \mathbb{C}$$

the corresponding graded trace. For a gauge field $A \in \Omega^1\mathcal{A}$, the Yang-Mills functional is defined by

$$YM(A) = \int_\tau (dA + A^2)^2$$

This action is gauge invariant in the sense that

$$YM(A) = YM(\gamma_u(A))$$

where

$$\gamma_u(A) = udu^* + uAu^*, \quad u \in A,\, uu^* = u^*u = 1$$

proof: It follows from

$$F(\gamma_u(A)) = uF(A)u^*$$

To obtain inequalities similar to classical inequalities for YM impose the condition:

A Hochschild cocycle $\tau \in Z^{2n}(\mathcal{A})$ is called positive if for all $\omega \in \Omega^n \mathcal{A}$

$$\int_\tau \omega\omega^* \geq 0$$

where

$$(a_0 da_1 \cdots da_n)^* = (-1)^n da_n^* \cdots da_1^* a_0^*$$

Simple example: a zero cocycle is positive iff the corresponding trace is positive.

Assuming τ is positive one can then how that

$$YM_\tau(A) \geq 0 \qquad \forall A \in \Omega^1$$

From complex structures to positive Hochschild cocycles: Let M be a compact Riemann surface. Then

$$\varphi(f^0, f^1, f^2) = \frac{i}{\pi} \int_M f^0 \partial f^1 \bar{\partial} f^2,$$

is a positive Hochschild 2-cocycle on $\mathcal{A} = C^\infty(M)$.

For another example notice that the Dixmier trace is positive in the sense that

$$Tr_\omega(T) \geq 0 \quad \forall T \in \mathcal{L}^{(1,\infty)}, \, T \geq 0$$

Let $(\mathcal{H}, D)$ be an even (n, ∞)-summable module over $\mathcal{A}$. Then for $n = 2m$ even,

$$\varphi_\omega(a^0, \cdots, a^n) =$$

$$\text{const.Tr}_\omega((1+\gamma)a^0[D,a^1]\cdots[D,a^n]D^{-n})$$

is a positive Hochschild cocycle on $\mathcal{A}$.

3. Chern-Simons action in NC gauge theory

Let ψ be a cyclic 3-cocycle on $\mathcal{A}$. For any gauge field $A \in \Omega^1\mathcal{A}$, the Chern-Simon action is defined by

$$CS_\psi(A) = \int_\psi AdA + \frac{2}{3}A^3.$$

Unlike the Yang-Mils action, Chern-Simons action is not invariant under gauge transformation $\gamma_u(A) = udu^* + uAu*$, but we have (Chamseddine-Connes):

$$CS_\psi(\gamma_u(A)) = CS_\psi(A) + \frac{1}{3} < \psi, u >$$

where $< \psi, u >$ denotes the pairing between $HC^3(\mathcal{A})$ and $K_1(\mathcal{A})$ given by

$$< \psi, u > = \int_\psi udu^*dudu^*.$$

Proof: (on board!)

Cyclic Homology

Cyclic cohomology is a contravariant functor on the category of algebras. There is a dual covariant theory called *cyclic homology*. The relation between the two is similar to the relation between currents and differential forms on manifolds. Through (continuous) cyclic homology we can recover the de Rham cohomology.

For each $n \geq 0$, let

$$C_n(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}.$$

Define the operators

$$
\begin{aligned}
b : C_n(\mathcal{A}) &\longrightarrow C_{n-1}(\mathcal{A}) \\
b' : C_n(\mathcal{A}) &\longrightarrow C_{n-1}(\mathcal{A}) \\
\lambda : C_n(\mathcal{A}) &\longrightarrow C_n(\mathcal{A}) \\
s : C_n(\mathcal{A}) &\longrightarrow C_{n+1}(\mathcal{A}) \\
N : C_n(\mathcal{A}) &\longrightarrow C_n(\mathcal{A}) \\
B : C_n(\mathcal{A}) &\longrightarrow C_{n+1}(\mathcal{A})
\end{aligned}
$$

by

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n (a_n a_0 \otimes a_1 \cdots \otimes a_{n-1})$$

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$

$$\lambda(a_0 \otimes \cdots \otimes a_n) = (-1)^n (a_n \otimes a_0 \cdots \otimes a_{n-1})$$

$$s(a_0 \otimes \cdots \otimes a_n) = (-1)^n (a_0 \otimes \cdots \otimes a_n \otimes 1)$$

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n$$

$$B = (1 - \lambda) s N$$

They satisfy the relations

$$b^2 = 0, \qquad b'^2 = 0, \qquad (1-\lambda)b' = b(1-\lambda)$$

$$b'N = Nb, \qquad B^2 = 0, \qquad bB + Bb = 0$$

The complex $(C_\bullet(\mathcal{A}), b)$ is the Hochschild complex of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A}$. The complex

$$C_n^\lambda(\mathcal{A}) := C_n(\mathcal{A})/\mathrm{Im}(1-\lambda)$$

is the *Connes complex* of $\mathcal{A}$. Its homology

$$HC_n(\mathcal{A}), \; n = 0, 1, \cdots$$

is the *cyclic homology* of $\mathcal{A}$.

It is clear that the space of cyclic cochains is the linear dual of the space of cyclic chains

$$C_\lambda^n(\mathcal{A}) \simeq \mathrm{Hom}\,(C_n^\lambda(\mathcal{A}), \mathbb{C})$$

and

$$HC^n(\mathcal{A}) \simeq HC_n(\mathcal{A})^*.$$

Similar to cyclic cohomology, there is a long exact sequence relating Hochschild and cyclic homologies, and also there is a spectral sequence from Hochschild to cyclic homology. In particular cyclic homology can be computed using the following bicomplex.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
\downarrow b & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array}$$

Example (Hochschild-Kostant-Rosenberg and Connes theorems) Let

$$\mathcal{A} \xrightarrow{d} \Omega^1\mathcal{A} \xrightarrow{d} \Omega^2\mathcal{A} \xrightarrow{d} \cdots$$

denote the de Rham complex of a commutative unital algebra $\mathcal{A}$. By definition

$$d : \mathcal{A} \to \Omega^1\mathcal{A}$$

is a *universal derivation* into a symmetric $\mathcal{A}$-bimodule, the *module of Kähler differentials*, and

$$\Omega^n\mathcal{A} := \bigwedge_{\mathcal{A}}^{n} \Omega^1\mathcal{A}$$

is the k-th exterior power of $\Omega^1\mathcal{A}$ over $\mathcal{A}$.

Alternatively:

$$\Omega^1\mathcal{A} = I/I^2,$$

where

$$I = Ker\ \{m : A \otimes A \to A\}$$

and

$$d(a) = a \otimes 1 - 1 \otimes a \quad \mathrm{mod}(I^2).$$

The universal derivation d has a unique extension to a graded derivation of degree one on $\Omega\mathcal{A}$, denoted by d.

The *antisymmetrization map:*

$$\varepsilon_n : \Omega^n\mathcal{A} \longrightarrow \mathcal{A}^{\otimes(n+1)},$$

$$\varepsilon_n(a_0 da_1 \wedge \cdots \wedge da_n) = \sum_{\sigma \in S_n} sgn(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}.$$

We also have a map

$$\mu_n : \mathcal{A}^{\otimes n} \longrightarrow \Omega^n\mathcal{A},$$

$$\mu_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 da_1 \wedge \cdots \wedge da_n.$$

The resulting maps

$(\Omega\mathcal{A}, 0) \to (C(\mathcal{A}), b)$, and, $(C(\mathcal{A}), b) \to (\Omega\mathcal{A}, 0)$

are morphisms of complexes, i.e.

$$b \circ \varepsilon_n = 0, \quad \mu_n \circ b = 0.$$

We have:

$$\mu_n \circ \varepsilon_n = n!\, \mathrm{Id}_n.$$

It follows that, for any commutative algebra $\mathcal{A}$, the antisymmetrization map induces an inclusion

$$\varepsilon_n : \Omega^n\mathcal{A} \hookrightarrow HH_n(\mathcal{A}),$$

for all n.

Hochschild-Kostant-Rosenberg theorem: if $\mathcal{A}$ is a *regular algebra*, e.g. the algebra of regular functions on a smooth affine variety, then ε_n defines an algebra isomorphism

$$\varepsilon_n : \Omega^n \mathcal{A} \simeq HH_n(\mathcal{A})$$

between Hochschild homology of $\mathcal{A}$ and the algebra of differential forms on $\mathcal{A}$.

To compute the cyclic homology of $\mathcal{A}$, we first show that under the map μ the operator B corresponds to the de Rham differential d. More precisely, for each integer $n \geq 0$ we have a commutative diagram:

$$\begin{array}{ccc} C_n(\mathcal{A}) & \xrightarrow{\mu} & \Omega^n \mathcal{A} \\ \downarrow B & & \downarrow d \\ C_{n+1}(\mathcal{A}) & \xrightarrow{\mu} & \Omega^{n+1} \mathcal{A} \end{array}$$

We have

$$
\begin{aligned}
&\mu B(f_0 \otimes \cdots \otimes f_n) = \\
&\mu \sum_{i=0}^{n} (-1)^{ni} (1 \otimes f_i \otimes \cdots \otimes f_{i-1} \\
&-(-1)^n f_i \otimes \cdots f_{i-1} \otimes 1) \\
&= \frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{ni} df_i \cdots df_{i-1} \\
&= \frac{1}{(n+1)!} (n+1) df_0 \cdots df_n \\
&= d\mu(f_0 \otimes \cdots \otimes f_n).
\end{aligned}
$$

It follows that μ defines a morphism of bicomplexes

$$\mathcal{B}(\mathcal{A}) \longrightarrow \Omega(\mathcal{A}),$$

where $\Omega(\mathcal{A})$ is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Omega^2\mathcal{A} & \stackrel{d}{\longleftarrow} & \Omega^1\mathcal{A} & \stackrel{d}{\longleftarrow} & \Omega^0\mathcal{A} \\
\downarrow 0 & & \downarrow 0 & & \\
\Omega^1\mathcal{A} & \stackrel{d}{\longleftarrow} & \Omega^0\mathcal{A} & & \\
\downarrow 0 & & & & \\
\Omega^0\mathcal{A} & & & &
\end{array}$$

Since μ induces isomorphisms on row homologies, it induces isomorphisms on total homologies as well. Thus we have (Connes):

$$HC_n(\mathcal{A}) \simeq \Omega^n\mathcal{A}/\mathrm{Im}\, d \oplus H_{dR}^{n-2}(\mathcal{A}) \oplus \cdots \oplus H_{dR}^k(\mathcal{A}),$$

where $k = 0$ if n is even and $k = 1$ if n is odd.

Using the same map μ acting between the corresponding periodic complexes, one concludes that the periodic cyclic homology of $\mathcal{A}$ is given by

$$HP_k(\mathcal{A}) \simeq \bigoplus_i H_{dR}^{2i+k}(\mathcal{A}), \quad k = 0, 1.$$

By a completely similar method one can compute the *continuous cyclic homology* of the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a smooth closed manifold M. Here by continuous cyclic homology we mean the homology of the cyclic complex where instead of algebraic tensor products $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$, one uses the topological *projective tensor product* $\mathcal{A} \hat{\otimes} \cdots \hat{\otimes} \mathcal{A}$. The continuous Hochschild homology of $\mathcal{A}$ can be computed using Connes' topological resolution for $\mathcal{A}$ as an $\mathcal{A}$-bimodule as in Example (**??**). The result is

$$HH_n^{cont}(C^\infty(M)) \simeq \Omega^n M$$

with isomorphism induced by the map

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \cdots df_n.$$

The rest of the computation of continuous cyclic homology follows the same pattern as in the case of regular algebras above. The end result is [**?**]:

$$HC_n^{\text{cont}}(C^\infty(M)) \simeq$$

$$\Omega^n M/\text{Im}\, d \oplus H_{dR}^{n-2}(M) \oplus \cdots \oplus H_{dR}^k(M),$$

and

$$HP_k^{cont}(C^\infty(M)) \simeq \bigoplus_i H_{dR}^{2i+k}(M), \quad k = 0, 1.$$

Cyclic Modules

Any (co)homology theory has two variables

$$H(A, M)$$

Typically M, the coefficient, belongs to a linear category but A, the main object of interest, lives in a highly non-linear category. For cyclic homology (resp. cohomology) $M = A$ (resp. $M = A^*$), though this is rather hidden at first.

As we saw before, cyclic cohomology of algebras was first defined by Connes through an explicit complex or bicomplex. Soon after he introduced the notion of *cyclic module* and defined its cyclic cohomology. Later developments proved that this extension was of great significance.

Apart from earlier applications, in recent work of Connes-Marcolli-Consani on Riemann zeta function, the abelian category of cyclic modules plays the role of the category of motives in noncommutative geometry.

Another recent example is the cyclic cohomology of Hopf algebras which cannot be defined as the cyclic cohomology of an algebra or a coalgebra but only as the cyclic cohomology of a cyclic module naturally attached to the given Hopf algebra.

The original motivation of Connes was to define cyclic cohomology of algebras as a derived functor. Since the category of algebras and algebra homomorphisms is not even an additive category (for the simple reason that the sum of two algebra homomorphisms is not an algebra homomorphism in general), the standard (abelian) homological algebra is not applicable. In Connes' approach, the category Λ_k of cyclic k-modules appears as an abelianization of the category of k-algebras. Cyclic cohomology is then shown to be the derived functor of the functor of traces, as we shall explain.

The *simplicial category* Δ:

objects: totally ordered sets

$$[n] = \{0 < 1 < \cdots < n\}, \quad n = 0, 1, 2, \cdots$$

morphisms: order preserving, i.e. monotone non-decreasing, maps $f : [n] \to [m]$

faces δ_i and *degeneracies* σ_i,

$$\delta_i : [n-1] \to [n], \quad \sigma_i : [n] \to [n-1], \qquad i = 1, 2, \cdots$$

δ_i = unique injective morphism missing i

σ_i = unique surjective morphism identifying i with $i+1$

simplicial identities:

$$\begin{aligned} \delta_j \delta_i &= \delta_{j-1} \delta_i \quad \text{if} \quad i < j, \\ \sigma_i \sigma_i &= \sigma_i \sigma_i \quad \text{if} \quad i < j, \\ \sigma_i \delta_i &= \begin{cases} \sigma_{j-1} \delta_i & i < j \\ \text{id} & i = j \text{ or } i = j+1 \\ \sigma_j \delta_{i-1} & i > j+1. \end{cases} \end{aligned}$$

Every morphism of Δ can be uniquely decomposed as a product of faces followed by a product of degeneracies.

cyclic category Λ: same objects as Δ, but more morphisms

Extra morphisms:

$$\tau_n : [n] \to [n], \quad n = 0, 1, 2, \cdots$$

extra relations:

$$\begin{aligned}
\tau_n \delta_i &= \delta_{i-1} \tau_{n-1} \qquad 1 \le i \le n \\
\tau_n \delta_0 &= \delta_n \\
\tau_n \sigma_i &= \sigma_{i-1} \tau_{n-1} \qquad 1 \le i \le n \\
\tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2 \\
\tau_n^{n+1} &= \text{id}.
\end{aligned}$$

cyclic object in a category: a functor

$$\Lambda^{\text{op}} \to \mathcal{C}.$$

cocyclic object:

$$\Lambda \to \mathcal{C}$$

Λ_k = category of cyclic objects in k-modules = aka cyclic modules

morphism of cyclic k-modules = natural transformation between corresponding functors.

Equivalently: sequences of k-linear maps $f_n : X_n \to Y_n$ compatible with the face, degeneracy, and cyclic operators.

Λ_k is an abelian category. Kernel and cokernel of a morphism f defined pointwise:

$$(\mathrm{Ker}\, f)_n = \mathrm{Ker}\, f_n : X_n \to Y_n$$

$$(\mathrm{Coker}\, f)_n = \mathrm{Coker}\, f_n : X_n \to Y_n$$

Let Alg_k = category of unital k-algebras and unital algebra homomorphisms. There is a functor

$$\natural : Alg_k \longrightarrow \Lambda_k, \quad A \mapsto A^\natural$$

where

$$A_n^\natural = A^{\otimes(n+1)}, n \geq 0$$

with face, degeneracy and cyclic operators

$$\begin{aligned}
\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \\
\sigma_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n \\
\tau_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \cdots \otimes a_{n-1}.
\end{aligned}$$

A unital algebra map $f : A \to B$ induces a morphism of cyclic modules $f^\natural : A^\natural \to B^\natural$ by

$$f^\natural(a_0 \otimes \cdots \otimes a_n) = f(a_0) \otimes \cdots \otimes f(a_n).$$

Example:

$$\text{Hom}_{\Lambda_k}(A^\natural,\ k^\natural) \simeq T(A),$$

where $T(A)$ is the space of traces $A \to k$. A trace τ is sent to the cyclic map $(f_n)_{n\geq 0}$,

$$f_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1 \cdots a_n), \qquad n \geq 0.$$

Theorem: (Connes) For any unital k-algebra A, there is a canonical isomorphism

$$HC^n(A) \simeq Ext^n_{\Lambda_k}(A^\natural, k^\natural), \qquad \text{for all } n \geq 0.$$

Now the above Example and Theorem, combined together, say that cyclic cohomology is the derived functor of the functor of traces $A \to T(A)$ where the word derived functor is understood to mean as above.

Motivated by the above theorem, one defines the cyclic cohomology and homology of any cyclic module M by

$$HC^n(M) := \mathsf{Ext}^n_{\Lambda_k}(M, k^\natural),$$

and

$$HC_n(M) := \mathsf{Tor}_n^{\Lambda_k}(M, k^\natural),$$

One can use the injective resolution used to prove the above Theorem to show that these Ext and Tor groups can be computed by explicit complexes and bicomplexes, similar to the situation with algebras. For example one has the following first quadrant bicomplex, called the *cyclic bicomplex* of M

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
M_2 & \xleftarrow{1-\lambda} & M_2 & \xleftarrow{N} & M_2 & \xleftarrow{1-\lambda} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_1 & \xleftarrow{1-\lambda} & M_1 & \xleftarrow{N} & M_1 & \xleftarrow{1-\lambda} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_0 & \xleftarrow{1-\lambda} & M_0 & \xleftarrow{N} & M_0 & \xleftarrow{1-\lambda} & \dots
\end{array}$$

whose total homology is naturally isomorphic to cyclic homology. Here the operator $\lambda : M_n \to M_n$ is defined by $\lambda = (-1)^n \tau_n$, while

$$b = \sum_{i=0}^{n} (-1)^i \delta_i, \qquad b' = \sum_{i=0}^{n-1} (-1)^i \delta_i,$$

and $N = \sum_{i=0}^{n} \lambda^i$. Using the simplicial and cyclic relations, one can check that $b^2 = b'^2 = 0$, $b(1 - \lambda) = (1 - \lambda)b'$ and $b'N = Nb'$. These relations amount to saying that the above is a bicomplex.

The (b, B)-bicomplex of a cyclic module is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\
\downarrow b & & \downarrow b & & \\
M_1 & \xleftarrow{B} & M_0 & & \\
\downarrow b & & & & \\
M_0 & & & &
\end{array}$$

whose total homology is again isomorphic to the cyclic homology of M (this time we have to assume that k is a field of characteristic 0). Here $B : M_n \to M_{n+1}$ is Connes' boundary operator defined by $B = (1 - \lambda)sN$, where $s = (-1)^n \sigma_n$.

A remarkable property of the cyclic category Λ, not shared by the simplicial category, is its *self-duality* in the sense that there is a natural isomorphism of categories $\Lambda \simeq \Lambda^{\text{op}}$. Roughly speaking, Connes' duality functor $\Lambda^{\text{op}} \longrightarrow \Lambda$ acts as the identity on objects of Λ and exchanges face and degeneracy operators while sending the cyclic operator to its inverse. Thus to a cyclic (resp. cocyclic) module one can associate a cocyclic (resp. cyclic) module by applying Connes' duality isomorphism. In the next section we shall see examples of cyclic modules in Hopf cyclic (co)homology that are dual to each other in the above sense.

Hochschild Cohomology

What we called the Hochschild cohomology of $\mathcal{A}$ and denoted by $HH^n(\mathcal{A})$ is in fact the Hochschild cohomology of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A}^*$. In general, given an $\mathcal{A}$-bimodule $\mathcal{M}$, the Hochschild complex of $\mathcal{A}$ with coefficients in the bimodule $\mathcal{M}$ is the complex

$$C^0(\mathcal{A}, \mathcal{M}) \xrightarrow{\delta} C^1(\mathcal{A}, \mathcal{M}) \xrightarrow{\delta} C^2(\mathcal{A}, \mathcal{M}) \longrightarrow \cdots$$

where

$$C^0(\mathcal{A}, \mathcal{M}) = \mathcal{M}, \quad C^n(\mathcal{A}, \mathcal{M}) = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, \mathcal{M})$$

is the space of n-linear functionals on $\mathcal{A}$ with values in $\mathcal{M}$. The differential δ is given by

$$\begin{aligned}
& (\delta\varphi)(a_1, \cdots, a_{n+1}) = \\
& a_1\varphi(a_2, \cdots, a_{n+1}) \\
+ & \sum_{i=1}^{n} (-1)^{i+1} \varphi(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\
+ & (-1)^{n+1} \varphi(a_1, \cdots, a_n) a_{n+1}.
\end{aligned}$$

Among all bimodules M over an algebra A, the following two bimodules play an important role:

1) M=A, with bimodule structure $a(b)c = abc$, for all a, b, c in A. The Hochschild complex $C^\bullet(A, A)$ is also known as the *deformation complex* , or *Gerstenhaber complex* of A. It plays an important role in deformation theory of associative algebras pioneered by Gerstenhaber. For example it is easy

to see that $H^2(A,A)$ is the space of *infinitesimal deformations* of A and $H^3(A,A)$ is the *space of obstructions* for deformations of A.

2) $M = A^* = Hom(A,k)$ with bimodule structure defined by

$$(afb)(c) = f(bca),$$

for all a,b,c in A, and f in A^*. This bimodule is relevant to cyclic cohomology. Indeed as we shall see the Hochschild groups $H^\bullet(A,A^*)$ and the cyclic cohomology groups $HC^\bullet(A)$ enter into a long exact sequence (Connes's long sequence). Using the identification

$$Hom(A^{\otimes n}, A^*) \simeq Hom(A^{\otimes(n+1)}, k), \quad f \mapsto \varphi,$$

$$\varphi(a_0, a_1, \cdots, a_n) = f(a_1, \cdots, a_n)(a_0),$$

the Hochschild differential δ is transformed into the differential b given by

$$\begin{aligned} b\varphi(a_0, \cdots, a_{n+1}) &= \sum_{i=0}^{n} (-1)^i \varphi(a_0, \cdots a_i a_{i+1}, \cdots, a_{n+1}) \\ &\quad + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \cdots, a_n). \end{aligned}$$

Thus for $n = 0, 1, 2$ we have the following formulas for b:

$$\begin{aligned} b\varphi(a_0, a_1) &= \varphi(a_0 a_1) - \varphi(a_1 a_0), \\ b\varphi(a_0, a_1, a_2) &= \varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, \\ b\varphi(a_0, a_1, a_2, a_3) &= \varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3) \\ &\quad + \varphi(a_0, a_1, a_2 a_3) - \varphi(a_3 a_0, a_1, a_2). \end{aligned}$$

We give a few examples of Hochschild cohomology in low dimensions.

Examples

1. $n = 0$. It is clear that

$$H^0(A, M) = \{m \in M; ma = am \quad \text{for all} \quad a \in A\}.$$

In particular for $M = A^*$,

$$H^0(A, A^*) = \{f : A \to k;\ f(ab) = f(ba) \ \text{ for all } a, b \in A\},$$

is the space of traces on A.

Exercise: For $A = k[x, \frac{d}{dx}]$, the algebra of differential operators with polynomial coefficients, show that $H^0(A, A^*) = 0$.

2. $n = 1$. A Hochschild *1-cocycle* $f \in C^1(A, M)$ is simply a *derivation*, i.e. a linear map $f : A \to M$

such that

$$f(ab) = af(b) + f(a)b,$$

for all a, b in A. A cocycle is a *coboundary* if and only if the corresponding derivation is *inner*, that is there exists m in M such that $f(a) = ma - am$ for all a in A. Therefore

$$H^1(A, M) = \frac{\text{derivations}}{\text{inner derivations}}$$

Sometimes this is called the space of *outer derivations* of A to M.

Exercise: 1) Show that any derivation on the algebra $C(X)$ of continuous functions on a compact Hausdorff space X is zero. (Hint: If $f = g^2$ and $g(x) = 0$ then $f'(x) = 0$.)

2) Show that any derivation on the matrix algebra $M_n(\mathbb{C})$ is inner. (This was proved by Dirac in 1925 in his first paper on quantum mechanics where derivations are called red*quantum differentials*.

3) Show that any derivation on the *Weyl algebra* $A = k[x, \frac{d}{dx}]$ is inner.

3. $n = 2$. We show, following Hochschild, that $H^2(A, M)$ classifies *abelian extensions* of A by M. Let A be a unital algebra over a field k. By definition, an abelian extension is an exact sequence of algebras

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0,$$

such that B is unital, M has trivial multiplication

($M^2 = 0$), and the induced A-bimodule structure on M coincides with the original bimodule structure. Let $E(A, M)$ denote the set of isomorphism classes of such extensions. We define a natural bijection

$$E(A, M) \simeq H^2(A, M)$$

as follows. Given an extension as above, let $s : A \to B$ be a linear splitting for the projection $B \to A$, and let $f : A \otimes A \to M$ be its *curvature* defined by,

$$f(a, b) = s(ab) - s(a)s(b),$$

for all a, b in A. One can easily check that f is a Hochschild 2-cocycle and its class is independent of the choice of splitting s. In the other direction, given a 2-cochain $f : A \otimes A \to M$, we try to define a multiplication on $B = A \oplus M$ via

$$(a, m)(a', m') = (aa', am' + ma' + f(a, a')).$$

It can be checked that this defines an associative multiplication if and only if f is a 2-cocycle. The extension associated to a 2-cocycle f is the extension

$$0 \longrightarrow M \longrightarrow A \oplus M \longrightarrow A \longrightarrow 0.$$

It can be checked that these two maps are bijective and inverse to each other.

Hochschild Cohomology is a Derived Functor

Let A^{op} denote the *opposite algebra* of A, where $A^{op} = A$ and the new multiplication is defined by $a.b := ba$. There is a one to one correspondence between A-bimodules and left $A \otimes A^{op}$-modules defined by

$$(a \otimes b^{op})m = amb.$$

Define a functor from the category of left $A \otimes A^{op}$ modules to k-modules by

$$M \mapsto Hom_{A\otimes A^{op}}(A, M) = \{m \in M; ma = am \, \forall a\}$$

$$= H^0(A, M).$$

To show that Hochschild cohomology is the derived functor of the functor $Hom_{A\otimes A^{op}}(A, -)$, we introduce the *bar resolution* of A. It is defined by

$$0 \longleftarrow A \xleftarrow{b'} B_1(A) \xleftarrow{b'} B_2(A) \cdots,$$

where $B_n(A) = A \otimes A^{op} \otimes A^{\otimes n}$ is the free left $A \otimes A^{op}$ module generated by $A^{\otimes n}$. The differential b' is defined by

$$\begin{aligned} &b'(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = aa_1 \otimes b \otimes a_2 \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i (a \otimes b \otimes a_1 \otimes \cdots a_i a_{i+1} \cdots \otimes a_n) \\ &+ (-1)^n (a \otimes a_n b \otimes a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

Define the operators $s : B_n(A) \to B_{n+1}(A)$, $n \geq 0$, by

$$s(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes b \otimes a \otimes a_1 \otimes \cdots \otimes a_n.$$

One checks that

$$b's + sb' = id$$

which shows that $(B_\bullet(A), b')$ is acyclic. Thus $(B_\bullet(A), b')$

is a projective resolution of A as an A-bimodule. Now, for any A-bimodule M we have

$$Hom_{A\otimes A^{op}}(B_\bullet(A),\, M) \simeq (C^\bullet(A,M),\, \delta),$$

which shows that Hochschild cohomology is a derived functor.

One can therefore use resolutions to compute Hochschild cohomology groups. Here are a few exercises

1. Let

$$A = T(V) = k \oplus V \oplus V^{\otimes 2} \oplus \cdots\,,$$

be the tensor algebra of a vector space V. Show that

$$0 \longleftarrow T(V) \stackrel{\delta}{\longleftarrow} T(V)\otimes T(V) \stackrel{\delta}{\longleftarrow} T(V)\otimes V\otimes T(V) \longleftarrow 0,$$

$$\delta(x \otimes y) = xy, \quad \delta(x \otimes v \otimes y) = xv \otimes y - x \otimes vy,$$

is a free resolution of $T(V)$. Conclude that A has Hochschild cohomological dimension 1 in the sense that $H^n(A, M) = 0$ for all M and all $n \geq 2$. Compute H^0 and H^1.

2. Let $A = k[x_1, \cdots, x_n]$ be the polynomial algebra in n variables over a field k of characteristic zero. Let V be an n dimensional vector space over k. Define a resolution of the form

$$0 \leftarrow A \leftarrow A \otimes A \leftarrow A \otimes V \otimes A \leftarrow \cdots A \otimes \wedge^i V \otimes A \cdots \leftarrow A \otimes \wedge^n V$$

by tensoring resolutions in 1) above for one dimensional vector spaces.

Conclude that for any symmetric A-bimodule M,

$$H^i(A, M) \simeq M \otimes \wedge^i V, \quad i = 0, 1, \cdots .$$

Hochschild Homology

The *Hochschild complex of A with coefficients in M*,

$$(C(A, M), \delta)$$

is defined by

$$C_0(A, M) = M, \qquad C_n(A, M) = M \otimes A^{\otimes n},$$

with $\delta : C_n(A, M) \longrightarrow C_{n-1}(A, M)$ defined by

$$\begin{aligned}
&\delta(m \otimes a_1 \otimes \cdots a_n) = ma_1 \otimes a_1 \cdots a_n \\
&+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes a_i a_{i+1} \cdots a_n \\
&+ (-1)^n a_n m \otimes a_1 \otimes \cdots a_n.
\end{aligned}$$

We denote this homology by

$$H_n(A, M).$$

Clearly,

$$H_0(A, M) = M/[A, M],$$

where $[A, M]$ is the subspace of M spanned by commutators $am - ma$ for a in A and m in M.

The following facts are easily established:

1. Hochschild homology $H_\bullet(A, M)$ is the derived functor of the functor

$$A \otimes A^{op} - Mod \longrightarrow k - Mod, \quad M \mapsto A \otimes_{A \otimes A^{op}} M,$$

i.e.

$$H_n(A, M) = Tor_n^{A \otimes A^{op}}(A, M).$$

For the proof one uses the bar resolution as before.

2. (Duality) Let $M^* = Hom(M, k)$. It is an A-bimodule via $(afb)(m) = f(bma)$. One checks that the natural isomorphism

$$Hom(A^{\otimes n}, M^*) \simeq Hom(M \otimes A^{\otimes n}, k), \quad n = 0, 1, \cdots$$

is compatible with differentials. Thus if k is field of characteristic zero, we have

$$H^\bullet(A, M^*) \simeq (H_\bullet(A, M))^*.$$

From now on we denote by $HH^n(A)$ the Hochschild group $H^n(A, A^*)$ and by $HH_n(A)$ the Hochschild group $H_n(A, A)$.

We give a few examples of Hochschild (co)homology computations. In particular we shall see that group homology and Lie algebra homology are instances of Hochschild homology. We start by recalling the classical results of Connes [**?**] and Hochschild-Kostant-Rosenberg [**?**] on the Hochschild homology of smooth commutative algebras.

Example (Commutative Algebras)
Let A be a commutative unital algebra over a ring k. We recall the definition of the *algebraic de Rham complex* of A. The module of 1-forms over A, denoted by $\Omega^1 A$, is defined to be a left A-module $\Omega^1 A$ with a universal derivation

$$d : A \longrightarrow \Omega^1 A.$$

This means that any other derivation $\delta : A \to M$ into a left A-module M, uniquely factorizes through

d. One usually defines $\Omega^1 A = I/I^2$ where I is the kernel of the multiplication map $A \otimes A \to A$. Note that since A is commutative this map is an algebra homomorphism. d is defined by

$$d(a) = a \otimes 1 - 1 \otimes a \quad \mathrm{mod}(I^2).$$

One defines the space of n-forms on A as the n-th exterior power of the A-module $\Omega^1 A$:

$$\Omega^n A := \wedge_A^n \Omega^1 A.$$

There is a unique extension of d to a graded derivation

$$d : \Omega^\bullet A \longrightarrow \Omega^{\bullet+1} A.$$

It satisfies the relation $d^2 = 0$. The *algebraic de Rham cohomology* of A is defined to be the cohomology of the complex $(\Omega^\bullet A, d)$.

Let M be a symmetric A-bimodule. We compare the complex $(M \otimes_A \Omega^\bullet A, 0)$ with the Hochschild complex of A with coefficients in M. Consider the *antisymmetrization map*

$$\varepsilon_n : M \otimes_A \Omega^n A \longrightarrow M \otimes A^{\otimes n}, \quad n = 0, 1, 2, \cdots,$$

$$\varepsilon_n(m \otimes da_1 \wedge \cdots \wedge da_n) = \sum_{\sigma \in S_n} sgn(\sigma) m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

where S_n denotes the symmetric group on n letters. We also have a map

$$\mu_n : M \otimes A^{\otimes n} \longrightarrow M \otimes_A \Omega^n A, \quad n = 0, 1, \cdots$$

$$\mu_n(m \otimes a_1 \otimes \cdots a_n) = m \otimes da_1 \wedge \cdots \wedge da_n.$$

One checks that both maps are morphisms of complexes, i.e.

$$\delta \circ \varepsilon_n = 0, \quad \mu_n \circ \delta = 0.$$

Moreover, one has

$$\mu_n \circ \varepsilon_n = n!\, Id_n.$$

It follows that if k is a field of characteristic zero then the antisymmetrization map

$$\varepsilon_n : M \otimes_A \Omega^n A \longrightarrow H_n(A, M),$$

is an inclusion. For $M = A$ we obtain a natural inclusion

$$\Omega^n A \longrightarrow HH_n(A).$$

The celebrated Hochschild-Kostant-Rosenberg theorem [**?**] states that if A is the algebra of regular functions on an smooth affine variety the above map is an isomorphism.

Let M be a smooth closed manifold and let $A =$

$C^\infty(M)$ be the algebra of smooth complex valued functions on M. It is a locally convex (in fact, Frechet) topological algebra. Fixing a finite atlas on M, one defines a family of seminorms

$$p_n(f) = sup\{|\partial^I(f)|;\ |I| \leq n\},$$

where the supremum is over all coordinate charts. It is easily seen that the induced topology is independent of the choice of atlas. In [**?**], using an explicit resolution, Connes shows that the canonical map

$$HH_n^{cont}(A) \to \Omega^n M, \qquad f_0 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \cdots df_n,$$

is an isomorphism. In fact the original, equivalent, formulation of Connes in [**?**] is for continuous Hochschild cohomology $HH^n(A)$ which is shown to be isomorphic to the continuous dual of $\Omega^n M$

(space of n-dimensional *de Rham currents*).

Example (Group Algebras)

It is clear from the original definitions that group (co)homology is an example of Hochschild (co)homolog[y]
Let G be a group and let M be a left G-module over the ground ring k. Recall that the standard complex for computing group cohomology is given by

$$M \xrightarrow{\delta} C^1(G,M) \xrightarrow{\delta} C^2(G,M) \xrightarrow{\delta} \cdots,$$

where

$$C^n(G,M) = \{f : G^n \longrightarrow M\}.$$

The differential δ is defined by

$$(\delta m)(g) = gm - m,$$

$$\begin{aligned}\delta f(g_1, \cdots, g_{n+1}) &= g_1 f(g_2, \cdots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \cdots g_i g_{i+1}, \cdots g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, g_2, \cdots, g_n).\end{aligned}$$

Let $A = kG$ denote the group algebra of the group G over k. Then M is a kG-bimodule via the two actions

$$g.m = g(m), \quad m.g = m,$$

for all g in G and m in M. It is clear that for all n,

$$C^n(kG, M) \simeq C^n(G, M),$$

and the two differentials are the same. It follows that the cohomology of G with coefficients in M coincides with the Hochschild cohomology of kG with coefficients in M.

Lie algebra (co)homology

We shall see that Lie algebra (co)homology is Hochschild (co)homology:

$$H_n^{Lie}(\mathfrak{g}, M) = H_n(U(\mathfrak{g}), M)$$

and dually for cohomology.

Let $\mathfrak{g}$ be a Lie algebra and M be a $\mathfrak{g}$-module given by a Lie algebra morphism

$$\mathfrak{g} \longrightarrow End_k(M).$$

The *Lie algebra homology* of $\mathfrak{g}$ with coefficients in M is the homology of the *Chevalley-Eilenberg complex*

$$M \xleftarrow{\delta} M \otimes \bigwedge^1 \mathfrak{g} \xleftarrow{\delta} M \otimes \bigwedge^2 \mathfrak{g} \xleftarrow{\delta} \cdots$$

with the differential

$$\delta(m \otimes X) = X(m),$$

$$\delta(m \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_n) =$$
$$\sum_{i<j}(-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \cdots \wedge \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_n$$
$$+ \sum_i (-1)^i X_i(m) \otimes X_1 \wedge \cdots \hat{X}_i \wedge \cdots \wedge X_n.$$

One checks that $\delta^2 = 0$.

Let $U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$. Given a $\mathfrak{g}$ module M we define a $U(\mathfrak{g})$-bimodule

$$M^{ad} = M$$

with left and right $U(\mathfrak{g})$-actions:

$$X \cdot m = X(m), \quad m \cdot X = 0.$$

Define a map

$$\varepsilon_n : C_n^{Lie}(\mathfrak{g}, M) \longrightarrow C_n(U(\mathfrak{g}), M^{ad}),$$

$$\varepsilon_n(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{\sigma \in S_n} sgn(\sigma) m \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}.$$

Fact: ε is a chain map (prove this!).

Claim: ε is a quasi-isomorphism, i.e. induces isomorphism on homology.

Sketch proof: use the Poincare-Birkhoff-Witt filtration on $U(\mathfrak{g})$ to define a filtration on $(C_\bullet(U(\mathfrak{g}), M), \delta)$. The associated E^1 term is the de Rham complex of the symmetric algebra $S(\mathfrak{g})$. The induced map is the antisymmetrization map

$$\varepsilon_n : M \otimes \wedge^n \mathfrak{g} \to M \otimes S(\mathfrak{g})^{\otimes n}.$$

By Hochschild-Kostant-Rosenberg theorem (for polynomial algebras), this map is a quasi-isomorphism hence the original map is a quasi-isomorphism.

Dual version: Lie algebra cohomology

$$C^n(\mathfrak{g}, M) = \mathrm{Hom}(\bigwedge^n \mathfrak{g}, M),$$

space of alternating n-linear maps $f(X_1, \cdots, X_n)$ on $\mathfrak{g}$ with values in M.

Chevalley-Eilenberg complex:

$$M \xrightarrow{\delta} C^1(\mathfrak{g}, M) \xrightarrow{\delta} C^2(\mathfrak{g}, M) \cdots$$

$$(\delta f)(X_1, \cdots, X_{n+1}) =$$
$$\sum_{i<j} (-1)^{i+j} f([X_i, X_j], \cdots, \widehat{X}_i \cdots, \widehat{X}_j \cdots, X_{n+1})$$
$$+ \sum_i (-1)^i X_i \cdot f(X_1, \cdots, \widehat{X}_i, \cdots, X_{n+1}).$$

Note: when $M = \mathbb{C}$, the ground field with trivial action, we shall denote the cohomology by $H^n_{Lie}(\mathfrak{g})$.

Relative Cohomology: Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. The relative complex:

$$C^n(\mathfrak{g}, \mathfrak{h}) = \{\varphi;\, i_X\varphi = i_X(\delta\varphi) = 0 \ \forall X \in \mathfrak{h}\}$$

Example 1: $\mathfrak{g}$ is abelian, i.e. $[X, Y] = 0$ for all X and Y and M is trivial. Then, clearly,

$$H_n^{Lie}(\mathfrak{g}) = \bigwedge^n \mathfrak{g}$$

for all n.

Example 2: $\mathfrak{g} = Lie(G)$, G a compact connected Lie group. Then

$$H^n_{Lie}(\mathfrak{g}) = H^n_{dR}(G).$$

Sketch proof:

$$(\bigwedge^n \mathfrak{g})^* = (\Omega^n G)^G$$

is the space of left invariant n-forms on G. Using Cartan's formula for d:

$$(d\omega)(X_1, \cdots, X_{n+1}) =$$

$$\sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i \cdots, \hat{X}_j \cdots, X_{n+1})$$

we see that the de Rham differential reduces to the Chevalley-Eilenberg differential and we have an inclusion of complexes:

$$(C(\mathfrak{g}), \delta) \hookrightarrow (\Omega G, d).$$

We are done using homotopy invariance and averaging over G (on board!).

Note: the principle used in this proof is very general and worth recording in a more abstract form. Let a compact connected topological group G act continuously on a topological chain complex $(C^\bullet, d)$. Then the inclusion of complexes

$$(C^\bullet, d)^G \hookrightarrow (C^\bullet, d)$$

is a quasi-isomorphism.

Note: The theorem fails if G is not compact. e.g. let $G = \mathbb{R}^n$, or, even better, consider the next example.

Example 3: Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. basis:

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Dual basis:

$$\varphi_0, \quad \varphi_1, \quad \varphi_2$$

Using

$$[X_0, X_1] = 2X_1, [X_0, X_2] = -2X_2, [X_1, X_2] = X_0$$

we obtain:

$$\delta\varphi_0 = -\varphi_1 \wedge \varphi_2, \quad \delta\varphi_1 = -2\varphi_0 \wedge \varphi_1, \quad \delta\varphi_2 = 2\varphi_0 \wedge \varphi_2$$

and

$$H^n_{Lie}(\mathfrak{sl}_2(\mathbb{R})) = \begin{cases} \mathbb{R} & n = 0, 3 \\ 0 & n \neq 0, 3, \end{cases}$$

Relative cohomology of the pair $\mathfrak{so}_2 \subset \mathfrak{sl}_2$: basis for $\mathfrak{so}_2$: $X = X_2 - X_1$.

$$i_X \varphi_0 = 0, \quad i_X \varphi_1 = -1, \imath_X \varphi_2 = 1$$

The only nontrivial relative cocycle: $\varphi_0 \wedge (\varphi_1 + \varphi_2)$ and hence

$$H^n_{Lie}(\mathfrak{sl}_2, \mathfrak{so}_2) = \begin{cases} \mathbb{R} & n = 0, 2 \\ 0 & n \neq 0, 2, \end{cases}$$

Example 4: Let $\mathfrak{g} = \mathsf{gl}_n(\mathbb{R})$. We have, with $l = [\frac{n}{2}]$,

$$H^*_{Lie}(\mathsf{gl}_n(\mathbb{R}) = H^*_{Lie}(\mathfrak{u}_n) = \bigwedge(u_1, u_3, \cdots, u_l)$$

Generators as left invariant differential forms:

$$u_k = [\mathsf{Tr}((g^{-1}dg)^k)]$$

Check: these forms are closed and invariant.

Example 4: (Gelfand-Fuks cohomology)

Let $\mathfrak{g} = Vect(M)$= Lie algebra of smooth vector fields on M. Topologize by: uniform convergence of partial derivatives of components of vector fields on compact subsets. Let

$$C^n_{GF}(\mathfrak{g}) = Hom_{con}(\bigwedge^n, \mathbb{R})$$

and

$$H^n_{GF}(Vect(M)) = H^n(C^n_{GF}(\mathfrak{g})$$

Amazing fact: For M compact, $H^n(Vect(M), \mathbb{R})$ are finite dimensional for all n.

Example: For $M = S^1$, Gelfand and Fuks have shown that $H^*(Vect(S^1))$, as an algebra, by 2 and 3 cocycles φ and ψ explicitly given by:

$$\varphi(f,g) = \int_{S^1} \left| \begin{array}{cc} f'(x) & f''(x) \\ g'(x) & g''(x) \end{array} \right|$$

$$\psi(f,g,h) = \int_{S^1} \begin{vmatrix} f(x) & f'(x) & f''(x) \\ g(x) & g'(x) & g''(x) \\ h(x) & h'(x) & h''(x) \end{vmatrix}$$

Formal vector fields

Hopf Cyclic Cohomology of Connes and Moscovici

In their study of index theory of transversally elliptic operators, Connes and Moscovici developed a cyclic cohomology theory for Hopf algebras which can be regarded, *post factum*, as the right noncommutative analogue of group homology and Lie algebra homology.

One of the main motivations was to obtain a noncommutative characteristic map

$$\chi_\tau : HC^*_{(\delta,\sigma)}(H) \longrightarrow HC^*(A),$$

for an action of a Hopf algebra H on an algebra A endowed with an "invariant trace" $\tau : A \to \mathbb{C}$. Here, the pair (δ, σ), called a modular pair in involution consists of a grouplike element $\sigma \in H$ and a character $\delta : H \to \mathbb{C}$, satisfying certain compatibility conditions explained below.

Later on the theory was extended, by Khalkhali-Rangipour and Hajac-Khalkhali-Rangipour-Sommerhaeu to a cyclic cohomology theory with coefficients for triples

$$(C, H, M)$$

where C is a coalgebra endowed with an action of a Hopf algebra H and M is an H-module and an H-comodule satisfying some compatibility conditions, called SAYD conditions (see below) The theory of Connes and Moscovici corresponds to $C = H$ equipped with the regular action of H and M a one dimensional SAYD H-module.

The idea was to view the Hopf-cyclic cohomology as the cohomology of the *invariant* part of certain natural complexes attached to (C, H, M). This is remarkably similar to interpreting the cohomology of the Lie algebra of a Lie group as the invariant part of the de Rham cohomology of the Lie group. The second main idea was to introduce *coefficients* into the theory. This also explained the important role played by modular pairs (δ, σ) in Connes-Moscovici's theory.

The module M is a noncommutative analogue of coefficients for Lie algebra and group homology theories. The periodicity condition

$$\tau_n^{n+1} = id$$

for the cyclic operator and the fact that all simplicial and cyclic operators have to descend to the invariant complexes, puts very stringent conditions on the type of the H-module M. This problem is solved by introducing the class of *stable anti-Yetter-Drinfeld modules* over a Hopf algebra.

The category of anti-Yetter-Drinfeld modules over a Hopf algebra H is a twisting, or 'mirror image' of the category of Yetter-Drinfeld H-modules. Technically it is obtained from the latter by replacing the antipode S by S^{-1} although this connection is hardly illuminating.

Coalgebras, Bialgebras, and Hopf Algebras

A coalgebra is a triple

$$(C,\, \Delta,\, \varepsilon)$$

where C is a linear space, and

$$\Delta : C \longrightarrow C \otimes C, \quad \varepsilon : C \longrightarrow k,$$

are linear maps called *comultiplication* and *counit*.

They satisfy the coassociativity and counit axioms:

$$(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta : C \longrightarrow C \otimes C \otimes C,$$

$$(\varepsilon \otimes I) \circ \Delta = (I \otimes \varepsilon) \circ \Delta = I.$$

C is called *cocommutative* if $\tau\Delta = \Delta$, where $\tau : C \otimes C \to C \otimes C$ is the *fillip* $x \otimes y \mapsto y \otimes x$.

Sweedler's notation with summation suppressed:

$$\Delta(c) = c^{(1)} \otimes c^{(2)}.$$

Coassociativity and counit axioms via Sweedler's notation:

$$c^{(1)} \otimes c^{(2)(1)} \otimes c^{(2)(2)} = c^{(1)(1)} \otimes c^{(1)(2)} \otimes c^{(2)},$$

$$\varepsilon(c^{(1)})(c^{(2)}) = c = (c^{(1)})\varepsilon(c^{(2)}).$$

Notation:

$$c^{(1)} \otimes c^{(2)} \otimes c^{(3)} := (\Delta \otimes I)\Delta(c).$$

Similarly, for *iterated comultiplication* maps

$$\Delta^n := (\Delta \otimes I) \circ \Delta^{n-1} : C \longrightarrow C^{\otimes(n+1)},$$

we write

$$\Delta^n(c) = c^{(1)} \otimes \cdots \otimes c^{(n+1)}.$$

Algebra notions have their dual analogues for coalgebras, like, *subcoalgebra*, (left, right, two sided) *coideal*, *quotient coalgebra*, and *morphism of coalgebras*.

A left C-*comodule* is a linear space M endowed with a left coaction of C, i.e. a linear map

$$\rho : M \longrightarrow C \otimes M$$

such that

$$(\rho \otimes 1)\rho = \Delta\rho \quad \text{and} \quad (\varepsilon \otimes 1)\rho = \rho$$

Notation:

$$\rho(m) = m^{(-1)} \otimes m^{(0)},$$

Similarly if M is a right C-comodule, we write

$$\rho(m) = m^{(0)} \otimes m^{(1)}$$

to denote its coaction $\rho : M \to M \otimes C$.

Convolution Product

Let C =coalgebra, A = algebra. Then

$$Hom(C, A)$$

is an associative unital algebra under the *convolution product* $f * g$ defined by

$$f * g = (f \otimes g)\Delta : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A,$$

or, equivalently, by

$$(f * g)(c) = f(c^{(1)})g(c^{(2)}).$$

Its unit is the map $e : C \to A$, $e(c) = \varepsilon(c)1_A$.

In particular the linear dual of a coalgebra C

$$C^* = Hom(C, k)$$

is an algebra.

Note: The linear dual of an algebra A is not a coalgebra unless it is finite dimensional. Ways to get around this: topologize A and consider just the continuous dual; or use topological tensor products...

Bialgebra

A bialgebra is an algebra equipped with a compatible coalgebra structure. Thus

$$\Delta : B \longrightarrow B \otimes B, \quad \varepsilon : B \longrightarrow k,$$

are morphisms of unital algebras.

Equivalently: multiplication and unit maps of B are morphisms of coalgebras.

Hopf Algebra

A Hopf algebra is a bilagebra endowed with an antipode, i.e. a linear map $S : H \to H$ with

$$S * I = I * S = \eta\varepsilon,$$

where $\eta : k \to H$ is the unit map of H. Equivalently,

$$S(h^{(1)})h^{(2)} = h^{(1)}S(h^{(2)}) = \varepsilon(h)1.$$

S is the inverse of the identity map $I : H \to H$ in the convolution algebra $Hom(H, H)$. This shows that the antipode is unique, if it exists at all.

The following properties of the antipode are well known:

1. If H is commutative or cocommutative then $S^2 = I$. The converse need not be true.

2. S is an anti-algebra map and an anti-coalgebra map. The latter means

$$S(h^{(2)}) \otimes S(h^{(1)}) = S(h)^{(1)} \otimes S(h)^{(2)},$$

for all $h \in H$.

Examples of Hopf algebras:

1. Commutative or cocommutative Hopf algebras are closely related to groups and Lie algebras, as we indicate below:

1.a. Let G be a discrete group (need not be finite), and $H = kG$ its group algebra. Let

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \text{and } \varepsilon(g) = 1$$

and linearly extend to H.

$$(H, \Delta, \varepsilon, S)$$

is a cocommutative Hopf algebra.

1.b. Let $\mathfrak{g}$ be a Lie algebra, and $H = U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Use the universal property of $U(\mathfrak{g})$ to define algebra homomorphisms

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad \varepsilon : U(\mathfrak{g}) \to k$$

and an anti-algebra map $S : U(\mathfrak{g}) \to U(\mathfrak{g})$, by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = -X,$$

for all $X \in \mathfrak{g}$. Then

$$(U(\mathfrak{g}), \Delta, \varepsilon, S)$$

is a cocommutative Hopf algebra. It is commutative iff $\mathfrak{g}$ is abelian, in which case $U(\mathfrak{g}) = S(\mathfrak{g}) =$ symmetric algebra of $\mathfrak{g}$.

1.c. Let G be a compact topological group. A continuous function $f : G \to \mathbb{C}$ is called representable if the set of left translates of f by all elements of G forms a finite dimensional subspace of $C(G)$. Clearly

$$H = Rep(G) \subset C(G)$$

is a subalgebra. Let $m : G \times G \to G$= multiplication of G and

$$m^* : C(G \times G) \to C(G),\ m^* f(x, y) = f(xy),$$

its dual map.

If f is representable, then

$$m^* f \in Rep(G) \otimes Rep(G) \subset C(G \times G).$$

Let e= the identity of G. Then

$$\Delta f = m^* f, \quad \varepsilon f = f(e), \quad (Sf)(g) = f(g^{-1}),$$

define a Hopf algebra structure on $Rep(G)$.

Alternatively: Let $Rep(G)$ = linear span of matrix coefficients of all finite dimensional complex reps of G. Peter-Weyl's Theorem: $Rep(G)$ is a dense subalgebra of $C(G)$. It is finitely generated (as an algebra) iff G is a Lie group.

1.d. Affine Group Scheme = Commutative Hopf

Algebra = Representable Functors

$$ComAlg_k \longrightarrow Groups$$

The coordinate ring of an affine algebraic group $H = k[G]$ is a commutative Hopf algebra. The maps Δ, ε, and S are the duals of the multiplication, unit, and inversion maps of G, respectively. More generally, for any commutative Hopf algebra H and a commutative algebra A,

$$Hom_{Alg}(H, A)$$

is a group under convolution product and

$$A \mapsto Hom_{Alg}(H, A)$$

is a functor from $ComAlg_k \rightarrow Groups$. Conversely, any representable functor

$$\mathsf{ComAlg}_k \rightarrow \mathsf{Groups}$$

is represented by a, unique up to isomorphism, commutative Hopf algebra.

2. Compact quantum groups and quantized enveloping algebras are examples of noncommutative and noncommutative Hopf algebras. We won't recall these examples here.

A very important example for noncommutative geometry and its applications to transverse geometry and number theory is the *Connes-Moscovici Hopf algebra* $\mathcal{H}_1$ which we recall now. Let $\mathfrak{g}_{aff}$ be the Lie algebra of the group of affine transformations of the line with linear basis X and Y and the relation

$$[Y, X] = X$$

Let $\mathfrak{g}$ be an abelian Lie algebra with basis

$$\{\delta_n; \quad n = 1, 2, \cdots\}$$

It is easily seen that $\mathfrak{g}_{aff}$ acts on $\mathfrak{g}$ via

$$[Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1},$$

for all n. Let

$$\mathfrak{g}_{CM} := \mathfrak{g}_{aff} \rtimes \mathfrak{g}$$

be the corresponding semidirect product Lie algebra. As an algebra, $\mathcal{H}_1$ coincides with the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{CM}$. Thus $\mathcal{H}_1$ is the universal algebra generated by

$$\{X, Y, \delta_n; n = 1, 2, \cdots\}$$

subject to relations

$$[Y,X]=X,\;\;[Y,\delta_n]=n\delta_n$$

$$[X,\delta_n]=\delta_{n+1},\;\;[\delta_k,\delta_l]=0,$$

for $n,k,l=1,2,\cdots$. We let the counit of $\mathcal{H}_1$ coincide with the counit of $U(\mathfrak{g}_{CM})$. Its coproduct and antipode, however, will be certain deformations of the coproduct and antipode of $U(\mathfrak{g}_{CM})$ as follows. Using the universal property of $U(\mathfrak{g}_{CM})$, one checks that the relations

$$\Delta Y=Y\otimes 1+1\otimes Y,\quad \Delta\delta_1=\delta_1\otimes 1+1\otimes\delta_1,$$

$$\Delta X=X\otimes 1+1\otimes X+\delta_1\otimes Y,$$

determine a unique algebra map $\Delta:\mathcal{H}_1\to\mathcal{H}_1\otimes\mathcal{H}_1$. Note that Δ is not cocommutative and it differs from the corrodent of the enveloping algebra $U(\mathfrak{g}_{CM})$. Similarly, one checks that there is a unique antialgebra map $S : \mathcal{H}_1 \to \mathcal{H}_1$ determined by the relations

$$S(Y) = -Y, \;\; S(X) = -X + \delta_1 Y, \;\; S(\delta_1) = -\delta_1.$$

Again we note that this antipode also differs from the antipode of $U(\mathfrak{g}_{CM})$, and in particular $S^2 \neq I$. In fact $S^n \neq I$ for all n.

In the next section we will show, following Connes-Moscovici, that $\mathcal{H}_1$ is a bicrossed product of Hopf algebras $U(\mathfrak{g}_{aff})$ and $U(\mathfrak{g})^*$, where $\mathfrak{g}$ is a pro-unipotent Lie algebra to be described in the next section.

In any Hopf algebra the set of grouplike elements G(H) is defined as

$$\Delta g = g \otimes g, \quad g \neq 0.$$

$G(H)$ is a subgroup of the multiplicative group of H.

e.g. for $H = kG$, $G(H) = G$.

Similarly primitive elements P(H) are defined as:

$$\Delta x = 1 \otimes x + x \otimes 1$$

$P(H)$ is a Lie algebra under the bracket $[x, y] := xy - yx$.

Poincare-Birkhoff-Witt $\Rightarrow P(U(\mathfrak{g})) = \mathfrak{g}$.

A Character is a unital algebra map

$$\delta : H \to k$$

e.g. the counit $\varepsilon : H \to k$ is a character. For a less trivial example, let G be a *non-unimodular* real Lie group and $H = U(\mathfrak{g})$ the universal enveloping algebra of the Lie algebra of G. The *modular function* of G, measuring the difference between the right and left Haar measures on G, is a smooth group homomorphism $\Delta : G \to \mathbb{R}^+$. Its derivative at identity defines a Lie algebra map $\delta : \mathfrak{g} \to \mathbb{R}$. We denote its natural extension by $\delta : U(\mathfrak{g}) \to \mathbb{R}$. It is obviously a character of $U(\mathfrak{g})$. For a concrete example, let $G = Aff(\mathbb{R})$ be the group of affine transformations of the real line. It is a non-unimodular group with modular homomorphism given by

$$\Delta \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = |a|.$$

The corresponding infinitesimal character on $\mathfrak{g}_{aff} = Lie(G)$ is given by

$$\delta(Y) = 1, \quad \delta(X) = 0,$$

where $Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are a basis for $\mathfrak{g}_{aff}$. We will see that this character plays an important role in constructing a *twisted antipode* for the Connes-Moscovici Hopf algebra $\mathcal{H}_1$.

If H is a Hopf algebra, by a left H-module (resp. left H-comodule), we mean a left module (resp. left comodule) over the underlying algebra (resp. the underlying coalgebra) of H.

Monoidal Categories

A monoidal category

$$(\mathcal{C},\, \otimes,\, U,\, a,\, l,\, r)$$

consists of a category $\mathcal{C}$, a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

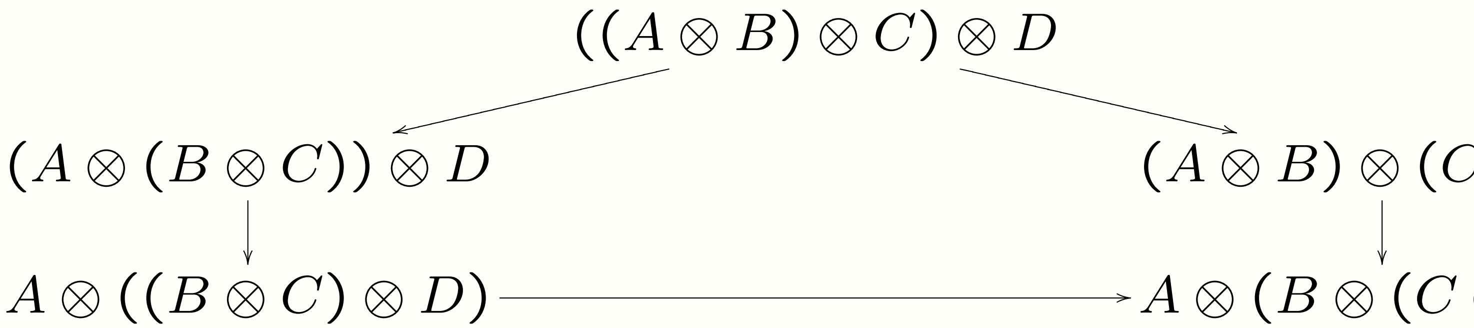
, an object $U \in \mathcal{C}$ (called the *unit object*), and natural isomorphisms

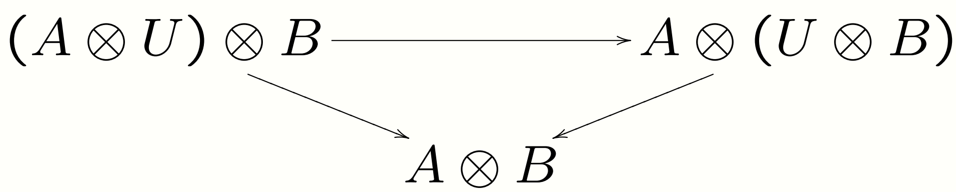
$$a = a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

$$l = l_A : U \otimes A \to A, \qquad r = r_A : A \otimes U \to A,$$

called associativity and unit constraints. Moreover the following

Pentagon and Triangle diagrams should commute:





Coherence Theorem (MacLane): all diagrams composed from a, l, r by tensoring, substituting and composing, commute.

A Braided Monoidal Category is a monoidal Category endowed with natural isomorphisms

$$c_{A,B} : A \otimes B \to B \otimes A,$$

called braiding such that the folowing diagram, the Hexagon Axiom, is commutative:

$$\begin{array}{ccccccc}
 & & A \otimes (B \otimes C) & \longrightarrow & (B \otimes C) \otimes A & & \\
 & \nearrow & & & & \searrow & \\
(A \otimes B) \otimes C & & & & & & (C \otimes B) \otimes \\
 & \searrow & & & & \nearrow & \\
 & & (B \otimes A) \otimes C & \longrightarrow & C \otimes (B \otimes A) & &
\end{array}$$

A braiding is called a *symmetry* if we have

$$c_{A,B} \circ c_{B,A} = I$$

for all A, B.

A *symmetric monoidal category* is a monoidal category endowed with a symmetry.

Example: Let H be a bialgebra. Using the comultiplication of H, the category

$$H - Mod$$

of left H-modules is a monoidal category via

$$h(m \otimes n) = h^{(1)}m \otimes h^{(2)}n,$$

and unit object $U = k$ with trivial H-action:

$$h(1) = \varepsilon(h)1,$$

If H is cocommutative, then one checks that the map

$$c_{M,N} : M \otimes N \to N \otimes M, \quad c_{M,N}(m \otimes n) = n \otimes m,$$

is a morphism of H-modules and is a symmetry operator on $H - Mod$, turning it into a symmetric monoidal category.

The category $H-Mod$ is not braided in general. For that to happen, one must either restrict the class of modules to what is called Yetter-Drinfeld modules, or restrict the class of Hopf algebras to quasitriangular Hopf algebras to obtain a braiding on $H-Mod$. We shall discuss the first scenario and will see that, quite unexpectedly, this question is closely related to Hopf cyclic cohomology!

Similarly, the category

$$H-Comod$$

of left H-comodules is a monoidal category:

$$\rho(m\otimes n)=m^{(-1)}n^{(-1)}\otimes m^{(0)}\otimes n^{(0)}.$$

If H is commutative then $H-Comod$ is a symmetric monoidal category. More generally, when H is coquasitriangular, $H - Comod$ can be endowed with a braiding.

Symmetry in Noncommutative Geometry

The idea of *symmetry* in noncommutative geometry is encoded by the *action* or *coaction* of a Hopf algebra on an algebra or on a coalgebra. Thus there are four possibilities in general that will be referred to as (Hopf-) *module algebra*, *module coalgebra*, *comodule algebra*, and *comodule coalgebra*.

For each type of symmetry there is a corresponding Hopf cyclic cohomology theory with coefficients. These theories in a certain sense are generalizations of equivariant de Rham cohomology with coefficients in an equivariant local system.

An algebra A is called a left H-module algebra if

-A is a left H-module

- multiplication and unit maps

$$A \otimes A \to A, \quad k \to A$$

are morphisms of H-modules, i.e.

$$h(ab) = h^{(1)}(a)h^{(2)}(b), \quad \text{and} \quad h(1) = \varepsilon(h)1.$$

Using the relations

$$\Delta g = g \otimes g, \quad \Delta x = 1 \otimes x + x \otimes 1$$

it is easily seen that in an H-module algebra, group-like elements act as unit preserving automorphisms while primitive elements act as derivations. E.g. for $H = kG$, H-module algebra structure on A is simply an action of G by unit preserving automorphisms on A. Similarly, we have a 1-1 correspondence between $U(\mathfrak{g})$-module algebra structures on

A and Lie actions of the Lie algebra $\mathfrak{g}$ by derivations on A.

Left H-comodule algebra B:

-B is a left H-comodule

- multiplication and unit maps of B are H-comodule maps.

Left H-module coalgebra C:

-C is a left H-module

comultiplication and counit maps of C

$$\Delta : C \to C \otimes C, \quad \varepsilon : C \to k$$

are H-module maps:

$$(hc)^{(1)} \otimes (hc)^{(2)} = h^{(1)} c^{(1)} \otimes h^{(2)} c^{(2)}, \; \varepsilon(hc) = \varepsilon(h)\varepsilon(c).$$

Example: the coproduct $\Delta : H \to H \otimes H$ turns H into a left (and right) H-comodule algebra. The product $H \otimes H \to H$ turns H into a left (and right) H-module coalgebra. These are noncommutative analogues of translation action of a group on itself. The *conjugation action* $H \otimes H \to H$,

$$g \otimes h \mapsto g^{(1)} h S(g^{(2)})$$

turns H into a left H-module algebra.

Example: An important feature of the Connes-Moscovici$\mathcal{H}_1$, and in fact its *raison d'être*, is that it acts as quantum symmetries of various objects of interest in noncommutative geometry, like the 'space' of leaves of codimension one foliations or the 'space' of modular forms modulo the action of Hecke correspondences. Let M be a one dimensional manifold and $A = C_0^\infty(F^+M)$ denote the algebra of smooth functions with compact support on the bundle of positively oriented frames on M. Given a discrete group $\Gamma \subset Diff^+(M)$ of orientation preserving diffeomorphisms of M, one has a natural prolongation of the action of Γ to $F^+(M)$ by

$$\varphi(y, y_1) = (\varphi(y), \varphi'(y)(y_1)).$$

Let

$$A_\Gamma = C_0^\infty(F^+M) \rtimes \Gamma$$

denote the corresponding crossed product algebra. Elements of A_Γ consist of finite linear combinations (over $\mathbb{C}$) of terms fU_φ^* with $f \in C_0^\infty(F^+M)$ and

$\varphi \in \Gamma$. Its product is defined by

$$fU_\varphi^* \cdot gU_\psi^* = (f \cdot \varphi(g))U_{\psi\varphi}^*.$$

There is an action of $\mathcal{H}_1$ on A_Γ given by [**?**, **?**]:

$$Y(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y_1} U_\varphi^*, \quad X(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y} U_\varphi^*,$$

$$\delta_n(fU_\varphi^*) = y_1^n \frac{d^n}{dy^n}(log \frac{d\varphi}{dy}) fU_\varphi^*.$$

Once these formulas are given, it can be checked, by a long computation, that A_Γ is indeed an $\mathcal{H}_1$-module algebra. In the original application, M is a transversal for a codimension one foliation and thus $\mathcal{H}_1$ acts via transverse differential operators.

Modular Hecke Algebras and the Action of $\mathcal{H}_1$

We recall, very briefly, the action of the Hopf algebra $\mathcal{H}_1$ on the so called *modular Hecke algebras*, discovered by Connes and Moscovici where a very intriguing dictionary comparing transverse geometry notions with modular forms notions can be found. For each $N \geq 1$, let

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z});\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

denote the level N congruence subgroup of $\Gamma(1) = SL(2,\mathbb{Z})$. Let $\mathcal{M}_k(\Gamma(N))$ denote the space of modular forms of level N and weight k and

$$\mathcal{M}(\Gamma(N)) := \oplus_k \mathcal{M}_k(\Gamma(N))$$

be the graded algebra of modular forms of level N.

Finally let

$$\mathcal{M} := \varinjlim_{N} \mathcal{M}(\Gamma(N))$$

denote the algebra of modular forms of all levels, where the inductive system is defined by divisibility. The group

$$G^+(\mathbb{Q}) := GL^+(2, \mathbb{Q}),$$

acts on $\mathcal{M}$ by its usual action on functions on the upper half plane (with corresponding weight):

$$(f, \alpha) \mapsto f|_k \alpha(z) = det(\alpha)^{k/2}(cz+d)^{-k} f(\alpha \cdot z),$$

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \alpha.z = \frac{az+b}{cz+d}.$$

The elements of the corresponding crossed-product algebra

$$\mathcal{A} = \mathcal{A}_{G^+(\mathbb{Q})} := \mathcal{M} \rtimes G^+(\mathbb{Q}),$$

are finite sums

$$\sum fU_\gamma^*, \quad f \in \mathcal{M}, \ \gamma \in G^+(\mathbb{Q}),$$

with a product defined by

$$fU_\alpha^* \cdot gU_\beta^* = (f \cdot g|\alpha)U_{\beta\alpha}^*.$$

$\mathcal{A}$ can be thought of as the algebra of 'noncommutative coordinates' on the 'noncommutative quotient space' of modular forms modulo Hecke correspondences.

Consider the operator X of degree two on the space of modular forms defined by

$$X := \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{12\pi i}\frac{d}{dz}(log\Delta) \cdot Y,$$

where

$$\Delta(z) = (2\pi)^{12}\eta^{24}(z) = (2\pi)^{12}q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi$$

and Y denotes the grading operator

$$Y(f) = \frac{k}{2} \cdot f, \quad \text{for all } f \in \mathcal{M}_k.$$

The following theorem is proved by Connes-Moscivici. It shows that $\mathcal{A}_{G+(\mathbb{Q})}$ is an $\mathcal{H}_1$-module algebra:

Theorem: There is a unique action of $\mathcal{H}_1$ on $\mathcal{A}_{G+(\mathbb{Q})}$ determined by

$$X(fU_\gamma^*) = X(f)U_\gamma^*, \quad Y(fU_\gamma^*) = Y(f)U_\gamma^*,$$

$$\delta_1(fU_\gamma^*) = \mu_\gamma \cdot f(U_\gamma^*),$$

where

$$\mu_\gamma(z) = \frac{1}{2\pi i}\frac{d}{dz}\log\frac{\Delta|\gamma}{\Delta}.$$

More generally, for any congruence subgroup Γ an algebra $A(\Gamma)$ is constructed that contains as subalgebras both the algebra of Γ-modular forms and the Hecke ring at level Γ. There is also a corresponding action of $\mathcal{H}_1$ on $A(\Gamma)$.

Bicrossed Products

For four types of symmetries there is a corresponding *crossed product* construction as well as a more elaborate version called *bicrossed product* construction. The Connes-Moscovici Hopf algebra $\mathcal{H}_1$ is a bicrossed product of two, easy to describe, Hopf algebras.

Crossed Product: Let $A =$ left H-module algebra. Crossed product algebra $A \rtimes H = A \otimes H$ with product

$$(a \otimes g)(b \otimes h) = a(g^{(1)}b) \otimes g^{(2)}h.$$

E.g. for $H = kG$, recover the crossed product algebra $A \rtimes G$.

Also, for $\mathfrak{g}$ acting by derivations on a commutative

A. $A \rtimes U(\mathfrak{g})$ is a subalgebra of the algebra of differential operators on A generated by derivations from $\mathfrak{g}$ and multiplication operators by elements of A.

Simple example: $A = k[x]$ and $\mathfrak{g} = k$ acting via $\frac{d}{dx}$ on A. Then $A \rtimes U(\mathfrak{g}) =$ *Weyl algebra* of differential operators on the line with polynomial coefficients.

Crossed Coproduct: let $D =$ right H-comodule coalgebra with coaction $d \mapsto d^{(0)} \otimes d^{(1)} \in D \otimes H$. $H \rtimes D = H \otimes D$ is a coalgebra with coproduct and counit:

$$\Delta(h \otimes d) = h^{(1)} \otimes (d^{(1)})^{(0)} \otimes h^{(2)}(d^{(1)})^{(1)} \otimes d^{(2)}$$

$$\varepsilon(h \otimes d) = \varepsilon(d)\varepsilon(h).$$

These constructions deform multiplication or comultiplication, of algebras or coalgebras, respectively. Thus to obtain a simultaneous deformation of multiplication and comultiplication of a Hopf algebra it stands to reason to try to apply both constructions simultaneously. This idea, going back to G. I. Kac in 1960's in the context of Kac-von Neumann Hopf algebras, has now found its complete generalization in the notion of *bicrossed product* of *matched pairs* of Hopf algebras. There are many variations of this construction of which the most relevant for the structure of the Connes-Moscovici Hopf algebra is the following:

Let U and F be Hopf algebras s.t.

- F is a left U-module algebra

- U is a right F-comodule coalgebra

$$\rho : U \to U \otimes F.$$

(U, F) is called a matched pair if

$$\epsilon(u(f)) = \epsilon(u)\epsilon(f),$$

$$\Delta(u(f)) = (u^{(1)})^{(0)}(f^{(1)}) \otimes (u^{(1)})^{(1)}(u^{(2)}(f^{(2)})),$$

$$\rho(1) = 1 \otimes 1,$$

$$\rho(uv) = (u^{(1)})^{(0)}v^{(0)} \otimes (u^{(1)})^{(1)}(u^{(2)}(v^{(1)})),$$

$$(u^{(2)})^{(0)} \otimes (u^{(1)}(f))(u^{(2)})^{(1)} =$$

$$(u^{(1)})^{(0)} \otimes (u^{(1)})^{(1)}(u^{(2)}(f)).$$

Given a matched pair as above, we define its bicrossed product Hopf algebra $F \rtimes U$ to be $F \otimes U$

with crossed product algebra structure and crossed coproduct coalgebra structure. Its antipode S is defined by

$$S(f \otimes u) = (1 \otimes S(u^{(0)}))(S(fu^{(1)}) \otimes 1).$$

It is a remarkable fact that, thanks to the above compatibility conditions, all the axioms of a Hopf algebra are satisfied for $F \rtimes U$.

Example: The simplest, and first, example of a bicrossed product Hopf algebra is as follows. Let

$$G = G_1 G_2$$

be a *factorization* of a finite group G. This means that G_1 and G_2 are subgroups of G and

$$G_1 \cap G_2 = \{e\} \quad \text{and} \quad G_1 G_2 = G$$

We denote the factorization of an element g by $g = g_1 g_2$. The relation

$$g \cdot h := (gh)_2$$

defines a left action of G_1 on G_2 and

$$g \bullet h := (gh)_1$$

defines a right action of G_2 on G_1. Let

$$F = F(G_2), \quad U = kG_1$$

The first action turns F into a left U-module algebra. The second action turns U into a right F-comodule coalgebra. The latter coaction is simply the dual of the map $F(G_1) \otimes kG_2 \to F(G_1)$ induced by the right action of G_2 on G_1.

Remark: By a theorem of Kostant, any cocommutative Hopf algebra H over an algebraically closed field k of characteristic zero is isomorphic (as a Hopf algebra) with a crossed product algebra

$$H = U(P(H)) \rtimes kG(H)$$

where $P(H)$ is the Lie algebra of primitive elements of H and $G(H)$ is the group of grouplike elements of H and $G(H)$ acts on $P(H)$ by inner automorphisms. The coalgebra structure of $H = U(P(H)) \rtimes kG(H)$ is simply the tensor product of the two coalgebras $U(P(H))$ and $kG(H)$.

Example: The Connes-Moscovici Hopf algebra $\mathcal{H}_1$ is a bicrossed product Hopf algebra. Let

$$G = Diff(\mathbb{R})$$

denote the group of diffeomorphisms of the real line. It has a factorization of the form

$$G = G_1 G_2,$$

where G_1 is the subgroup of diffeomorphisms that satisfy

$$\varphi(0) = 0, \quad \varphi'(0) = 1,$$

and G_2 is the $ax + b$- group of affine transformations.

The first Hopf algebra, F, is formally speaking, the *algebra of polynomial functions* on the pro-unipotent group G_1. It can also be defined as the "continuous dual" of the enveloping algebra of the Lie algebra of G_1. It is a commutative Hopf algebra generated by functions δ_n, $n = 1, 2, \ldots$, defined by

$$\delta_n(\varphi) = \frac{d^n}{dt^n}(log(\varphi'(t))|_{t=0}.$$

The second Hopf algebra, U, is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_2$ of the $ax+b$-group. It has generators X and Y and one relation $[X,Y]=X$.

F is a right U-module algebra:

$$\delta_n(X) = -\delta_{n+1}, \quad \text{and } \delta_n(Y) = -n\delta_n.$$

U is a left F-comodule coalgebra:

$$X \mapsto 1 \otimes X + \delta_1 \otimes X, \quad \text{and } Y \mapsto 1 \otimes Y.$$

One can check that they are a matched pair of Hopf algebras and the resulting bicrossed product Hopf algebra is isomorphic to the Connes-Moscovici Hopf algebra $\mathcal{H}_1$.

Example: The Drinfeld double $D(H)$ of a finite dimensional Hopf algebra H is a bicrossed product:

$$D(H) = H \rtimes H^*$$

Modular Pair in Involution

Let $\delta : H \to k$ be a character and $\sigma \in H$ a group-like element. Following Connes-Moscovici, (δ, σ) is called a *modular pair* if

$$\delta(\sigma) = 1$$

and a *modular pair in involution* if in addition:

$$\widetilde{S}_\delta^2 = Ad_\sigma, \quad \text{or,} \quad \widetilde{S}_\delta^2(h) = \sigma h \sigma^{-1}$$

Here the δ- twisted antipode $\widetilde{S}_\delta : H \to H$ is defined by $\widetilde{S}_\delta = \delta * S$, i.e.

$$\widetilde{S}_\delta(h) = \delta(h^{(1)}) S(h^{(2)}),$$

for all $h \in H$.

The notion of an invariant trace for actions of groups and Lie algebras can be extended to the

Hopf setting. For applications to transverse geometry and number theory, it is important to formulate a notion of ‘invariant trace’ twisted by a modular pair (δ, σ) as follows.

Let A be an H-module algebra. A linear map $\tau : A \to k$ is called δ-*invariant* if for all $h \in H$ and $a \in A$,

$$\tau(h(a)) = \delta(h)\tau(a).$$

τ is called a σ-*trace* if for all a, b in A,

$$\tau(ab) = \tau(b\sigma(a)).$$

For the following formula the fact that A is unital is crucial. For $a, b \in A$, let

$$< a, b >:= \tau(ab).$$

Integration by parts formula: Let τ be a σ-trace on A. Then τ is δ-invariant if and only if the *integration by parts formula* holds:

$$< h(a), b > = < a, \widetilde{S}_\delta(h)(b) >,$$

for all $h \in H$ and $a, b \in A$.

Loosely speaking, the lemma says that the formal adjoint of the differential operator h is $\widetilde{S}_\delta(h)$.

Example 1: For any Hopf algebra H, the pair $(\varepsilon, 1)$ is modular. It is involutive if and only if $S^2 = id$. This happens, for example, when H is a commutative or cocommutative Hopf algebra.

Example 2: The original non-trivial example of a modular pair in involution is the pair $(\delta, 1)$ for

Connes-Moscovici Hopf algebra $\mathcal{H}_1$. Let δ denote the unique extension of the modular character

$$\delta : \mathfrak{g}_{aff} \to \mathbb{R}, \quad \delta(X) = 1,\ \delta(Y) = 0,$$

to a character $\delta : U(\mathfrak{g}_{aff}) \to \mathbb{C}$. There is a unique extension of δ to a character, denoted by the same symbol $\delta : \mathcal{H}_1 \to \mathbb{C}$. The relations

$$[Y, \delta_n] = n\delta_n$$

show that

$$\delta(\delta_n) = 0$$

for $n = 1, 2, \cdots$. One can then check that these relations are compatible with the algebra structure of $\mathcal{H}_1$.

The algebra

$$A_\Gamma = C_0^\infty(F^+(M) \rtimes \Gamma$$

admits a δ-invariant trace $\tau : A_\Gamma \to \mathbb{C}$ given by:

$$\tau(fU_\varphi^*) = \int_{F^+(M)} f(y, y_1) \frac{dy dy_1}{y_1^2}, \quad \text{if } \varphi = 1,$$

and $\tau(fU_\varphi^*) = 0$, otherwise.

Example 3: Let $H = A(SL_q(2,k)) =$ Hopf algebra of functions on quantum $SL(2,k)$. As an algebra it is generated by symbols x, u, v, y, with the following relations:

$$ux = qxu, \ \ vx = qxv, \ \ yu = quy, \ \ yv = qvy,$$

$$uv = vu, \ \ xy - q^{-1}uv = yx - quv = 1.$$

The coproduct, counit and antipode of $\mathcal{H}$ are defined by

$$\Delta(x) = x \otimes x + u \otimes v, \quad \Delta(u) = x \otimes u + u \otimes y,$$

$$\Delta(v) = v \otimes x + y \otimes v, \quad \Delta(y) = v \otimes u + y \otimes y,$$

$$\epsilon(x) = \epsilon(y) = 1, \quad \epsilon(u) = \epsilon(v) = 0,$$

$$S(x) = y, \;\; S(y) = x, \;\; S(u) = -qu, \;\; S(v) = -q^{-1}v.$$

Define a character $\delta : H \to k$ by:

$$\delta(x) = q, \;\; \delta(u) = 0, \;\; \delta(v) = 0, \;\; \delta(y) = q^{-1}.$$

One checks that $\widetilde{S}_\delta^2 = id$. This shows that $(\delta, 1)$ is a modular pair for H.

More generally, Connes-Moscovici show that coribbon Hopf algebras and compact quantum groups are endowed with canonical modular pairs of the form $(\delta, 1)$ and, dually, ribbon Hopf algebras have canonical modular pairs of the type $(1, \sigma)$. Notice that a pair (1, σ) with σ a group-like element is an MPI iff for all $h \in H$

$$S^2(h) = \sigma h \sigma^{-1}$$

Example 4: We shall see later that modular pairs in involution are in fact one dimensional cases of stable anti-Yetter-Drinfeld modules, i.e. they are one dimensional noncommutative local systems over a quantum group.

Anti-Yetter-Drinfeld Modules

An important question: to identify the most general class of coefficients allowable in cyclic (co)homology of Hopf algebras and Hopf (co) module (co)algebras in general.

This problem is completely solved by Hajac, Rangipour, Sommerhauser, and M.K., following the work of M.K. and B. Rangipour. It was shown that the most general coefficients are the class of so called stable anti-Yetter-Drinfeld modules.

It is quite surprising that when the general formalism of cyclic cohomology theory, namely the theory of (co)cyclic modules is applied to Hopf algebras, variations of such standard notions like

Yetter-Drinfeld (YD) modules appear naturally. The so called anti-Yetter-Drinfeld modules are twistings, by modular pairs in involution, of YD modules. This means that the category of anti-Yetter-Drinfeld modules is a “mirror image” of the category of YD modules.

Yetter-Drinfeld Modules

Motivation: How to define a *braiding* on the monoidal category $H-Mod$ or a subcategory of it?

To define a braiding one should either restrict to special classes of Hopf algebras, or, to special classes of modules. Drinfeld showed that when H is a *quasitriangular*, $H-Mod$ is a braided monoidal category. Dually, when H is *coquasitriangular*, $H-Comod$ is a braided monoidal category. Yetter shows that to obtain a braiding on a subcategory of $H-Mod$, for an arbitrary H, one has essentially one choice and that is restricting to the class of Yetter-Drinfeld modules as we explain now.

Definition: Let M be a left H-module and a left H-comodule. M is a left-left *Yetter-Drinfeld* H-

module if the two structures on M are compatible in the sense that

$$\rho(hm) = h^{(1)}m^{(-1)}S(h^{(3)}) \otimes h^{(2)}m^{(0)}.$$

We denote the category of left-left YD modules over H by ${}^H_H\mathcal{YD}$.

Facts about ${}^H_H\mathcal{YD}$:

1. The tensor product $M \otimes N$ of two YD modules is a YD module. Its module and comodule structure are the standard ones:

$$h(m \otimes n) = h^{(1)}m \otimes h^{(1)}n$$

$$(m \otimes n) \mapsto m^{(-1)}n^{(-1)} \otimes m^{(0)} \otimes n^{(0)}$$

This shows that the category ${}^H_H\mathcal{YD}$ is a monoidal subcategory of the monoidal category $H - Mod$.

2. The category ${}^H_H\mathcal{YD}$ is braided under the braiding

$$c_{M,N} : M \otimes N \longrightarrow N \otimes M$$

$$m \otimes n \mapsto m^{(-1)} \cdot n \otimes m^{(0)}$$

Yetter proved a strong inverse to this statement: for any small strict monoidal category $\mathcal{C}$ endowed with a monoidal functor $F : \mathcal{C} \to Vect_f$ to the category of finite dimensional vector spaces, there is a Hopf algebra H and a monoidal functor $\tilde{F} : \mathcal{C} \to {}^H_H\mathcal{YD}$ such that the following diagram comutes

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & {}^H_H\mathcal{YD} \\ \downarrow & & \downarrow \\ Vect_f & \longrightarrow & Vect \end{array}$$

3. ${}^H_H\mathcal{YD}$ is the center of $H - Mod$. Recall that the (left) center $\mathcal{ZC}$ of a monoidal category is a

category whose objects are pairs $(X, \sigma_{X,-})$, where X is an object of $\mathcal{C}$ and

$$\sigma_{X,-} : X \otimes - \to - \otimes X$$

is a natural isomorphism satisfying certain compatibility axioms with associativity and unit constraints. It can be shown that the center of a monoidal category is a braided monoidal category and

$$\mathcal{Z}(H - Mod) =^H_H \mathcal{YD}$$

4. If H is finite dimensional then the category ${}^H_H\mathcal{YD}$ is isomorphic to the category of left modules over the Drinfeld double $D(H)$.

Example 1: 1. Let $H = kG$ be the group algebra of a discrete group G. A left kG-comodule is simply a G-graded vector space

$$M = \bigoplus_{g \in G} M_g$$

where the coaction is defined by

$$m \in M_g \mapsto g \otimes m.$$

An action of G on M defines a YD module structure iff for all $g, h \in G$,

$$hm \in M_{hgh^{-1}}.$$

This example can be explained as follows. Let $\mathcal{G}$ be a groupoid whose objects are G and its morphisms are defined by

$$Hom(g, h) = \{k \in G;\ kgk^{-1} = h\}.$$

Recall that an *action* of a groupoid $\mathcal{G}$ on the category $Vect$ of vector spaces is simply a functor $F : \mathcal{G} \to Vect$. Then it is easily seen that we have a one-one correspondence between YD modules for kG and actions of $\mathcal{G}$ on $Vect$. This example clearly shows that one can think of an YD module over kG as an *'equivariant sheaf'* on G and of YD modules as noncommutative analogues of equivariant sheaves on a topological group.

Example 2: If H is cocommutative then any left H-module M can be turned into a left-left YD module via the coaction $m \mapsto 1 \otimes m$. Similarly, when H is cocommutative then any left H-comodule M can be turned into a YD module via the H-action $h \cdot m := \varepsilon(h)m$.

Example 3: Any Hopf algebra acts on itself via *conjugation action* $g \cdot h := g^{(1)}hS(g^{(2)})$ and coacts via translation coaction $h \mapsto h^{(1)} \otimes h^{(2)}$. It can be checked that this endows $M = H$ with a YD module structure.

Stable anti-Yetter-Drinfeld modules

This class of modules for Hopf algebras were introduced for the first time by Hajac-Khalkhali-Rangipour-Sommerhauser. Unlike Yetter-Drinfeld modules, its definition, however, was entirely motivated and dictated by cyclic cohomology theory: the anti-Yetter-Drinfeld condition guarantees that the simplicial and cyclic operators are well defined on invariant complexes and the stability condition implies that the crucial periodicity condition for cyclic modules are satisfied.

Definition: A left-left anti-Yetter-Drinfeld H-module is a left H-module and left H-comodule such that

$$\rho(hm) = h^{(1)}m^{(-1)}S(h^{(3)}) \otimes h^{(2)}m^{(0)},$$

for all $h \in H$ and $m \in M$. We say M is stable if in addition we have

$$m^{(-1)}m^{(0)} = m,$$

for all $m \in M$.

Notice that by changing S to S^{-1} in the above equation, we obtain the compatibility condition for a Yetter-Drinfeld module.

The following lemma shows that 1-dimensional SAYD modules correspond to Connes-Moscovici's modular pairs in involution:

Lemma: There is a one-one correspondence between modular pairs in involution (δ, σ) on H and SAYD module structure on $M = k$, defined by

$$h.r = \delta(h)r, \quad r \mapsto \sigma \otimes r.$$

We denote this module by $M = {}^{\sigma}k_{\delta}$.

Let ${}^{H}_{H}\mathcal{AYD}$ denote the category of left-left anti-Yetter-Drinfeld H-modules, where morphisms are H-linear and H-colinear maps. Unlike YD modules, anti-Yetter-Drinfeld modules do not form a monoidal category under the standard tensor product. This can be checked easily on 1-dimensional modules given by modular pairs in involution. The following result of HKRS, however, shows that the tensor product of an anti-Yetter-Drinfeld module with a Yetter-Drinfeld module is again anti-Yetter-Drinfeld.

Lemma: Let M be a Yetter-Drinfeld module and N be an anti-Yetter-Drinfeld module (both left-left). Then $M \otimes N$ is an anti-Yetter-Drinfeld module under the diagonal action and coaction:

$$h(m \otimes n) = h^{(1)}m \otimes h^{(1)}n,$$

$$(m \otimes n) \mapsto m^{(-1)}n^{(-1)} \otimes m^{(0)} \otimes n^{(0)}.$$

In particular, using a modular pair in involution (δ, σ), we obtain a functor

$${}^{H}_{H}\mathcal{YD} \to {}^{H}_{H}\mathcal{AYD}, \quad M \mapsto \bar{M} = {}^{\sigma}k_{\delta} \otimes M.$$

This result clearly shows that AYD modules can be regarded as the *twisted analogue* or *mirror image* of YD modules, with twistings provided by modular pairs in involution. This result was later strengthened by the following result, pointed out to us by M. Staic. It shows that if the Hopf algebra has a modular pair in involution then the category of

YD modules is equivalent to the category of AYD modules:

Proposition: Let H be a Hopf algebra, (δ, σ) a modular pair in involution and M an anti-Yetter-Drinfeld module. If we define $m \cdot h = mh^{(1)}\delta(S(h^{(2)}))$ and $\rho(m) = \sigma^{-1}m^{(-1)} \otimes m^{(0)}$, then $(M, \cdot, \rho)$ is an Yetter-Drinfeld module. This defines an isomorphism between the categories of AYD and YD modules.

It follows that tensoring with ${}^{\sigma^{-1}}k_{\delta \circ S}$ turns the anti-Yetter-Drinfeld modules to Yetter-Drinfeld modules and this is the inverse for the operation of tensoring with ${}^{\sigma}k_{\delta}$.

Example 1: For Hopf algebras with $S^2 = I$, e.g. commutative or cocommutative Hopf algebras, there is no distinction between YD and AYD modules. This applies in particular to $H = kG$ and to $H = U(\mathfrak{g})$. The stability condition $m^{(-1)}m^{(0)} = m$ is equivalent to

$$g \cdot m = m, \quad \text{for all } g \in G, m \in M_g.$$

Example 2: Hopf-Galois extensions are noncommutative analogues of principal bundles in (affine) algebraic geometry. Following HKRS we show that they give rise to large classes of examples of SAYD modules. Let P be a right H-comodule algebra, and let

$$B := P^H = \{p \in P;\ \rho(p) = p \otimes 1\}$$

be the space of coinvariants of P. It is easy to see that B is a subalgebra of P. The extension $B \subset P$ is called a Hopf-Galois extension if the map

$$can : P \otimes_B P \to B \otimes H, \quad p \otimes p' \mapsto p\rho(p'),$$

is bijective. (Note that in the commutative case this corresponds to the condition that the action of the structure group on fibres is free). The bijectivity assumption allows us to define the translation map $T : H \to P \otimes_B P$,

$$T(h) = can^{-1}(1 \otimes h) = h^{(\bar{1})} \otimes h^{(\bar{2})}.$$

It can be checked that the centralizer $Z_B(P) = \{p \mid bp = pb \quad \forall b \in B\}$ of B in P is a subcomodule of P. There is an action of H on $Z_B(P)$ defined by $ph = h^{(1)} p h^{(2)}$ called the Miyashita-Ulbrich action. It is shown that this action and coaction satisfy the Yetter-Drinfeld compatibility condition. On the other hand if B is central, then by defining the new action $ph = (S^{-1}(h))^{(2)}p(S^{-1}(h))^{(1)}$ and the right coaction of P we have a SAYD module.

Example 3: Let $M = H$. Then with conjugation action $g \cdot h = g^{(1)}hS(g^{(2)})$ and comultiplication $h \mapsto h^{(1)} \otimes h^{(2)}$ as coaction, M is an SAYD module.

Hopf-cyclic cohomology

We first recall the approach by Connes and Moscovici towards the definition of their cyclic cohomology theory for Hopf algebras. The characteristic map χ_τ plays an important role here. Then we switch to the point of view adopted by HKRS based on invariant complexes. The resulting Hopf-cyclic cohomology theories include all known examples of cyclic theory discovered so far. Very recently A. Kaygun has extended the Hopf cyclic cohomology to a cohomology for bialgebras with coefficients in stable modules. For Hopf algebras it reduces to HKRS.

Connes-Moscovici's breakthrough

Without going into details we formulate one of the problems that was faced and solved by Connes and Moscovici in the course of their study of an index problem on foliated manifolds. This led them to a new cohomology theory for Hopf algebras that is the quintessential example of Hopf cyclic cohomology.

The local index formula of Connes and Moscovici gives the Chern character $Ch(A, h, D)$ of a regular spectral triple $(A, \mathcal{H}, D)$ as a cyclic cocycle in the (b, B)-bicomplex of the algebra A. For spectral triples of interest in transverse geometry this cocycle is *differentiable* in the sense that it is in the image of the Connes-Moscovici characteristic map

χ_τ,

$$\chi_\tau : H^{\otimes n} \longrightarrow Hom(A^{\otimes(n+1)},\ k),$$

defined below, with $H = \mathcal{H}_1$ and $A = \mathcal{A}_\Gamma$. To identify this class in terms of characteristic classes of foliations, it would be extremely helpful to show that it is the image of a cocycle for a cohomology theory for Hopf algebras. This is rather similar to the situation for classical characteristic classes which are pull backs of group cohomology classes.

We can formulate this problem abstractly as follows: Let H be a Hopf algebra endowed with a modular pair in involution (δ, σ), and A be an H-module algebra. Let $\tau : A \to k$ be a δ- invariant σ-trace on A as we defined before. Consider the

Connes-Moscovici *characteristic map*

$$\chi_\tau : H^{\otimes n} \longrightarrow Hom(A^{\otimes(n+1)},\ k),$$

$$\chi_\tau(h_1 \otimes \cdots \otimes h_n)(a_0 \otimes \cdots \otimes a_n) \ = \ \tau(a_0 h_1(a_1) \cdots h_n(a_n)) \qquad .$$

Now the burning question is: can we promote the collection of spaces $\{H^{\otimes n}\}_{n\geq 0}$ to a *cocyclic module* such that the characteristic map χ_τ turns into a morphism of cocyclic modules? We recall that the face, degeneracy, and cyclic operators for

$$Hom(A^{\otimes(n+1)},\ k)$$

are defined by:

$$\begin{aligned}
\delta_i^n \varphi(a_0, \cdots, a_{n+1}) &= \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\
\delta_{n+1}^n \varphi(a_0, \cdots, a_{n+1}) &= \varphi(a_{n+1} a_0, a_1, \cdots, a_n) \\
\sigma_i^n \varphi(a_0, \cdots, a_n) &= \varphi(a_0, \cdots, a_i, 1, \cdots, a_n) \\
\tau_n \varphi(a_0, \cdots, a_n) &= \varphi(a_n, a_0, \cdots, a_{n-1})
\end{aligned}$$

The relation

$$h(ab) = h^{(1)}(a) h^{(2)}(b)$$

shows that in order for χ_τ to be compatible with face operators, the face operators on $H^{\otimes n}$ must involve the coproduct of H. In fact if we define, for $0 \leq i \leq n$, $\delta_i^n : H^{\otimes n} \to H^{\otimes (n+1)}$, by

$$\begin{aligned}
\delta_0^n (h_1 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n, \\
\delta_i^n (h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_i^{(1)} \otimes h_i^{(2)} \otimes \cdots \otimes h_n, \\
\delta_{n+1}^n (h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma,
\end{aligned}$$

then we have, for all n and i,

$$\chi_\tau \delta_i^n = \delta_i^n \chi_\tau.$$

Similarly, the relation $h(1_A) = \varepsilon(h)1_A$, shows that the degeneracy operators on $H^{\otimes n}$ should involve the counit of H. We thus define

$$\sigma_i^n(h_1 \otimes \cdots \otimes h_n) = h_1 \otimes \cdots \otimes \varepsilon(h_i) \otimes \cdots \otimes h_n.$$

The most difficult part in this regard is to guess the form of the cyclic operator

$$\tau_n : H^{\otimes n} \to H^{\otimes n}.$$

Compatibility with χ_τ demands that

$$\tau(a_0 \tau_n(h_1 \otimes \cdots \otimes h_n)(a_1 \otimes \cdots \otimes a_n)) =$$

$$\tau(a_n h_1(a_0) h_2(a_1) \cdots h_n(a_{n-1})),$$

for all a_i's and h_i's. Now integration by parts formula combined with the σ-trace property of τ, gives us:

$$\tau(a_1 h(a_0)) = \tau(h(a_0)\sigma(a_1)) = \tau(a_0 \tilde{S}_\delta(h)(\sigma(a_1)).$$

This suggests that we should define $\tau_1 : H \to H$ by

$$\tau_1(h) = \tilde{S}_\delta(h)\sigma.$$

Note that the condition $\tau_1^2 = I$ is equivalent to the involutive condition $\tilde{S}_\delta^2 = Ad_\sigma$.

For any n, integration by parts formula together with the σ-trace property shows that:

$$\begin{aligned} \tau(a_n h_1(a_0) \cdots h_n(a_{n-1})) &= \\ \tau(h_1(a_0) \cdots h_n(a_{n-1})\sigma(a_n)) &= \\ \tau(a_0 \tilde{S}_\delta(h_1)(h_2(a_1) \cdots h_n(a_{n-1})\sigma(a_n))) &= \end{aligned}$$
$$\tau(a_0 \tilde{S}_\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma)(a_1 \otimes \cdots \otimes a_n).$$

This suggests that the *Hopf-cyclic operator* τ_n : $H^{\otimes n} \to H^{\otimes n}$ should be defined as

$$\tau_n(h_1 \otimes \cdots \otimes h_n) = \tilde{S}_\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma),$$

where $\cdot$ denotes the diagonal action defined by

$$h \cdot (h_1 \otimes \cdots \otimes h_n) := h^{(1)} h_1 \otimes h^{(2)} h_2 \otimes \cdots \otimes h^{(n)} h_n.$$

We let $\tau_0 = I : H^{\otimes 0} = k \to H^{\otimes 0}$, be the identity map.

The remarkable fact, proved by Connes and Moscovici, is that endowed with the above face, degeneracy, and cyclic operators,

$$\{H^{\otimes n}\}_{n \geq 0}$$

is a cocyclic module. The resulting cyclic cohomology groups are denoted by

$$HC^n_{(\delta,\sigma)}(H), \quad n = 0, 1, \cdots$$

and we obtain the desired characteristic map

$$\chi_\tau : HC^n_{(\delta,\sigma)}(H) \to HC^n(A).$$

As with any cocyclic module, cyclic cohomology can also be defined in terms of cyclic cocycles. In this case a cyclic n-cocycle is an element $x \in H^{\otimes n}$ satisfying the conditions

$$bx = 0, \quad (1 - \lambda)x = 0,$$

where $b : H^{\otimes n} \to H^{\otimes (n+1)}$ and $\lambda : H^{\otimes n} \to H^{\otimes n}$ are defined by

$$b(h^1 \otimes \cdots \otimes h^n) = 1 \otimes h_1 \otimes \cdots \otimes h_n$$
$$+ \sum_{i=1}^{n} (-1)^i h_1 \otimes \cdots \otimes h_i^{(1)} \otimes h_i^{(2)} \otimes \cdots \otimes h_n$$
$$+(-1)^{n+1} h_1 \otimes \cdots \otimes h_n \otimes \sigma,$$

$$\lambda(h_1 \otimes \cdots \otimes h_n) = (-1)^n \tilde{S}_\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma).$$

The cyclic cohomology groups $HC^n_{(\delta,\sigma)}(H)$ have been computed in several cases. Of particular interest for applications to transverse index theory and number theory is the (periodic) cyclic cohomology of the Connes-Moscovici Hopf algebra $\mathcal{H}_1$. Connes and Moscovici have shown that that the periodic groups $HP^n_{(\delta,1)}(\mathcal{H}_1)$ are canonically isomorphic to the Gelfand-Fuks cohomology of the Lie algebra of formal vector fields on the line:

$$H^*(\mathfrak{a}_1, \mathbb{C}) = HP^*_{(\delta,1)}(\mathcal{H}_1).$$

Calculation of the unstable groups is an interesting open problem.

The following interesting elements have already been identified. It can be directly checked that the elements

$$\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \quad \text{and} \quad \delta_1$$

are cyclic 1-cocycles on $\mathcal{H}_1$, and

$$F := X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y$$

is a cyclic 2-cocycle. See [**?**] for detailed calculations and relations between these cocycles and the Schwarzian derivative, Godbillon-Vey cocycle, and the transverse fundamental class of Connes respectively.

Hopf-cyclic cohomology: type A, B, and C theories

We recall the definitions of the three cyclic cohomology theories that were defined by HKRS. We call them A, B and C theories. In the first case the algebra A is endowed with an action of a Hopf algebra; in the second case the algebra B is equipped with a coaction of a Hopf algebra; and finally in theories of type C, we have a coalgebra endowed with an action of a Hopf algebra. In all three theories the module of coefficients is a stable anti-Yetter-Drinfeld (SAYD) module over the Hopf algebra and we attach a cocyclic module to the given data. Along the same lines one can define a Hopf-cyclic cohomology theory for comodule coalgebras as well (type D theory). Since so far we have found no applications of such a theory we won't give its definition here. We also show that all known examples of cyclic cohomology theories that are introduced so far such as ordinary cyclic cohomology for algebras, Connes-Moscovici's cyclic cohomology for Hopf algebras, twisted and equivariant cyclic cohomology are special cases of these theories.

Let A be a left H-module algebra and M be a left-right SAYD H-module. Then the spaces

$$M \otimes A^{\otimes(n+1)}$$

are right H-modules via the diagonal action

$$(m \otimes \widetilde{a})h := mh^{(1)} \otimes S(h^{(2)})\widetilde{a},$$

where the left H-action on $\widetilde{a} \in A^{\otimes(n+1)}$ is via the left diagonal action of H.

We define the space of *equivariant cochains on A with coefficients in M* by

$$\mathcal{C}_H^n(A, M) := Hom_H(M \otimes A^{\otimes(n+1)}, k).$$

More explicitly, $f : M \otimes A^{\otimes(n+1)} \longrightarrow k$ is in $\mathcal{C}_H^n(A, M)$, if and only if

$$f((m \otimes a_0 \otimes \cdots \otimes a_n)h) = \varepsilon(h) f(m \otimes a_0 \otimes \cdots \otimes a_n),$$

for all $h \in H, m \in M$, and $a_i \in A$.

It can be shown that the following operators define a cocyclic module structure on $\{\mathcal{C}^n_H(A,M)\}_{n\in\mathbb{N}}$:

$$(\delta_i f)(m\otimes a_0\otimes\cdots\otimes a_n) =$$
$$f(m\otimes a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n),$$
$$(\delta_n f)(m\otimes a_0\otimes\cdots\otimes a_n) =$$
$$f(m^{(0)}\otimes (S^{-1}(m^{(-1)})a_n)a_0\otimes a_1\otimes\cdots\otimes a_{n-1}),$$
$$(\sigma_i f)(m\otimes a_0\otimes\cdots\otimes a_n) =$$
$$f(m\otimes a_0\otimes\cdots\otimes a_i\otimes 1\otimes\cdots\otimes a_n),$$
$$(\tau_n f)(m\otimes a_0\otimes\cdots\otimes a_n) =$$
$$f(m^{(0)}\otimes S^{-1}(m^{(-1)})a_n\otimes a_0\otimes\cdots\otimes a_{n-1}).$$

We denote the resulting cyclic cohomology theory by $HC^n_H(A,M), n=0,1,\cdots$.

Example 1: For $H=k=M$ we obviously recover the standard cocyclic module of the algebra A. The resulting cyclic cohomology theory is the ordinary cyclic cohomology of algebras.

Example 2: For $M = H$ and H acting on M by conjugation and coacting via coproduct, we obtain the equivariant cyclic cohomology theory of Akbarpour and Khalkhali For H-module algebras.

Example 3: For $H = k[\sigma, \sigma^{-1}]$ the Hopf algebra of Laurent polynomials, where σ acts by automorphisms on an algebra A, and $M = k$ is a trivial module, we obtain the so called *twisted cyclic cohomology* of A with respect to σ. A *twisted cyclic n-cocycle* is a linear map $f : A^{\otimes(n+1)} \to k$ satisfying:

$$f(\sigma a_n, a_0, \cdots, a_{n-1}) = (-1)^n f(a_0, \cdots, a_n), \quad b_\sigma f = 0,$$

where b_σ is the twisted Hochschild boundary defined by

$$b_\sigma f(a_0, \cdots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1} f(\sigma(a_{n+1}) a_0, a_1, \cdots, a_n).$$

Example 4: It is easy to see that for $M = {}^\sigma k_\delta$, the SAYD module attached to a modular pair in involution (δ, σ), $HC_H^0(A, M)$ is the space of δ-invariant σ-traces on A in the sense of Connes-Moscovici.

Next, let B be a right H-comodule algebra and M be a right-right SAYD H-module. Let

$$\mathcal{C}^{n,H}(B, M) := Hom^H(B^{\otimes(n+1)}, M),$$

denote the space of right H-colinear $(n+1)$-linear functionals on B with values in M. Here $B^{\otimes(n+1)}$ is considered a right H-comodule via the diagonal coaction of H:

$$b_0 \otimes \cdots \otimes b_n \mapsto (b_0^{(0)} \otimes \cdots \otimes b_n^{(0)}) \otimes (b_0^{(1)} b_1^{(1)} \cdots b_n^{(1)}).$$

It can be shown that, thanks to the invariance property imposed on our cochains and the SAYD condition on M, the following maps define a cocyclic module structure on $\{\mathcal{C}^{n,H}(B, M)\}_{n\in\mathbb{N}}$:

$$(\delta_i f)(b_0, \cdots, b_{n+1}) = f(b_0, \cdots, b_i b_{i+1}, \cdots, b_{n+1}),$$
$$(\delta_n f)(b_0, \cdots, b_{n+1}) = f(b_{n+1}^{(0)} b_0, b_1, \cdots, b_n) b_{n+1}^{(1)},$$
$$(\sigma_i f)(b_0, \cdots, b_{n-1}) = f(b_0, \cdots, b_i, 1, \cdots b_{n-1}),$$
$$(\tau_n f)(b_0, \cdots, b_n) = f(b_n^{(0)}, b_0, \cdots, b_{n-1}) b_n^{(1)}.$$

We denote the resulting cyclic cohomology groups by $HC^{n,H}(B, M)$, $n = 0, 1, \cdots$.

Example 1: For $B = H$, equipped with comultiplication as coaction, and $M = {}^{\sigma}k_{\delta}$ associated to a modular pair in involution, we obtain the dual Hopf cyclic cohomology of Hopf algebras (Rangipour+ M. K.). This theory is different from Connes-Moscovici's theory for Hopf algebras. It is dual, in the sense of Hopf algebras and not Hom dual, to Connes-Moscovici's theory. It is computed in the following cases: $H = kG$, $H = U(\mathfrak{g})$, where it is isomorphic to group cohomology and Lie algebra cohomology, respectively; $H = SL_2(q, k)$, and $H = U_q(sl_2)$.

Example 2:. For $H = k$, and $M = k$ a trivial module, we obviously recover the cyclic cohomology of the algebra B.

Finally we describe theories for module coalgebras and their main examples. As we shall see, Connes-Moscovici's original example of Hopf-cyclic cohomology belong to this class of theories.

Let C be a left H-module coalgebra, and M be a right-left SAYD H-module. Let

$$\mathcal{C}^n(C, M) := M \otimes_H C^{\otimes(n+1)} \quad n \in \mathbb{N}.$$

It can be checked that, thanks to the SAYD condition on M, the following operators are well defined and define a cocyclic module, denoted $\{\mathcal{C}^n(C, M)\}_{n\in\mathbb{N}}$. In particular the crucial periodicity conditions

$$\tau_n^{n+1} = id, \quad n = 0, 1, 2 \cdots$$

are satisfied:

$$\delta_i(m \otimes c_0 \otimes \cdots \otimes c_{n-1}) =$$
$$m \otimes c_0 \otimes \cdots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes c_{n-1},$$
$$\delta_n(m \otimes c_0 \otimes \cdots \otimes c_{n-1}) =$$
$$m^{(0)} \otimes c_0^{(2)} \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes m^{(-1)} c_0^{(1)},$$
$$\sigma_i(m \otimes c_0 \otimes \cdots \otimes c_{n+1}) =$$
$$m \otimes c_0 \otimes \cdots \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_{n+1},$$
$$\tau_n(m \otimes c_0 \otimes \cdots \otimes c_n) =$$
$$m^{(0)} \otimes c_1 \otimes \cdots \otimes c_n \otimes m^{(-1)} c_0.$$

Example 1: For $H = k = M$, we recover the cocyclic module of a coalgebra which defines its cyclic cohomology.

Example 2: For $C = H$ and $M = {}^{\sigma}k_{\delta}$, the cocyclic module $\{\mathcal{C}_H^n(C, M)\}_{n\in\mathbb{N}}$ is isomorphic to the cocyclic module of Connes-Moscovici, attached to a Hopf algebra endowed with a modular pair in involution. This example is truly fundamental and started the whole theory.