The Music of Quantum Spheres

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Project Outline

- Spectral Geometry: can one hear the shape of a drum?
- What is a noncommutative space and how to measure its shape?
- What is a quantum sphere?
- Is quantum sphere curved at all?

Spectral Geometry: can one hear the shape of a drum?

- What do we hear when we play a drum? We hear different modes of vibrations with different frequencies.

Figure : Is there a relation between the shape of a drum and its frequencies?

Fundamental frequencies; the spectrum

- It is a mathematical theorem that fundamental frequencies of any object/shape form a sequence

$$\nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \to \infty$$

- This is often called the spectrum of the drum. The spectrum of a shape contains a huge amount of information about its geometry.
- But there are isospectral figures that are not isometric.

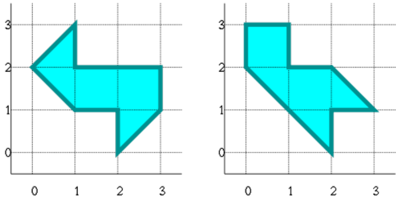
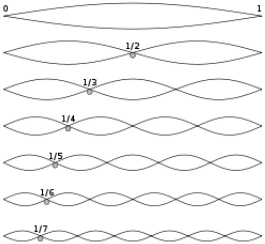


Figure : Isospectral but not isometric

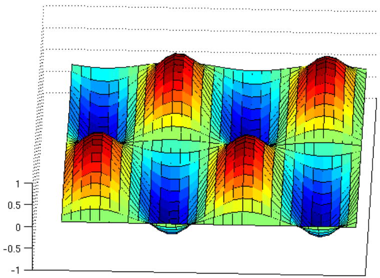
Frequencies of a vibrating string

- Only for a few simple shapes (rectangles, circles, equilateral triangles) the spectrum is explicitly known!



$$\nu_n = c\frac{n}{L}, \quad n = 1, 2, 3, \cdots$$

Frequencies of rectangular drums



$$\lambda = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad m, n \in \mathbb{Z}$$

Spectrum in Math: Eigenvalues and Eigenvectors

- The spectrum of an operator (matrix) A is the set of its eigenvalues

$$\{\lambda_1, \lambda_2, \cdots \lambda_n\}$$

  defined by

$$Av = \lambda v, \quad v \neq 0$$

  A can be extremely complicated, but a wealth of information about A can be obtained from its spectrum (linear algebra).

- An operator on an infinite dimensional space can have a more complicated spectrum.

Geometry and spectrum

- Every shape (domain, manifold) has some natural operators attached to it:

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \qquad \text{Laplacian}$$

- Let $\Omega \subset \mathbb{R}^2$ be our drum: a compact connected domain with a piecewise smooth boundary.

$$\begin{cases} \Delta u = \lambda u \\ u|\partial\Omega = 0 \end{cases}$$

$$\text{Spectrum}\,(\Omega): \;\; 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

One can hear the area of a drum!

- Let $N(\lambda) = \#\{\lambda_i \leq \lambda\}$. In 1912 Hermann Weyl proved:

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{\text{Area}(\Omega)}{4\pi}$$

  That is one can hear the area of any drum of arbitrary shape! This is a remarkable result in its universality and generality!

- In genera for an n-dimensional drum $\Omega \subset \mathbb{R}^n$

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} = \frac{\omega_n \text{Vol}(\Omega)}{(2\pi)^n}$$

- Weyl Law: One can hear the dimension and volume of a shape.

Going beyond volumes

- Using more refined techniques one can show that the total curvature of a space can also be heard:

- Poles and residues of spectral zeta functions:

$$\zeta_\triangle(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{n}{2}$$

- Heat kernel asymptotic expansion:

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-n}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad\qquad (t \to 0)$$

Enter Noncommutative Geometry

- Why geometry, the science of measuring shapes, should have anything to do with commutativity $xy = yx$ or lack of it $xy \neq yx$, algebra?

- A great idea of Descartes: use coordinates! Then geometry becomes (commutative) algebra!

Geometry = Commutative Algebra

- : Examples:

Algebra	Geometry
$x^2 + y^2 = 1$	circle
$x^2 + y^2 + z^2 = 1$	sphere
$f(x_1, \cdots, x_n) = 0$	hypersurface

Figure : Descartes explains his idea to Queen Christina of Sweden

Geometry goes Noncommutative: Connes' Program

- A modern version of Descartes's idea (Hilbert, Gelfand, Grothendieck, Connes): replace a geometric object by the algebra of functions defined on it:

$$C^\infty(M) \Leftrightarrow M$$

- By the same token, a noncommutative algebra can be regarded as the 'algebra of functions' on a mysterious, magical, noncommutative space! But there is no magic. A noncommutative space is as real as an *n*-dimensional space!

- Simplest noncommutative algebras (spaces)

$$A = M_n(\mathbb{C})$$

  or their limits

$$M_1(\mathbb{C}) \hookrightarrow M_2(\mathbb{C}) \hookrightarrow M_3(\mathbb{C}) \hookrightarrow \cdots$$

Links with the quantum world

- Noncommutative spaces are quite often referred to as quantum space. Why quantum?

- The essence of quantum mechanics in noncommutativity: Heisenberg uncertainty relation

  $$qp - pq = i\hbar.$$

  replaces the commutation relation $qp - pq = 0$. Thus q and p generate a noncommutative algebra.

Hydrogen lines: Mathematics meets Physics though spectrum

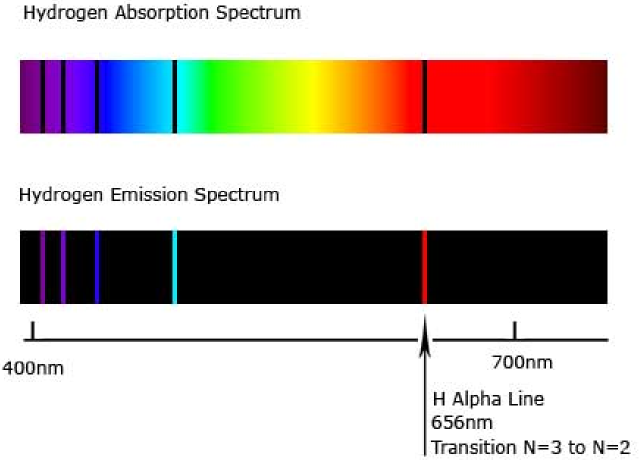


Figure : Hydrogen spectral lines in the visible range

From Spheres to Quantum Spheres

- 2-sphere $x^2 + y^2 + z^2 = 1$. Let $B = x + iy$, $B^* = x - iy$, $A = z$. Then
$$BB^* + A^2 = 1$$
- Equations for quantum sphere S_q^2, $0 < q < 1$.
$$AB = q^2 BA, \qquad AB^* = q^{-2} B^* A,$$
$$BB^* = q^{-2} A(1 - A), \qquad B^* B = A(1 - q^2 A).$$
- In the limit $q = 1$ this algebra is commutative and is isomorphic to algebra of functions on S^2. But for $q < 1$ it is a limit of matrix algebras:
$$S_q^2 = \lim_{n \to \infty} M_n(\mathbb{C}) \oplus \mathbb{C}$$

Fundamental frequencies of the quantum sphere

- The spectrum of S_q^2 has exponential growth:

$$\pm \frac{q^{l+1/2} - q^{-(l+1/2)}}{q - q^{-1}}, \quad l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$$

- This means it is a zero dimensional object! But it has a positive volume!.

- Note: As $q \to 1$, the above numbers approach to the spectrum of the sphere S^2.

Summary

- We learned that the concept of spectrum plays such an important role in mathematics and physics.
- Spectral geometry teaches us how to extract information about shape from a knowledge of spectrum.
- There is a remarkable duality at work in mathematics between geometry and algebra. This makes noncommutative geometry possible.
- In this project we want to study the geometry of a quantum sphere (largely unknown) starting from its spectrum (known).

It could be better, but it is good enough now!