The Music of Quantum Spheres

Masoud Khalkhali

Fields Institute Undergraduate Summer Research Program

Toronto, May 2014
Project Outline

- Spectral Geometry: can one hear the shape of a drum?
- What is a noncommutative space and how to measure its shape?
- What is a quantum sphere?
- Is quantum sphere curved at all?
Spectral Geometry: can one hear the shape of a drum?

- What do we hear when we play a drum? We hear different modes of vibrations with different frequencies.

Figure: Is there a relation between the shape of a drum and its frequencies?
Fundamental frequencies; the spectrum

- It is a mathematical theorem that fundamental frequencies of any object/shape form a sequence

\[ \nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \rightarrow \infty \]

- This is often called the spectrum of the drum. The spectrum of a shape contains a huge amount of information about its geometry.

- But there are isospectral figures that are not isometric.

Figure: Isospectral but not isometric
Frequencies of a vibrating string

- Only for a few simple shapes (rectangles, circles, equilateral triangles) the spectrum is explicitly known!

\[ \nu_n = c \frac{n}{L}, \quad n = 1, 2, 3, \ldots \]
Frequencies of rectangular drums

\[ \lambda = c \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad m, n \in \mathbb{Z} \]
The spectrum of an operator (matrix) $A$ is the set of its eigenvalues

$$\{\lambda_1, \lambda_2, \ldots \lambda_n\}$$

defined by

$$Av = \lambda v, \quad v \neq 0$$

$A$ can be extremely complicated, but a wealth of information about $A$ can be obtained from its spectrum (linear algebra).

An operator on an infinite dimensional space can have a more complicated spectrum.
Geometry and spectrum

- Every shape (domain, manifold) has some natural operators attached to it:

\[ \Delta = - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \]  

Laplacian

- Let \( \Omega \subset \mathbb{R}^2 \) be our drum: a compact connected domain with a piecewise smooth boundary.

\[ \begin{cases} 
\Delta u = \lambda u \\
 u|_{\partial \Omega} = 0 
\end{cases} \]

Spectrum (\( \Omega \)) : \( 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \rightarrow \infty \)
One can hear the area of a drum!

- Let $N(\lambda) = \# \{ \lambda_i \leq \lambda \}$. In 1912 Hermann Weyl proved:

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{\text{Area}(\Omega)}{4\pi}$$

That is one can hear the area of any drum of arbitrary shape! This is a remarkable result in its universality and generality!

- In genera for an $n$-dimensional drum $\Omega \subset \mathbb{R}^n$

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} = \frac{\omega_n \text{Vol}(\Omega)}{(2\pi)^n}$$

- Weyl Law: One can hear the dimension and volume of a shape.
Going beyond volumes

- Using more refined techniques one can show that the total curvature of a space can also be heard:

- Poles and residues of spectral zeta functions:

\[
\zeta_\triangle(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \quad \text{Re}(s) > \frac{n}{2}
\]

- Heat kernel asymptotic expansion:

\[
\text{Trace}(e^{-t\triangle}) \sim (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} a_j t^j \quad (t \to 0)
\]
Enter Noncommutative Geometry

- Why geometry, the science of measuring shapes, should have anything to do with commutativity $xy = yx$ or lack of it $xy \neq yx$, algebra?

- A great idea of Descartes: use coordinates! Then geometry becomes (commutative) algebra!

  **Geometry = Commutative Algebra**

- : Examples:

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + y^2 = 1$</td>
<td>circle</td>
</tr>
<tr>
<td>$x^2 + y^2 + z^2 = 1$</td>
<td>sphere</td>
</tr>
<tr>
<td>$f(x_1, \cdots, x_n) = 0$</td>
<td>hypersurface</td>
</tr>
</tbody>
</table>
Figure: Descartes explains his idea to Queen Christina of Sweden
Geometry goes Noncommutative: Connes’ Program

- A modern version of Descartes’s idea (Hilbert, Gelfand, Grothendieck, Connes): replace a geometric object by the algebra of functions defined on it:

\[ C^\infty(M) \Leftrightarrow M \]

- By the same token, a noncommutative algebra can be regarded as the ‘algebra of functions’ on a mysterious, magical, noncommutative space! But there is no magic. A noncommutative space is as real as an \( n \)-dimensional space!

- Simplest noncommutative algebras (spaces)

\[ A = M_n(\mathbb{C}) \]

or their limits

\[ M_1(\mathbb{C}) \hookrightarrow M_2(\mathbb{C}) \hookrightarrow M_3(\mathbb{C}) \hookrightarrow \cdots \]
Links with the quantum world

- Noncommutative spaces are quite often referred to as quantum space. Why quantum?

- The essence of quantum mechanics in noncommutativity: Heisenberg uncertainty relation

\[ qp - pq = i\hbar. \]

replaces the commutation relation \( qp - pq = 0 \). Thus \( q \) and \( p \) generate a noncommutative algebra.
Hydrogen lines: Mathematics meets Physics though spectrum

Figure: Hydrogen spectral lines in the visible range
From Spheres to Quantum Spheres

- 2-sphere $x^2 + y^2 + z^2 = 1$. Let $B = x + iy$, $B^* = x - iy$, $A = z$. Then
  $$BB^* + A^2 = 1$$

- Equations for quantum sphere $S^2_q$, $0 < q < 1$.
  $$AB = q^2 BA, \quad AB^* = q^{-2} B^* A,$$
  $$BB^* = q^{-2} A(1 - A), \quad B^* B = A(1 - q^2 A).$$

- In the limit $q = 1$ this algebra is commutative and is isomorphic to algebra of functions on $S^2$. But for $q < 1$ it is a limit of matrix algebras:
  $$S^2_q = \lim_{n \to \infty} M_n(\mathbb{C}) \oplus \mathbb{C}$$
The spectrum of $S^2_q$ has exponential growth:

$$\pm \frac{q^{l+1/2} - q^{-(l+1/2)}}{q - q^{-1}}, \quad l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

This means it is a zero dimensional object! But it has a positive volume!.

Note: As $q \to 1$, the above numbers approach to the spectrum of the sphere $S^2$. 

Fundamental frequencies of the quantum sphere
We learned that the concept of spectrum plays such an important role in mathematics and physics.

Spectral geometry teaches us how to extract information about shape from a knowledge of spectrum.

There is a remarkable duality at work in mathematics between geometry and algebra. This makes noncommutative geometry possible.

In this project we want to study the geometry of a quantum sphere (largely unknown) starting from its spectrum (known).
It could be better, but it is good enough now!