Quillen’s Metric and Determinant Line Bundle in Noncommutative Geometry

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Warm up: zeta regularized determinants

▶ Given a sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ of eigenvalues of the Laplacian $\Delta$.

How one defines $\prod \lambda_i = \det \Delta$?

▶ Define the spectral zeta function:

$$\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \quad \Re(s) \gg 0$$

Assume:

$\zeta_\Delta(s)$ has meromorphic extension to $\mathbb{C}$ and is regular at 0.

▶ Zeta regularized determinant:

$$\prod \lambda_i := e^{-\zeta'_\Delta(0)} = \det \Delta$$
Warm up: zeta regularized determinants

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- Zeta regularized determinant:

\[ \prod \lambda_i := e^{-\zeta_\Delta'(0)} = \det \Delta \]
Holomorphic determinants

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- Quillen's approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.
Curved noncommutative tori $A_\theta$

\[ A_\theta = C(\mathbb{T}_\theta^2) = \text{universal } C^*\text{-algebra generated by unitaries } U \text{ and } V \]

\[ VU = e^{2\pi i \theta} UV. \]

\[ A^\infty_{\theta} = C^\infty(\mathbb{T}_\theta^2) = \{ \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n : (a_{m,n}) \text{ Schwartz class} \}. \]
**Differential operators** \( \delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty \)

\[
\begin{align*}
\delta_1(U) &= U, & \delta_1(V) &= 0 \\
\delta_2(U) &= 0, & \delta_2(V) &= V
\end{align*}
\]

**Integration** \( \varphi_0 : A_\theta \to \mathbb{C} \) on smooth elements:

\[
\varphi_0 \left( \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n \right) = a_{0,0}.
\]

**Complex structures:** Fix \( \tau = \tau_1 + i\tau_2, \quad \tau_2 > 0 \). Dolbeault operators

\[
\partial := \delta_1 + \tau \delta_2, \quad \partial^* := \delta_1 + \bar{\tau} \delta_2.
\]
Conformal perturbation of the metric (Connes-Tretkoff)

- Fix \( h = h^* \in A_\theta^\infty \). Replace the volume form \( \varphi_0 \) by \( \varphi : A_\theta \to \mathbb{C} \),
  \[ \varphi(a) := \varphi_0( ae^{-h} ). \]

- It is a twisted trace (KMS state):
  \[ \varphi(ab) = \varphi(b \Delta(a)), \]
where
  \[ \Delta(x) = e^{-h}xe^h. \]
Perturbed Dolbeault operator

- Hilbert space \( \mathcal{H}_\varphi = L^2(A_\theta, \varphi) \), GNS construction.

- Let \( \partial \varphi = \delta_1 + \tau \delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)} \),
  \[
  \partial^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi.
  \]
  and \( \Delta = \partial^* \partial \varphi \), perturbed non-flat Laplacian.
Scalar curvature for $A_\theta$

- Gilkey-De Witt-Seeley formulae in spectral geometry motivates the following definition:

The scalar curvature of the curved nc torus $(A_\theta, \tau, h)$ is the unique element $R \in A_{\infty}^\theta$ satisfying

$$\text{Trace} \left( a \triangle^{-s} \right)_{s=0} + \text{Trace} (aP) = \varphi_0 (aR), \quad \forall a \in A_{\infty}^\theta,$$

where $P$ is the projection onto the kernel of $\triangle$. 
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  where $P$ is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansion for the kernel of the operator $e^{-t\triangle}$, using Connes’ pseudodifferential calculus for nc tori.
Theorem: The scalar curvature of \((A_\theta, \tau, k)\), up to an overall factor of \(-\frac{\pi}{\tau_2}\), is equal to

\[
R_1(\log \Delta)(\Delta_0(\log k)) + \ R_2(\log \Delta_1, \log \Delta_2)
\left(\delta_1(\log k)^2 + |\tau|^2 \delta_2(\log k)^2 + \tau_1 \left\{ \delta_1(\log k), \delta_2(\log k) \right\} \right) + \\
iW(\log \Delta_1, \log \Delta_2)
\left(\tau_2 \left[ \delta_1(\log k), \delta_2(\log k) \right] \right)
\]
where

\[ R_1(x) = -\frac{1}{2} - \frac{x}{\sinh^2(x/4)}, \]

\[ R_2(s, t) = (1 + \cosh((s + t)/2)) \times \]

\[-t(s + t) \cosh s + s(s + t) \cosh t - (s - t)(s + t + \sinh s + \sinh t - \sinh(s + t)) \]

\[
\frac{st(s + t) \sinh(s/2) \sinh(t/2) \sinh^2((s + t)/2)}{st \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)}.
\]

\[ W(s, t) = -\frac{(-s - t + t \cosh s + s \cosh t + \sinh s + \sinh t - \sinh(s + t))}{st \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)}. \]
Holomorphic determinants

- Logdet is not a holomorphic function. How to define a holomorphic determinant \( \det : \mathcal{A} \to \mathbb{C} \).
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- **Recall:** Space of Fredholm operators:

\[
F = \text{Fred}(H_0, H_1) = \{T : H_0 \to H_1; \ T \text{ is Fredholm}\}
\]

\[
K_0(X) = [X, F], \quad \text{classifying space for K-theory}
\]
The determinant line bundle

- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.
The determinant line bundle

- Let $\lambda = \wedge^{\text{max}}$ denote the top exterior power functor.

- **Theorem (Quillen)** 1) There is a holomorphic line bundle $\text{DET} \rightarrow F$ s.t.
  \[(\text{DET})_T = \lambda(\text{Ker}T)^* \otimes \lambda(\text{Ker}T^*)\]
The determinant line bundle

- Let $\lambda = \wedge^{\max}$ denote the top exterior power functor.

- **Theorem (Quillen) 1)** There is a holomorphic line bundle $\text{DET} \to F$ s.t.
  $$(\text{DET})_T = \lambda(\text{Ker}T)^* \otimes \lambda(\text{Ker}T^*)$$

  2) There map $\sigma : F_0 \to \text{DET}$

  $$\sigma(T) = \begin{cases} 
  1 & T \text{ invertible} \\
  0 & \text{otherwise} 
  \end{cases}$$

  is a holomorphic section of DET over $F_0$. 
Cauchy-Riemann operators on $A_\theta$

- Families of spectral triples
  \[ A_\theta, \ H_0 \oplus H^{0,1}, \ \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix}, \]
  with $\alpha \in A_\theta$, $\bar{\partial} = \delta_1 + \tau \delta_2$.

- Let $A =$ space of elliptic operators $D = \bar{\partial} + \alpha$. 
Cauchy-Riemann operators on $A_\theta$

- Families of spectral triples

$$A_\theta, \quad H_0 \oplus H^{0,1}, \quad \begin{pmatrix} 0 & \bar\partial^* + \alpha^* \\ \bar\partial + \alpha & 0 \end{pmatrix},$$

with $\alpha \in A_\theta$, $\bar\partial = \delta_1 + \tau \delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar\partial + \alpha$.

- Pull back DET to a holomorphic line bundle $\mathcal{L} \to \mathcal{A}$ with

$$\mathcal{L}_D = \lambda(KerD)^* \otimes \lambda(KerD^*)$$
If $\mathcal{L}$ admits a **canonical global holomorphic frame** $s$, then

$$\sigma(D) = \det(D)s$$

defines a holomorphic determinant function $\det(D)$. A canonical frame is defined once we have a canonical flat holomorphic connection.
Quillen's metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $\text{Index}(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta'_{\Delta}(0)) = \det\Delta, \quad \Delta = D^* D.$$ 

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$. 

A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from $\bar{\partial} \partial \log ||s||^2$, where $s$ is any local holomorphic frame.
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Connes’ pseudodifferential calculus

To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.

Symbols of order $m$: smooth maps $\sigma : \mathbb{R}^2 \rightarrow A_\theta^\infty$ with

$$\| \delta^{(i_1,i_2)} \partial^{(j_1,j_2)} \sigma(\xi) \| \leq c(1 + |\xi|)^{m-j_1-j_2}.$$ 

The space of symbols of order $m$ is denoted by $S^m(A_\theta)$. 
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The space of symbols of order $m$ is denoted by $S^m(A_\theta)$.

- To a symbol $\sigma$ of order $m$, one associates an operator

$$P_\sigma(a) = \int \int e^{-is \cdot \xi} \sigma(\xi) \alpha_s(a) \, ds \, d\xi.$$  

The operator $P_\sigma : A_\theta \to A_\theta$ is said to be a pseudodifferential operator of order $m$. 

Classical symbols

▶ Classical symbol of order $\alpha \in \mathbb{C}$:

$$
\sigma \sim \sum_{j=0}^{\infty} \sigma_{\alpha - j} \quad \text{ord } \sigma_{\alpha - j} = \alpha - j.
$$

$$
\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi) \sigma_{\alpha - j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.
$$

▶ We denote the set of classical symbols of order $\alpha$ by $S_{cl}^\alpha(A_\theta)$ and the associated classical pseudodifferential operators by $\Psi_{cl}^\alpha(A_\theta)$. 
A cutoff integral

- Any pseudo $P_{\sigma}$ of order $<-2$ is trace-class with

$$\text{Tr}(P_{\sigma}) = \varphi_0 \left( \int_{\mathbb{R}^2} \sigma(\xi) d\xi \right).$$
A cutoff integral

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$$\text{Tr}(P_\sigma) = \varphi_0 \left( \int_{\mathbb{R}^2} \sigma(\xi) d\xi \right).$$

- For $\text{ord}(P) \geq -2$ the integral is divergent, but, assuming $P$ is classical, and of non-integral order, one has an asymptotic expansion as $R \to \infty$

$$\int_{B(R)} \sigma(\xi) d\xi \sim \sum_{j=0, \alpha-j+2 \neq 0}^{\infty} \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma) \log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi = \text{Wodzicki residue of } P$ (Fathizadeh).
The Kontsevich-Vishik trace

- The cut-off integral of a symbol \( \sigma \in S^\alpha_{cl}(A_\theta) \) is defined to be the constant term in the above asymptotic expansion, and we denote it by \( \int \sigma(\xi) d\xi \).
The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in S^\alpha_{cl}(A_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $\int \sigma(\xi)d\xi$.

- The canonical trace of a classical pseudo $P \in \Psi^\alpha_{cl}(A_\theta)$ of non-integral order $\alpha$ is defined as

$$\text{TR}(P) := \varphi_0 \left( \int \sigma_P(\xi)d\xi \right).$$
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- NC residue in terms of TR:

$$\text{Res}_{z=0}\text{TR}(AQ^{-z}) = \frac{1}{q} \text{Res}(A).$$
Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^z)$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

\[
\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,
\]

with $\sigma_{\alpha-j,l}$ positively homogeneous in $\xi$ of degree $\alpha - j$. 

Example: $\log Q$ where $Q \in \Psi_{q_{cl}}(A^{\theta})$ is a positive elliptic pseudodifferential operator of order $q > 0$. 

Wodzicki residue: $\operatorname{Res}(A) = \varphi_0(\operatorname{res}(A))$, $\operatorname{res}(A) = \int_{|\xi|=1} \sigma^{-2}_{0,0}(\xi) d\xi$. 
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Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section $\sigma$,

$$\|\sigma\|^2 = e^{-\zeta_{\Delta_\alpha}(0)}$$
Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section $\sigma$,
  \[ \|\sigma\|^2 = e^{-\zeta'_{\Delta}(0)} \]

- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$. Want to compute
  \[ \bar{\partial} \partial \log \|\sigma\|^2 = \delta_{\bar{w}} \delta_w \zeta'_{\Delta}(0) = \delta_{\bar{w}} \delta_w \frac{d}{dz} \text{TR}(\Delta^{-z})|_{z=0}. \]
The second variation of logDet

- **Prop 1:** For a holomorphic family of Cauchy-Riemann operators $D_w$, the second variation of $\zeta'(0)$ is given by:

$$\delta \bar{w} \delta w \zeta'(0) = \frac{1}{2} \varphi_0 \left( \delta_w D \delta \bar{w} \text{res}(\log \Delta D^{-1}) \right).$$
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$$

- **Prop 2**: The residue density of $\log \Delta D^{-1}$:

$$
\sigma_{-2,0}(\log \Delta D^{-1}) = \frac{(\alpha + \alpha^*) \xi_1 + (\bar{\tau} \alpha + \tau \alpha^*) \xi_2}{(\xi_1^2 + 2\Re(\tau)\xi_1 \xi_2 + |\tau|^2 \xi_2^2)(\xi_1 + \tau \xi_2)}
$$

$$
- \log \left( \frac{\xi_1^2 + 2\Re(\tau)\xi_1 \xi_2 + |\tau|^2 \xi_2^2}{|\xi|^2} \right) \frac{\alpha}{\xi_1 + \tau \xi_2},
$$

and

$$
\delta \bar{w} \text{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi \Im(\tau)}(\delta w D)^*.
$$
Curvature of the determinant line bundle

Theorem (A. Fathi, A. Ghorbanpour, MK.): The curvature of the determinant line bundle for the noncommutative two torus is given by

$$\delta \bar{w} \delta w \zeta'(0) = \frac{1}{4\pi \Im(\tau)} \varphi_0 \left( \delta w D(\delta w D)^* \right).$$

Remark: For $\theta = 0$ this reduces to Quillen’s theorem (for elliptic curves).
A holomorphic determinant a la Quillen

- Modify the metric to get a flat connection:

\[ ||s||^2_f = e^{||D-D_0||^2} ||s||^2 \]
A holomorphic determinant a la Quillen

- Modify the metric to get a flat connection:
  \[ |s|_f^2 = e^{||D-D_0||^2} |s|^2 \]

- Get a flat holomorphic global section. This gives a holomorphic determinant function
  \[ \det(D, D_0) : \mathcal{A} \to \mathbb{C} \]
  It satisfies
  \[ |\det(D, D_0)|^2 = e^{||D-D_0||^2} \det_\zeta(D^*D) \]