Holomorphic Structures on the Quantum Projective Line

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- Pre-complex structres on $*$-algebras
- Holomorphic structures on NC vector bundles
- A holomorphic structure on the quantum projective line $\mathbb{C}P_q^1$
- The failure of the GAGA principle in the NC case
- Canonical line bundles on $\mathbb{C}P_q^1$ and their holomorphic structure
- The quantum homogeneous ring of $\mathbb{C}P_q^1$
- Positive Hoschschild cocycles and uniqueness

Noncommutative pre-complex structures

Initial data: $\mathcal{A}$ an $*$-algebra over $\mathbb{C}$, $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$ an *involutive differential calculus* over $\mathcal{A}$.

$$\Omega^0(\mathcal{A}) = \mathcal{A}, \qquad d(a^*) = (da)^*$$

Definition: A *pre-complex structure* on $\mathcal{A}$ for the differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$ is a bigraded differential $*$- algebra $\Omega^{(\bullet,\bullet)}(\mathcal{A})$ with differentials (derivations)

$$\partial : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{p+1,q}(\mathcal{A}),$$

$$\bar{\partial} : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p,q+1)}(\mathcal{A})$$

s.t.

$$\Omega^n(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{(p,q)}(\mathcal{A})$$

$$\partial(a)^* = \bar{\partial}(a^*), \quad \mathsf{d} = \partial + \bar{\partial}$$

Also, $*$ maps $\Omega^{(p,q)}(\mathcal{A})$ to $\Omega^{(q,p)}(\mathcal{A})$.

Motivating example: the de Rham complex of an almost complex manifold.

NC examples: Let L be a *real* Lie algebra with a *complex structure*:

$$L^{\mathbb{C}} = L_0 \oplus \overline{L_0}$$

Given $L \to \mathrm{Der}(\mathcal{A}, \mathcal{A})$, an action of L by $*$-derivations on $\mathcal{A}$, then

$$\Omega^{\bullet}\mathcal{A} = \mathrm{Hom}_{\mathbb{C}}(\Lambda^{\bullet} L^{\mathbb{C}}, \mathcal{A})$$

is a differential calculus for $\mathcal{A}$, and

$$\Omega^{(p,q)}\mathcal{A} = \mathrm{Hom}_{\mathbb{C}}(\Lambda^{p} L_0 \otimes \Lambda^{q} \overline{L_0}, \mathcal{A})$$

defines a pre-complex structure.

NC torus $\mathcal{A}_\theta$: Generators U_1, U_2 with

$$U_1 U_2 = e^{2\pi \mathrm{i}\theta} U_2 U_1$$

Basic derivations:

$$\delta_j(U_k) = 2\pi \mathrm{i}\, \delta_{jk} U_k, \quad j,k = 1,2$$

define an action of $\mathbb{R}^2$ on $\mathcal{A}_\theta$. Any $\tau \in \mathbb{C} \setminus \mathbb{R}$ defines a complex structure on $\mathcal{A}_\theta$:

$$\mathbb{R}^2 \otimes \mathbb{C} = L_0 \oplus \overline{L_0}$$

with $L_0 := e_1 + \tau e_2$.

Holomorphic functions:

$$\mathcal{O}(\mathcal{A}) := \ker \left\{ \bar{\partial} : \mathcal{A} \to \Omega^{(0,1)}(\mathcal{A}) \right\}.$$

Holomorphic structures on modules

Definition: Let $(A, \bar{\partial})$ be an algebra with a pre-complex structure and $\mathcal{E}$ a left $\mathcal{A}$-module. A holomorphic structure on $\mathcal{E}$ with respect to $(\mathcal{A}, \bar{\partial})$ is a flat $\bar{\partial}$-connection, i.e. a connection

$$\overline{\nabla} : \mathcal{E} \to \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$

s.t.

$$F(\overline{\nabla}) = \overline{\nabla}^2 = 0$$

If in addition $\mathcal{E}$ is a finitely generated projective $\mathcal{A}$-module, we call the pair $(\mathcal{E}, \overline{\nabla})$ a holomorphic vector bundle.

Since $\overline{\nabla}$ is a flat connection, we have a complex of vector spaces:

$$\mathcal{E} \xrightarrow{\overline{\nabla}} \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\overline{\nabla}} \Omega^{(0,2)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \cdots$$

Definition: The zeroth cohomology group of the above complex is the space of holomorphic sections of $\mathcal{E}$ and denoted by $H^0(\mathcal{E}, \overline{\nabla})$. It is a left $\mathcal{O}(\mathcal{A})$-module.

Holomorphic structures on bimodules

Let $\mathcal{E}$ be an $\mathcal{A}$-bimodule.

Definition: A bimodule connection on $\mathcal{E}$ is a *left* connection $\nabla : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ for which there is a bimodule isomorphism

$$\sigma(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E},$$

such that

$$\nabla(\xi a) = \nabla(\xi)a + \sigma(\nabla)\,(\xi \otimes \mathrm{d}a)$$

for all $\xi \in \mathcal{E}, a \in \mathcal{A}$.

In particular, this definition applies to the differential calculus $(\Omega^{(0,\bullet)}(\mathcal{A}), \bar{\partial})$ thus giving a notion of *holomorphic structures on bimodules*.

Tensor products of holomorphic vector bundles

Suppose we are given two $\mathcal{A}$-bimodules $\mathcal{E}_1, \mathcal{E}_2$ with two bimodule connections ∇_1, ∇_2, respectively. Let

$$\sigma := (\sigma_1 \otimes 1) \circ (1 \otimes \sigma_2) : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to$$

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$$

Lemma: The map

$$\nabla : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \mapsto \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$$

defined by

$$\nabla = \nabla_1 \otimes 1 + (\sigma_1 \otimes \mathrm{id})(1 \otimes \nabla_2)$$

defines a σ-compatible connection on the $\mathcal{A}$-bimodule $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$.

Note: We would like the flatness condition on holomorphic structures to survive under taking this tensor product. This is not the case in higher dimensions in the NC world! A possible way out might be the use of a more exotic tensor product.

The quantum Hopf fibration

$$S^1 \hookrightarrow S_q^3 \hookrightarrow S_q^2$$

A quantum homogeneous space:

$$\mathcal{A}(S_q^2) \to \mathcal{A}(S_q^3) \to \mathcal{A}(S^1)$$

$$\mathsf{S}_q^3 = \mathsf{SU}_q(2), \quad 0 < q \leq 1$$

$\mathcal{A}(\mathsf{SU}_q(2))$:= $*$-algebra generated by a and c, with relations

$$UU^* = U^*U = 1$$

$$U = \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix}$$

Hopf algebra structure on $\mathcal{A}(\mathsf{SU}_q(2))$:

$$\Delta U = U \otimes U$$

$$S(U) = U^*$$

$$\varepsilon(U) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The quantum enveloping algebra $\mathcal{U}_q(\mathrm{su}(2))$: It is the Hopf dual of $\mathrm{SU}_q(2)$. Generators: K, K^{-1}, E, F

Hopf pairing:

$$\mathcal{U}_q(\mathrm{su}(2)) \otimes \mathcal{A}(\mathrm{SU}_q(2)) \to \mathbb{C}$$

$$\langle K, a \rangle = q^{-1/2}, \quad \langle K^{-1}, a \rangle = q^{1/2}$$

$$\langle K, a^* \rangle = q^{1/2}, \quad \langle K^{-1}, a^* \rangle = q^{-1/2}$$

$$\langle E, c \rangle = 1, \quad \langle F, c^* \rangle = -q^{-1}$$

Left and right actions (infinitesimal symmetries):

$$\mathcal{U}_q(\mathrm{su}(2)) \otimes \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2)),$$

$$(X, f) \mapsto X \triangleright f.$$

$$\mathcal{A}(\mathsf{SU}_q(2)) \otimes \mathcal{U}_q(\mathsf{su}(2)) \to \mathcal{A}(\mathsf{SU}_q(2)),$$

$$(f, X) \mapsto f \triangleleft X$$

Uniquely fixed by:

$$\langle X, Y \triangleright f \rangle = \langle XY, f \rangle \,, \quad \langle X, f \triangleleft Y \rangle = \langle YX, f \rangle \,,$$

These right and left actions are mutually commuting.

The quantum projective line

There is a *quantum principal $U(1)$-bundle*:

$$\rho : \mathcal{A}(\mathsf{SU}_q(2)) \mapsto \mathcal{A}(\mathsf{SU}_q(2)) \otimes \mathcal{A}(\mathsf{U}(1))$$

$$\rho = (\mathrm{id} \otimes \pi) \circ \Delta$$

$$\pi : \mathcal{A}(\mathsf{SU}_q(2)) \to \mathcal{A}(\mathsf{U}(1)),$$

where

$$\pi \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix}$$

is a surjective Hopf algebra homomorphism, so that $\mathcal{A}(\mathsf{U}(1))$ becomes a quantum subgroup of $SU_q(2)$.

Coinvariants: A subalgebra of $\mathcal{A}(\mathsf{SU}_q(2))$:

$$\mathcal{A}(\mathsf{S}^2_q) := \{a \in \mathcal{A}(\mathsf{SU}_q(2));\ \rho(a) = a \otimes 1\}$$

The coordinate algebra of the Podleś sphere S^2_q = the underlying topological space of the quantum projective line $\mathbb{C}\mathsf{P}^1_q$.

The canonical line bundles on $\mathbb{C}P^1_q$

The action of the group-like element $K \Longrightarrow$ a decomposition:

$$\mathcal{A}(\mathrm{SU}_q(2)) = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$$

where,

$$\mathcal{L}_n := \{f \in \mathcal{A}(\mathrm{SU}_q(2)) \ : \ K \triangleright f = q^{n/2} f\}$$

Notice:

$$\mathcal{A}(\mathrm{S}^2_q) = \mathcal{L}_0, \quad \mathcal{L}^*_n \subset \mathcal{L}_{-n}, \quad \mathcal{L}_n\mathcal{L}_m \subset \mathcal{L}_{n+m}$$

$\mathcal{L}_n$: $\mathcal{A}(\mathrm{S}^2_q)$-bimodule; finite projective as a left module; analogues of canonical line bundles $\mathcal{O}(n)$ on $\mathbb{C}P^1$ of degree (monopole charge) $-n$.

A covariant differential calculus for $\mathsf{SU}_q(2)$

Left covariant calculus: $(\mathcal{A}, \Omega, H)$

- $\Omega = \bigoplus_{i \geq 0} \Omega^i$ is a DGA with $\Omega_0 = \mathcal{A}$
- Ω is a left DG H-comodule algebra, i.e. there is a morphism of DGA's

$$\rho : \Omega \longrightarrow H \otimes \Omega$$

s.t. Ω is a left DG H-comodule under ρ.

Example (Woronowicz): Let $H = \mathcal{A}(\mathsf{SU}_q(2))$ and

$$\Omega^i = \mathcal{A}(\mathsf{SU}_q(2)) \otimes \bigwedge^i \{\omega_+, \omega_-, \omega_z\} \quad 0 \leq i \leq 3$$

$\bigoplus \bigwedge^i\{\omega_+, \omega_-, \omega_z\}$ = The q-Grassmann algebra:

$$\omega_+ \wedge \omega_+ = \omega_- \wedge \omega_- = \omega_z \wedge \omega_z = 0$$

$$\omega_- \wedge \omega_+ + q^{-2}\omega_+ \wedge \omega_- = 0$$

$$\omega_z \wedge \omega_- + q^4\omega_- \wedge \omega_z = 0,$$

$$\omega_z \wedge \omega_+ + q^{-4}\omega_+ \wedge \omega_z = 0.$$

unique top form: $\omega_- \wedge \omega_+ \wedge \omega_z$.

differential $\mathrm{d} : \mathcal{A}(\mathrm{SU}_q(2)) \to \Omega^1(\mathrm{SU}_q(2))$:

$$\mathrm{d}f = (X_+ \triangleright f)\,\omega_+ + (X_- \triangleright f)\,\omega_- + (X_z \triangleright f)\,\omega_z,$$

where

$$X_z = \frac{1 - K^4}{1 - q^{-2}}, \qquad X_- = q^{-1/2}FK$$

$$X_+ = q^{1/2}EK$$

The holomorphic calculus on $\mathbb{C}P^1_q$

The 'cotangent bundle'

$$\Omega^1(\mathsf{S}^2_q) : \mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_2\omega_+$$

The differential d:

$$\mathrm{d}f = (X_- \triangleright f)\,\omega_- + (X_+ \triangleright f)\,\omega_+$$

where $X_- = q^{-1/2}F$ and $X_+ = q^{1/2}E$.

Break d into a holomorphic and an anti-holomorphic part, $\mathrm{d} = \bar{\partial} + \partial$, with:

$$\bar{\partial}f = (X_- \triangleright f)\,\omega_-, \quad \partial f = \left(X_+ \triangleright f\right)\omega_+$$

The above shows that:

$$\Omega^1(\mathsf{S}^2_q) = \Omega^{(0,1)}(\mathsf{S}^2_q) \oplus \Omega^{(1,0)}(\mathsf{S}^2_q)$$

where

$$\Omega^{(0,1)}(\mathsf{S}^2_q) \simeq \mathcal{L}_{-2} \simeq \bar{\partial}(\mathcal{A}(\mathsf{S}^2_q)),$$

$$\Omega^{(1,0)}(\mathsf{S}^2_q) \simeq \mathcal{L}_{+2} \simeq \partial(\mathcal{A}(\mathsf{S}^2_q))$$

These modules are not free.

2-forms: Let $\omega = \omega_- \wedge \omega_+$. We have $\omega f = f\omega$, for all $f \in \mathcal{A}(\mathsf{S}^2_q)$.

$$\Omega^2(\mathsf{S}^2_q) := \omega\mathcal{A}(\mathsf{S}^2_q) = \mathcal{A}(\mathsf{S}^2_q)\omega$$

Proposition: The 2D differential calculus on the sphere S^2_q is given by:

$$\Omega^\bullet(\mathsf{S}^2_q) = \mathcal{A}(\mathsf{S}^2_q) \oplus \Big(\mathcal{L}_{-2} \oplus \mathcal{L}_{+2}\Big) \oplus \mathcal{A}(\mathsf{S}^2_q)\omega_+ \wedge \omega_-$$

with the exterior differential $\mathsf{d} = \bar{\partial} + \partial$:

$$f \mapsto (q^{-1/2} F \triangleright f,\ q^{1/2} E \triangleright f)$$

$$(x, y) \mapsto q^{-1/2}(E \triangleright x - q^{-1} F \triangleright y)$$

for $f \in \mathcal{A}(\mathsf{S}^2_q)$, $(x, y) \in \mathcal{L}_{-2} \oplus \mathcal{L}_{+2}$.

Holomorphic functions on $\mathbb{C}\mathrm{P}^1_q$

$$\bar{\partial} : \mathcal{A}(\mathbb{C}\mathrm{P}^1_q) \to \Omega^{(0,1)}(\mathbb{C}\mathrm{P}^1_q)$$

We shall use the q-*number notation*:

$$[s] = [s]_q := \frac{q^s - q^{-s}}{q - q^{-1}}$$

Proposition: There are no non-trivial holomorphic *polynomial* functions on $\mathbb{C}\mathrm{P}^1_q$.

Proof:

$\bar{\partial} f = 0$ iff $F \triangleright f = 0$. Write f in PBW-basis $\{a^m c^k c^{*l}\}$ of $\mathcal{A}(SU_q(2))$,

$$f = \sum_{k,l \geq 0} f_{kl}\ a^{l-k} c^k c^{*l},$$

where $a^{-m} := a^{*m}$. The monomials $a^{l-k} c^k c^{*l}$ are the only K-invariant elements in the PBW-basis.

The vanishing of $F \triangleright f$ implies the following relations between f_{kl} with $0 \leq l < k$:

$$f_{kl} q^{-l} [k] = f_{k+1,l+1} q^{-k-1} [l+1]$$

the solutions of which are given by

$$\begin{aligned} f_{kl} &= \frac{[k-1][k-2] \cdots [k-l] q^k q^{k-1} \cdots q^{k-l+1}}{[l]! q^{l-1} q^{l-2} \cdots q^0} \tilde{f}_{k-l} \\ &= q^{(k-l+1)l} \begin{bmatrix} k-1 \\ l \end{bmatrix}_q \tilde{f}_{k-l} \end{aligned}$$

where $\tilde{f}_{k-l}$ are arbitrary. Clearly, the only polynomial solution is when $f_{kl} = 0$ for $(k,l) \neq (0,0)$. $\square$

The above expression for f_{kl} shows that in any reasonable smooth closure of $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$ there are non-trivial elements in the kernel of $\bar{\partial}$. In fact, we have

Proposition (failure of GAGA): For any $N > 0$, the series $f_{(N)}$ defined by

$$f_{(N)} = \sum_{l=0}^{\infty} q^{(N+1)l} \begin{bmatrix} N+l-1 \\ l \end{bmatrix}_q a^{*N} c^{N+l} c^{*l}$$

has coefficients of exponential decay and satisfies $\bar{\partial} f_{(N)} = 0$.

Proof: With the usual estimates $q^{-n-1}[n]^{-1} < 1$ and $q^n[n-1] < q$ we obtain

$$\left| q^{(N+1)l} \begin{bmatrix} N+l-1 \\ l \end{bmatrix}_q \right| < q^l$$

which gives a sequence of exponential decay. The vanishing of $\bar{\partial} f_{(N)}$ follows from the last proof since the coefficients of $f_{(N)}$ coincide with the f_{kl} with $N = k - l$. $\square$

Example: For $N = 1$ we have

$$f_{(1)} = \sum_{l=0}^{\infty} q^{2l} a^* c^{l+1} c^{*l}.$$

These holomorphic functions $f_{(N)}$ are in the C^*-algebra $C(\mathbb{C}\mathsf{P}^1_q) \simeq \mathcal{K} \oplus \mathbb{C}1$. In fact the C^*-norm of the generators a and c of $C(\mathsf{SU}_q(2))$ is less than one since $a^*a + c^*c = 1$. It follows that

$$\|f_{(N)}\| < \frac{1}{1-q}.$$

Proposition The algebra $\mathcal{O}(\mathbb{C}\mathsf{P}^1_q)$ is isomorphic to the polynomial algebra $\mathbb{C}[x]$.

proof: We have

$$f_{(N)} f_{(N')} = q^{NN'} f_{(N+N')}.$$

In fact $f_{(N)}$ is proportional to a^{*N}, and (up to scaling) it is the unique solution to $\bar{\partial} f$ with that property. Thus, because $f_{(N)}f_{(N')}$ is proportional to $a^{*(N+N')}$, it should be a complex multiple of $f_{(N+N')}$. By comparing the terms corresponding to $l = 0$ on both sides, we get $q^{NN'}$.

Now $x \mapsto f_{(1)}$ gives the desired isomorphism between $\mathbb{C}[x]$ and $\mathcal{O}(\mathbb{C}\mathsf{P}^1_q)$. $\square$

Notice that for $q = 1$ the series defining f_N do not converge unless $N = 0$ which yield the constant functions as the only holomorphic functions on $\mathbb{C}\mathsf{P}^1_{q=1}$.

Holomorphic vector bundles on $\mathbb{C}\mathsf{P}^1_q$

The 'line bundle' $\mathcal{L}_n$ is represented by a projection $\mathfrak{p}_n$ in $M_{|n|+1}(\mathcal{A}(\mathsf{S}^2_q))$. So we have a Grassmannian connection on $\mathcal{L}_n = (\mathcal{A}(\mathsf{S}^2_q))^{|n|+1}\mathfrak{p}_n$.

Equivalently, a connection is defined by a covariant splitting

$$\Omega^1(\mathsf{SU}_q(2)) = \Omega^1_{\mathsf{ver}}(\mathsf{SU}_q(2)) \oplus \Omega^1_{\mathsf{hor}}(\mathsf{SU}_q(2))$$

Let: ω_z to be vertical, and $\omega_\pm$ to be horizontal.

Now let $\mathcal{E} = \mathcal{L}_n$. We have:

$$\nabla\phi = \left(X_+ \triangleright \phi\right)\omega_+ + \left(X_- \triangleright \phi\right)\omega_-$$

$$= q^{-n-2}\omega_+\left(X_+ \triangleright \phi\right) + q^{-n+2}\omega_-\left(X_- \triangleright \phi\right)$$

Split ∇ into holomorphic and anti-holomorphic parts:

$$\nabla = \nabla^{\partial} + \nabla^{\bar{\partial}}$$

with

$$\nabla^{\partial}\phi = q^{-n-2}\omega_{+}\left(X_{+}\triangleright\phi\right)$$

$$\nabla^{\bar{\partial}}\phi = q^{-n+2}\omega_{-}\left(X_{-}\triangleright\phi\right)$$

Definition: The *standard holomorphic structure* on $\mathcal{L}_n$ is given by

$$\overline{\nabla} := \nabla^{\bar{\partial}} = q^{-n+2}\omega_{-}\left(X_{-}\triangleright -\right)$$

the anti-holomorphic part of ∇.

Theorem (failure of GAGA for canonical bundles): With notation as above,

1. For n positive, $H^0(\mathcal{L}_n, \overline{\nabla})$ is an $\mathcal{O}(\mathbb{C}\mathrm{P}^1_q)$-module

of rank 1.

2. For n negative, $H^0(\mathcal{L}_n, \overline{\nabla})$ is an $\mathcal{O}(\mathbb{C}\mathrm{P}^1_q)$-module of rank $|n| + 1$. $\square$

Remark: These spaces are certainly not finite dimensional since $\mathcal{O}(\mathbb{C}\mathrm{P}^1_q)$ is not finite dimensional. If we restrict to polynomial sections, then we obatin finite dimensional vector spaces whose dimensions are equal to the classical case.

We next study the tensor product of two noncommutative holomorphic line bundles.

Proposition: For any integer n there is a 'twisted flip' isomorphism

$$\Phi_{(n)} : \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \Omega^{(0,1)} \xrightarrow{\sim} \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \mathcal{L}_n$$

as $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$-bimodules.

Proof: $\Omega^{(0,1)}$ is generated (as a $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$-module) by $a^2\omega_-$, $ac\omega_-$ and $c^2\omega_-$. Define

$$\Phi_{(n)}(\phi_1 \otimes\ a^2\omega_- + \phi_2 \otimes ac\omega_- + \phi_3 \otimes c^2\omega_-)$$

$$= q^{-n}\left(a^2\omega_- \otimes \tilde{\phi}_1 + ac\omega_- \otimes \tilde{\phi}_2 + c^2\omega_- \otimes \tilde{\phi}_3\right)$$

with $\tilde{\phi}_1$ satisfying $\phi_1 a^2 = a^2\tilde{\phi}_1$ as elements of $\mathcal{A}(\mathrm{SU}_q(2))$ and similarly for $\tilde{\phi}_2, \tilde{\phi}_3$.

Proposition: The holomorphic structure $\overline{\nabla}$ on $\mathcal{L}_n$ is a bimodule connection with $\sigma(\overline{\nabla}) = \Phi_{(n)}$, i.e. it satisfies the left Leibniz rule and the *twisted right Leibniz rule*:

$$\overline{\nabla}(\xi f) = \overline{\nabla}(\xi)f + \Phi_{(n)}(\xi \otimes \bar{\partial} f)$$

for all $\xi \in \mathcal{L}_n$, $f \in \mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$).

So now we can consider the tensor product of these holomorphic line bundles $(\mathcal{L}_{n_i}, \overline{\nabla}_{n_i}), i = 1, 2$.

Proposition: The tensor product connection

$$\overline{\nabla}_{n_1} \otimes 1 + (\Phi_{(n_1)} \otimes 1)(1 \otimes \overline{\nabla}_{n_2})$$

coincides with the standard holomorphic structure on $\mathcal{L}_{n_1} \otimes_{\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)} \mathcal{L}_{n_2}$ when identified with $\mathcal{L}_{n_1+n_2}$.

The quantum homogeneous coordinate ring

Classical situation: X a projective variety and L a *very ample line bundle* on X. The *homogeneous coordinate ring* of (X, L) is the graded algebra

$$R_L = \bigoplus_{n \geq 0} H^0(X, L^{\otimes n})$$

For the quantum projective line $\mathbb{C}P^1_q$, using the line bundles $\mathcal{L}_n$, we define

$$\mathcal{R} := \bigoplus_{n \geq 0} H^0(\mathcal{L}_{-n}, \overline{\nabla})$$

Where now we consider only the algebraic sections, and hence the n-th component has dimension $n+1$. Notice that thanks to the twisting maps $\phi_{(n)}$, $\mathcal{R}$ is an algebra. What is the structure of this algebra?

Describe $\mathcal{L}_{-n}$: right $\mathcal{A}(S^2_q)$-module basis:

$$a^{-n-\mu}c^{\mu}, \qquad \mu = 0, 1, \cdots, n$$

Describe $H^0(\mathcal{L}_{-n}, \overline{\nabla})$: $\{a^{-n-\mu}c^{\mu}\}$ form a basis over $\mathbb{C}$

$\mathcal{R}$ is generated by a, c in degree one with one relation

$$ac = qca$$

which is one of the defining relation of the quantum group $\mathrm{SU}_q(2)$

Corollary: The homogeneous coordinate ring of $\mathbb{C}P^1_q$ is isomorphic to the coordinate ring of the quantum plane.

Twisted positivity

An approach to NC complex geometry suggested by Alain Connes [Book, 1994]: Let $\mathcal{A}$ be an $*$-algebra, A Hochschild $2m$-cocycle $\varphi \in Z^{2m}(\mathcal{A}, \mathcal{A}^*)$ is called positive if

$$\langle \omega, \eta \rangle := \int_\varphi \omega \eta^*$$

is a positive sesquilinear form on $\Omega^m \mathcal{A}$. Let

$$Z_+^{2m}(\mathcal{A}, \mathcal{A}^*) \subset Z^{2m}(\mathcal{A}, \mathcal{A}^*)$$

denote the set of positive $2m$-Hochschild cocycles on $\mathcal{A}$. It is a convex cone.

Let M= 2-dimensional compact oriented manifold, $\mathcal{A} = C^\infty(M)$, and define a 2-current $\mathbf{C}$ on M by

$$\mathbf{C}(f^0 df^1 df^2) = \frac{-1}{2\pi i} \int f^0 df^1 df^2$$

Let

$$C \subset C^2(\mathcal{A}, \mathcal{A}^*)$$

denote the *Hochschild class* representing the current $\mathbf{C}$. It is an affine subspace of $C^2(\mathcal{A}, \mathcal{A}^*)$.

Theorem (Connes; Book, 1994): There is a 1-1 correspondence between conformal structures on M and the extreme points of $Z^2_+ \cap C$ defined by $g \mapsto \varphi_g$, where

$$\varphi_g(f^0, f^1, f^2) = \frac{-1}{\pi i} \int_M f^0 \partial f^1 \bar{\partial} f^2$$

How can we extend all this to our $\mathbb{C}P^1_q$? There are no interesting 2-dimensional Hochschild classes on $\mathcal{A}(S^2_q)$ (dimension drop in quantization), but there are interesting *twisted cocycles*. In general Let σ :

$A \to A$ be an automorphism of A. Twisted n-cochains on (A, σ):

$$\varphi : A^{\otimes(n+1)} \to \mathbb{C}$$

$$\varphi(a_0, \cdots, a_n) = \varphi(\sigma(a_0), \cdots, \sigma(a_n))$$

Twisted Hochschild coboundary

$$b_\sigma : C_\sigma^n(A) \to C_\sigma^{n+1}(A)$$

$$b_\sigma \varphi(a_0, \cdots, a_{n+1}) =$$

$$\sum_{i=0}^{n} (-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1})$$

$$+(-1)^{n+1} \varphi(\sigma(a_{n+1}) a_n, a_0, \cdots, a_n).$$

Let $h : \mathcal{A}(SU_q(2)) \to \mathbb{C}$ denote the normalized Haar state of $SU_q(2)$. It is a positive twisted trace:

$$h(xy) = h(\sigma(y)x)$$

where $\sigma : SU_q(2) \to SU_q(2)$ is the modular automorphism

$$\sigma(x) = K^{-2} \triangleright x \triangleleft K^2.$$

Notice that, restricted to S_q^2, it induces an automorphism $\sigma : \mathcal{A}(S_q^2) \to \mathcal{A}(S_q^2)$, given by $\sigma(x) = x \triangleleft K^2$.

Define $\int : \Omega^2(S_q^2) \to \mathbb{C}$ by

$$\int x\omega_+ \wedge \omega_- = h(x)$$

Let $\varphi : \mathcal{A}(S_q)^{\otimes 3} \to \mathbb{C}$ be defined by

$$\varphi(a^0, a^1, a^2) = \int a^0 \bar{\partial} a^1 \partial a^2$$

Claim: φ is a twisted positive Hochschild 2-cocycle on $\mathcal{A}(S_q^2)$:

$$b_\sigma \varphi = 0$$

$$\int a^0 \partial a^1 (a^0 \partial a^1)^* \geq 0$$

for all $a^0, a^1 \in A(S_q^2)$.

Writing $\partial a^1 = y\omega_+$, we have $(\partial a^1)^* = \omega_- y^* = y^* \omega_-$, and

$$\int a^0 \partial a^1 (a^0 \partial a^1)^* = \int a^0 y \omega_+ y^* \omega_- (a^0)^*$$

$$= h(a^0 y y^* (a^0)^*) = h((a^0 y)(a^0 y)^*) \geq 0.$$

Another surprise: The above twisted cocycle φ is trivial (though it is the Hochschild class of a non-trivial twisted cyclic 2-cocycle on $\mathcal{A}(S_q^2)$). There is a non-trivial twisted Hochschild 2-cocycle on $\mathcal{A}(S_q^2)$, which seems to have nothing to do with our complex structure.

Open problems

• Uniqueness of the holomorphic structure on $\mathcal{L}_n$:

The holomorphic structure on the line bundle $\mathcal{O}(n)$ on the Riemann sphere is unique. Is this true in the q-deformed case as well?

Any connection on $\mathcal{L}_n$ can be written as $\overline{\nabla}+A$ with

$$A \in \mathrm{Hom}(\mathcal{L}_m, \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \mathcal{L}_m)$$

Now, since

$$\Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \mathcal{L}_n \simeq \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \Omega^{(0,1)}$$

as $\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)$-bimodules via the flip isomorphism, the connection one-form A can in fact be considered as an element of

$$\mathrm{End}_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)}(\mathcal{L}_n) \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \Omega^{(0,1)} \simeq \Omega^{(0,1)}$$

Now $\overline{\nabla}$ and $\overline{\nabla} + A$ are (gauge) equivalent iff for some $g \in C^\infty(\mathbb{C}\mathsf{P}^1_q)^\times$

$$g^{-1}\bar{\partial}g = A$$

After identifying $\Omega^{(0,1)}$ with $\mathcal{L}_{-2}\omega_-$, and writing $A = \phi\omega_-$ for ϕ a section of $\mathcal{L}_{-2}$, this equation is equivalent to

$$F \triangleright g = g\phi \tag{1}$$

The following surjectivity result should play a central role in establishing that for a given $\phi \in \mathcal{L}_{-2}$ there always exists a $g \in C^\infty(\mathbb{C}\mathsf{P}^1_q)$ solving (1), and hence in the uniqueness result for holomorphic structures on line bundles over $\mathbb{C}\mathsf{P}^1_q$:

Lemma The map $F : \mathcal{L}_0 \to \mathcal{L}_{-2}$ is surjective.

- Birkhoff-Grothendieck type theorem:

For every holomorphic vector bundle $\mathcal{E}$ on $\mathbb{CP}^1$ we have

$$\mathcal{E} \simeq \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_k)$$

where the integers $n_1, \cdots, n_k$ are unique up to permutation.

Is there a NC analogue? Note that we have a *topological* classification of vector bundles over S_q^2 in terms of monopole charge, as in the classical case.