The Gauss-Bonnet Theorem for the Noncommutative Torus

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Weyl's Law (1911-1915)

Consider the *Dirichlet boundary value problem* for $\Omega \subset \mathbb{R}^2$:

$$\Delta u = \lambda u, \qquad u|\partial\Omega = 0.$$

$$\Delta = -(\partial_x^2 + \partial_y^2) \qquad \text{Laplacian}$$

Eigenvalues:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

Eigenfunctions: $\{u_n\}_{n\geq 1}$ form an o.n. basis for $L^2(\Omega)$

Weyl's Law:

$$\lim \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|}, \qquad n \to \infty$$

Physics of the Weyl's Law

Laplcian Δ appears on the right hand side of most famous equations of both classical and quantum physics:
The Heat equation

$$\partial_t u = \Delta_x u$$

The wave equation

$$\partial_t^2 u = c^2 \Delta_x u$$

The Schroedinger equation (for a free particle)

$$ih\partial\psi = \Delta_x \psi$$

Musical interpretation (the theory of sound):
shows that the eigenfrequencies (pure tones) that a drum with clamped edge can produce

$$\nu_n \sim \sqrt{\lambda_n}$$

. Thus Weyl's law says that *one can hear the area of a drum!*
Why is this significant?
Introduce the *Eigenvalue Counting Function*:

Weyl's Law is equivalent to

$$N(\lambda) = \frac{|\Omega|}{4\pi}\lambda + o(\lambda) \qquad \lambda \to \infty$$

General statement: Let (M, g) be a closed, oriented, n-dimensional Riemannain manifold. Let $\Delta = d^*d$ be the Laplacian of (M, g). Consider the eigenvalue problem

$$\Delta u = \lambda u$$

It has a discrete set of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

Weyl's Law:

$$N(\lambda) = \frac{|\Omega|}{4\pi}\lambda + o(\lambda) \qquad \lambda \to \infty$$

Complex structure

Fix

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0,$$

and define

$$\partial = \delta_1 + \tau\delta_2, \qquad \partial^* = \delta_1 + \bar{\tau}\delta_2.$$

Define the Hilbert space (analogue of $(1,0)$-forms)

$$\mathcal{H}^{(1,0)} \subset \mathcal{H}_0$$

as the completion of the subspace spanned by finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$.

Connes and Tretkoff consider $\tau = i$.

Laplacian

View

$$\partial = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$$

as an unbounded operator with the adjoint given by

$$\partial^* = \delta_1 + \bar{\tau}\delta_2.$$

Define the Laplacian

$$\triangle := \partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

Conformal perturbation of the metric

To investigate the analogue of the Gauss-Bonnet theorem, vary the conformal class of the metric by $h = h^* \in A_\theta^\infty$: Define a postive linear functional $\varphi : A_\theta \to \mathbb{C}$ by

$$\varphi(a) = \tau_0(ae^{-h}), \qquad a \in A_\theta.$$

It is a twisted trace

$$\varphi(ba) = \varphi(a\sigma_i(b))$$

which is the KMS condition at $\beta = 1$ for 1PG of automorphisms $\sigma_t : A_\theta \to A_\theta,\ t \in \mathbb{R}$,

$$\sigma_t(x) = e^{ith} x e^{-ith}.$$

In fact

$$\sigma_t = \Delta^{-it}$$

with the modular operator

$$\Delta(x) = e^{-h} x e^{h}.$$

The perturbed Laplacian

Let $\mathcal{H}_\varphi =$ completion of A_θ w.r.t. $\langle , \rangle_\varphi$, where

$$\langle a, b\rangle_\varphi = \varphi(b^* a), \qquad a, b \in A_\theta.$$

Let

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

It has a formal adjoint ∂_φ^* given by

$$\partial_\varphi^* = R(e^h)\partial^*$$

where $R(e^h)$ is the right multiplication operator by e^h $(R(e^h)(x) = e^h x)$.
Define the new Laplacian:

$$\triangle' = \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi.$$

Lemma (Connes-Tretkoff; continues to hold)

$\triangle'$ is anti-unitarily equivalent to the positive unbounded operator $k\Delta k$ acting on $\mathcal{H}_0$, where $k = e^{h/2}$.

Spectral zeta function

$$\zeta(s) = \sum \lambda_i^{-s} = \text{Trace}\,(\triangle'^{-s}), \qquad \text{Re}(s) > 1.$$

Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt$$

gives us

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Trace}^+(e^{-t\triangle'}) t^{s-1} dt,$$

where

$$\text{Trace}^+(e^{-t\Delta'}) = \text{Trace}\,(e^{-t\Delta'}) - \text{Dim Ker}(\Delta').$$

ζ has a moromorphic extension to $\mathbb{C} \setminus 1$ with a simple pole at $s = 1$.

The Gauss-Bonnet theorem

Theorem (Gauss-Bonnet for classical Riemann surface)

Let $\Sigma =$ *compact connected oriented Riemann surface with metric* g*. Then*

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R = \frac{1}{6}\chi(\Sigma),$$

where ζ *is the zeta funtion associated to the Laplacian* $\triangle_g = d^*d$*, and* R *is the (scalar) curvature. In particular* $\zeta(0)$ *is a topological invariant; e.g. is invariant under conformal perturbations of the metric* $g \mapsto e^f g$*.*

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ *be an invertible positive element. Then the value* $\zeta(0)$ *of the zeta function* ζ *of the operator* $\triangle' \sim k\triangle k$ *is independent of* k*.*

Pseudodifferential calculus

Recall: Connes (1980), Baaj (1988).
Differential operators of order n:

$$P : A_\theta^\infty \to A_\theta^\infty, \quad P = \sum_j a_j \delta_1^{j_1} \delta_2^{j_2}$$

with $a_j \in A_\theta^\infty,\ j = (j_1, j_2)\, |j| \leq n$.
Operator valued symbols of order $n \in \mathbb{Z}$: smooth maps

$$\rho : \mathbb{R}^2 \to A_\theta^\infty$$

s.t.

$$||\delta_1^{i_1} \delta_2^{i_2} (\partial_1^{j_1} \partial_2^{j_2} \rho(\xi))|| \leq c(1 + |\xi|)^{n-|j|},$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$, and ρ is homogeneous of order n at infinity:

$$\lim \lambda^{-n} \rho(\lambda \xi_1, \lambda \xi_2), \qquad \lambda \to \infty$$

exists and is smooth.

Given a symbol ρ, define a pseudodifferential operator

$$P_\rho : A_\theta^\infty \to A_\theta^\infty$$

by

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi,$$

where

$$\alpha_s(U^n V^m) = e^{is.(n,m)} U^n V^m.$$

For pseudodifferential operators P, Q, with symbols $\sigma(P) = \rho, \sigma(Q) = \rho'$:

$$\sigma(PQ) \sim \sum \frac{1}{\ell_1! \ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2}(\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2}(\rho'(\xi)).$$

Elliptic Symbols: A symbol $\rho(\xi)$ of order n is called elliptic if $\rho(\xi)$ is invertible for $\xi \neq 0$, and, for $|\xi|$ large enough,

$$||\rho(\xi)^{-1}|| \leq c(1+|\xi|)^{-n}.$$

Example:

$$\triangle = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$$

is an elliptic operator with an elliptic symbol

$$\sigma(\Delta) = \xi_1^2 + 2\tau_1\xi_1\xi_2 + |\tau|^2\xi_2^2.$$

Computing $\zeta(0)$

Recall:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\text{Trace}(e^{-t\triangle'})t^{s-1} - 1)dt,$$

$1 = \text{Dim Ker}(\triangle')$.
Cauchy integral formula:

$$e^{-t\Delta'} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(\Delta' - \lambda 1)^{-1} d\lambda$$

gives the asymptotic expansion as $t \to 0^+$:

$$\text{Trace}(e^{-t\triangle'}) \sim t^{-1} \sum_0^\infty B_{2n}(\triangle')t^n.$$

It follows that:

$$\zeta(0) = B_2(\triangle'),$$

$$B_2(\Delta') = \frac{1}{2\pi i}\int_{\mathbb{R}^2}\int_C e^{-\lambda}\tau_0(b_2(\xi,\lambda))d\lambda d\xi$$

where

$$(b_0(\xi,\lambda) + b_1(\xi,\lambda) + b_2(\xi,\lambda) + \cdots)\sigma(\Delta' - \lambda) \sim 1,$$

$$b_j(\xi,\lambda) \,\text{is a symbol of order}\, -2-j.$$

Can assume $\lambda = -1$:

$$\zeta(0) = -\int \tau_0(b_2(\xi,-1))d\xi.$$

$$\sigma(\Delta' + 1) = \sigma(k\Delta k + 1) = (a_2 + 1) + a_1 + a_0$$

where

$$a_2 = k^2\xi_1^2 + 2\tau_1 k^2 \xi_1\xi_2 + |\tau|^2 k^2 \xi_2^2$$

$$a_1 = (2k\delta_1(k) + 2\tau_1 k\delta_2(k))\xi_1 +$$

$$(2\tau_1 k\delta_1(k) + 2|\tau|^2 k\delta_2(k))\xi_2$$

$$a_0 = k\delta_1^2(k) + 2\tau_1 k\delta_1\delta_2(k) + |\tau|^2 k\delta_2^2(k).$$

Using the calculus for symbols:

$$b_0 = (a_2 + 1)^{-1}$$

$$b_1 = -(b_0 a_1 b_0 + \partial_i(b_0)\delta_i(a_2)b_0)$$

$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_i(b_0)\delta_i(a_1)b_0$$

$$+\partial_i(b_1)\delta_i(a_2)b_0 + (1/2)\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0).$$

Integrating $b_2(\xi, -1)$ over the plane

Pass to these cordinates:

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta$$

$$\xi_2 = \frac{r}{\tau_2}\sin\theta$$

where θ ranges from 0 to 2π and r ranges from 0 to ∞.
After integrating $\int_0^{2\pi} \bullet \, d\theta$ we have terms such as

$$4\tau_1 r^3 b_0^3 k^2 \delta_2(k)\delta_1(k),$$

$$2r^3 b_0^2 k^2 \delta_1(k) b_0 \delta_1(k),$$

$$-2r^5 b_0^2 k^2 \delta_1(k) b_0^2 k^2 \delta_1(k),$$

where

$$b_0 = (1 + r^2 k^2)^{-1}.$$

Lemma (Connes-Tretkoff)

For $\rho \in A_\theta^\infty$ and every non-negative integer m:

$$\int_0^\infty \frac{k^{2m+2}u^m}{(k^2u+1)^{m+1}}\rho\frac{1}{(k^2u+1)}du = \mathcal{D}_m(\rho)$$

where

$$\mathcal{D}_m = \mathcal{L}_m(\Delta),$$

$$\Delta = \textit{the modular automorphism},$$

$$\mathcal{L}_m(u) = \int_0^\infty \frac{x^m}{(x+1)^{m+1}}\frac{1}{(xu+1)}dx =$$

$$(-1)^m(u-1)^{-(m+1)}\big(\log u - \sum_{j=1}^{m}(-1)^{j+1}\frac{(u-1)^j}{j}\big)$$

(modified logarithm).

Lemma

Let k be an invertible positive element of A_θ^∞. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is given by

$$\zeta(0)+1=\frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k))+\frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k))+$$

$$\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_2(k))+\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_1(k)),$$

where $\varphi(x)=\tau_0(xk^{-2})$, τ_0 is the unique trace on A_θ, Δ is the modular automorphism, and

$$f(u)=\frac{1}{6}u^{-1/2}-\frac{1}{3}+\mathcal{L}_1(u)-2(1+u^{1/2})\mathcal{L}_2(u)+(1+u^{1/2})^2\mathcal{L}_3(u).$$

($\mathcal{L}_m$ is the modified logarithm.)

Theorem (Gauss-Bonnet for NC torus)

Let $k \in A_\theta^\infty$ be an invertible positive element. Then the value $\zeta(0)$ of the zeta function ζ of the operator $\triangle' \sim k\triangle k$ is independent of k.

Proof.

$$\varphi(f(\Delta)(\delta_j(k))\delta_j(k)) = 0 \ \ for j = 1, 2,$$

$$\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) = -\varphi(f(\Delta)(\delta_2(k))\delta_1(k)).$$

Therefore

$$\zeta(0) + 1 = \frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k)) + \frac{2\pi|\tau|^2}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_2(k)) +$$

$$\frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) + \frac{2\pi\tau_1}{\tau_2}\varphi(f(\Delta)(\delta_2(k))\delta_1(k))$$

$$= 0$$

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□