Spectral Zeta Functions, Scalar Curvature, and Einstein-Hilbert action for Noncommutative Four Tori

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Related Works on $\mathbb{T}^2_\theta$

A. Connes, P. Tretkoff, *The Gauss-Bonnet Theorem for the noncommutative two torus* (2009).

F. Fathizadeh, M. Khalkhali, *The Gauss-Bonnet Theorem for Noncommutative Two Tori With a General Conformal Structure* (2010), Journal of Noncommutative Geometry.

F. Fathizadeh, M. Khalkhali, *Scalar Curvature for the Noncommutative Two Torus* (2011), Journal of Noncommutative Geometry.

F. Fathizadeh, M. Khalkhali, *Weyl's Law and Connes' Trace Theorem for Noncommutative Two Tori* (2011), LMP.

A. Connes, H. Moscovici, *Modular Curvature for Noncommutative Two-Tori* (2011).

F. Fathizadeh, M. Khalkhali, *Scalar Curvature for Noncommutative Four-Tori*, Preprint, 2013.

F. Fathizadeh, M. Khalkhali, A. Moatadelro, *A Riemann-Roch Theorem for Noncommutative Two Torus*, Preprint, 2012.

Laplace spectrum; commutative background

- (M, g) = closed Riemannian manifold. Laplacian on forms

  $$\triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

  has pure point spectrum:

  $$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Fact: Dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of M are fully determined by the spectrum of Δ (on all p-forms).

Method of proof: bring in the heat kernel

- Heat equation for functions: $\partial_t + \triangle = 0$
- $k(t, x, y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, \triangle) + a_1(x, \triangle)t + a_2(x, \Delta)t^2 + \cdots)$$

- $a_i(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in curvature tensor R and its covariant derivatives:

$$
\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}
$$

Heat trace asymptotics

Compute $\mathrm{Trace}(e^{-t\triangle})$ in two ways:

$$\text{Spectral Sum} = \text{Geometric Sum}.$$

$$\sum e^{-t\lambda_i} = \int_M k(t,x,x)d\mathrm{vol}_x \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \longrightarrow 0).$$

Hence

$$a_j = \int_M a_j(x,\triangle)d\mathrm{vol}_x,$$

are manifestly spectral invariants:

$$a_0 = \int_M d\mathrm{vol}_x = \mathrm{Vol}(M), \qquad \Longrightarrow \text{Weyl's law}$$

$$a_1 = \frac{1}{6}\int_M S(x)d\mathrm{vol}_x, \qquad \text{total scalar curvature}$$

Tauberian theory and $a_0 = 1$, implies Weyl's law:

$$N(\lambda) \sim \frac{\text{Vol }(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} \qquad \lambda \to \infty,$$

where

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

is the eigenvalue counting function.

Simplest example: flat tori

- $\Gamma \subset \mathbb{R}^m$ a cocompact lattice; $M = \mathbb{R}^m/\Gamma$

$$\mathrm{spec}(\triangle) = \{4\pi^2||\gamma^*||^2;\ \ \gamma^* \in \Gamma^*\}$$

- Then:

$$K(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \sum_{\gamma\in\Gamma} e^{-4\pi^2||x-y+\gamma||^2/4t}$$

- Poisson summation formula $\Longrightarrow$

$$\sum_{\gamma^*\in\Gamma^*} e^{-4\pi^2||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \sum_{\gamma\in\Gamma} e^{-4\pi^2||\gamma||^2/4t}$$

- And from this we obtain the asymptotic expansion of the heat trace near $t = 0$

$$\mathrm{Tr} e^{-t\Delta} \sim \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \qquad (t \to 0)$$

Application 1: heat equation proof of the Atiyah-Singer index theorem

- Dirac operator

$$D : C^\infty(S_+) \to C^\infty(S_-)$$

McKean-Singer formula:

$$\mathrm{Index}(D) = \mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*}), \qquad \forall t > 0$$

Heat trace asymptotics $\Longrightarrow$

$$\mathrm{Index}(D) = \int_M a_n(x)dx,$$

where $a_n(x) = a_n^+(x) - a_n^-(x), m = 2n$, can be explicitly computed and recovers the A-S integrand (The simplest proof is due to Getzler).

Application 2: meromorphic extension of spectral zeta functions

$$\zeta_{\triangle}(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

$$\begin{aligned} \zeta_{\triangle}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle}) - \mathrm{Dim\,Ker}\,\triangle) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \{ \int_0^c \cdots + \int_c^\infty \cdots \} \end{aligned}$$

The second term defines an entire function, while the first term has a meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Simplest example: For $M = S^1$ with round metric, we have

$$\zeta_\triangle(s) = 2\zeta(2s) \qquad \text{Riemann zeta function}$$

Scalar curvature

The spectral invariants a_i in the heat asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_{\triangle}(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x) dvol_x$, we obtain a formula for scalar curvature density as follows:

Let $\zeta_f(s) := \mathrm{Tr}\,(f\triangle^{-s}),\;\; f \in C^\infty(M)$.

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi}\int_M fS(x)dvol_x - \mathrm{Tr}(fP) \qquad m = 2$$

$\log\det(\triangle) = -\zeta'(0)$, Ray-Singer regularized determinant

Noncommutative Geometry: Spectral Triples $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A}$= involutive unital algebra, $\mathcal{H}$ = Hilbert space,

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \qquad D : \mathcal{H} \to \mathcal{H}$$

  D has compact resolvent and all commutators $[D, \pi(a)]$ are bounded.

- An asymptotic expansion holds

$$\mathrm{Trace}\,(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

- Let $\triangle = D^2$. Spectral zeta function

$$\zeta_D(s) = \mathrm{Tr}\,(|D|^{-s}) = \mathrm{Tr}\,(\Delta^{-s/2}), \quad \mathrm{Re}(s) \gg 0.$$

Noncommutative 4-torus $\mathbb{T}^4_\theta$

$C(\mathbb{T}^4_\theta)$ is the universal C^*-algebra generated by 4 unitaries

$$U_1, U_2, U_3, U_4,$$

satisfying

$$U_k U_\ell = e^{2\pi i \theta_{k\ell}} U_\ell U_k,$$

for a skew symmetric matrix

$$\theta = (\theta_{k\ell}) \in M_4(\mathbb{R}).$$

Action of $\mathbb{T}^4 = (\mathbb{R}/2\pi\mathbb{Z})^4$ on $C(\mathbb{T}^4_\theta)$

$$\mathbb{R}^4 \ni s \mapsto \alpha_s \in \mathrm{Aut}\Big(C(\mathbb{T}^4_\theta)\Big),$$

$$\alpha_s(U^m) := e^{is\cdot m}\, U^m, \quad U^m := U_1^{m_1}\, U_2^{m_2}\, U_3^{m_3}\, U_4^{m_4}, \quad m_j \in \mathbb{Z}.$$

$$\delta_j = \frac{\partial}{\partial s_j}\Big|_{s=0} \alpha_s \ : C^\infty(\mathbb{T}^4_\theta) \to C^\infty(\mathbb{T}^4_\theta),$$

$$\begin{aligned} \delta_j(U_k) &:= U_k \qquad \text{if} \qquad k = j, \\ &:= 0 \qquad \text{if} \qquad k \neq j. \end{aligned}$$

Complex Structure on $\mathbb{T}^4_\theta$

$$\partial = \partial_1 \oplus \partial_2, \qquad \bar{\partial} = \bar{\partial}_1 \oplus \bar{\partial}_2,$$

$$\partial_1 = \frac{1}{2}\,(\delta_1 - i\delta_3), \qquad \partial_2 = \frac{1}{2}\,(\delta_2 - i\delta_4),$$

$$\bar{\partial}_1 = \frac{1}{2}\,(\delta_1 + i\delta_3), \qquad \bar{\partial}_2 = \frac{1}{2}\,(\delta_2 + i\delta_4).$$

Volume form on ${\mathbb{T}_\theta}^4$

$$\varphi_0 : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi_0(1) := 1,$$

$$\varphi_0(U_1^{m_1}\, U_2^{m_2}\, U_3^{m_3}\, U_4^{m_4}) := 0, \qquad (m_1, m_2, m_3, m_4) \neq (0,0,0,0).$$

$$\varphi_0(a\, b) = \varphi_0(b\, a), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

$$\varphi_0(a^*\, a) > 0, \qquad a \neq 0.$$

Conformal perturbation (Connes-Tretkoff)

Let $h = h^* \in C^\infty(\mathbb{T}^4_\theta)$ and replace the trace φ_0 by

$$\varphi : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi(a) := \varphi_0(a\, e^{-2h}), \qquad a \in C(\mathbb{T}^4_\theta).$$

φ is a KMS state with the modular group

$$\sigma_t(a) = e^{2ith}\, a\, e^{-2ith}, \qquad a \in C(\mathbb{T}^4_\theta),$$

and the modular automorphism

$$\Delta(a) := \sigma_i(a) = e^{-2h}\, a\, e^{2h}, \qquad a \in C(\mathbb{T}^4_\theta).$$

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

Perturbed Laplacian on $\mathbb{T}^4_\theta$

$$d = \partial \oplus \bar{\partial} : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)},$$

$$\triangle_\varphi := d^* d.$$

Remark. If $h = 0$ then $\varphi = \varphi_0$ and

$$\triangle_{\varphi_0} = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = \partial^* \partial$$

(the underlying manifold is Kähler).

Scalar Curvature for $\mathbb{T}^4_\theta$

It is the unique element $R \in C^\infty(\mathbb{T}^4_\theta)$ such that

$$\mathrm{Res}_{s=1}\,\zeta_a(s) = \varphi_0(a\,R), \qquad a \in C^\infty(\mathbb{T}^4_\theta),$$

$$\zeta_a(s) := \mathrm{Trace}(a\,\triangle_\varphi^{-s}), \qquad \Re(s) \gg 0.$$

Lemma. Up to an anti-unitary equivalence $\triangle_\varphi$ is given by

$$e^h\bar\partial_1 e^{-h}\partial_1 e^h + e^h\,\partial_1 e^{-h}\bar\partial_1\,e^h + e^h\bar\partial_2 e^{-h}\,\partial_2 e^h + e^h\partial_2 e^{-h}\bar\partial_2 e^h,$$

where ∂_1, ∂_2 are analogues of the Dolbeault operators.

Mellin Transform and Asymptotic Expansions

$$\mathrm{Trace}(a\, e^{-t\triangle_\varphi}) \sim_{t\to 0^+} \; t^{-2} \sum_{n=0}^{\infty} B_n(a, \triangle_\varphi)\, t^{n/2}.$$

Approximate $e^{-t\triangle_\varphi^2}$ by pseudodifferential operators:

$$e^{-t\triangle_\varphi} = \frac{1}{2\pi i} \int_C e^{-t\lambda}\, (\triangle_\varphi - \lambda)^{-1}\, d\lambda,$$

$$B_\lambda\, (\triangle_\varphi - \lambda) \approx 1,$$

$$\sigma(B_\lambda) = b_0 + b_1 + b_2 + \cdots .$$

Connes' Pseudodifferential Calculus (1980)

- Symbols: $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$,

$$P_\rho(a) = (2\pi)^{-4} \int \int e^{-is.\xi} \, \rho(\xi) \, \alpha_s(a) \, ds \, d\xi, \qquad a \in C^\infty(\mathbb{T}^4_\theta).$$

- Differential operators:

$$\rho(\xi) = \sum a_\ell \, \xi^\ell, \quad a_\ell \in C^\infty(\mathbb{T}^4_\theta) \quad \Rightarrow \quad P_\rho = \sum a_\ell \, \delta^\ell.$$

- ΨDO's on $\mathbb{T}^4_\theta$ form an algebra:

$$\sigma(P\,Q) \sim \sum_{\ell \in \mathbb{Z}^4_{\geq 0}} \frac{1}{\ell!} \, \partial_\xi^\ell \rho(\xi) \, \delta^\ell(\rho'(\xi)).$$

Connes' Rearrangement Lemma

For any $m = (m_0, m_1, \ldots, m_\ell) \in \mathbb{Z}_{>0}^{\ell+1}$, $\rho_1, \ldots, \rho_\ell \in C^\infty(\mathbb{T}_\theta^4)$

$$\int_0^\infty \frac{u^{|m|-2}}{(e^h u + 1)^{m_0}} \prod_1^\ell \rho_j \, (e^h u + 1)^{-m_j} \, du$$

$$= e^{-(|m|-1)h} \, F_m(\Delta, \ldots, \Delta)\Big(\prod_1^\ell \rho_j \Big),$$

where

$$F_m(u_1, \ldots, u_\ell) = \int_0^\infty \frac{x^{|m|-2}}{(x+1)^{m_0}} \prod_1^\ell \Big(x \prod_1^j u_k + 1 \Big)^{-m_j} \, dx.$$

Final Formula for the Scalar Curvature of $\mathbb{T}^4_\theta$

Theorem.

$$R = e^{-h}\, k(\nabla)\Big(\sum_{i=1}^{4} \delta_i^2(h)\Big) + e^{-h}\, H(\nabla,\nabla)\Big(\sum_{i=1}^{4} \delta_i(h)^2\Big),$$

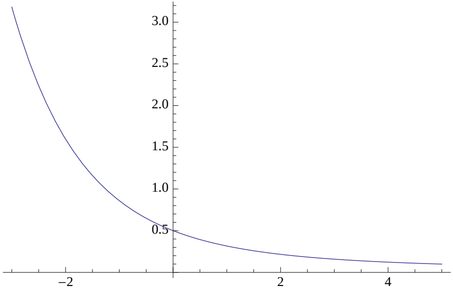
where

$$\nabla(a) = [-h, a], \qquad a \in C(\mathbb{T}^4_\theta),$$

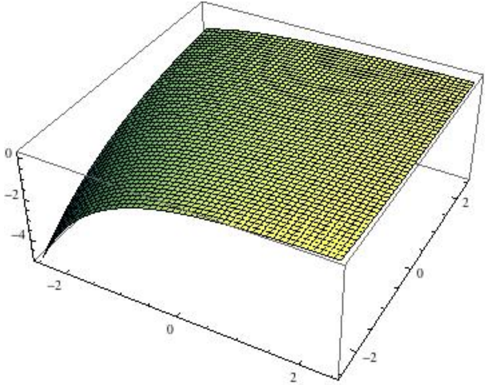
$$k(s) = \frac{1-e^{-s}}{2s},$$

$$H(s,t) = -\frac{e^{-s-t}\left(\left(-e^s-3\right) s \left(e^t-1\right) + \left(e^s-1\right)\left(3e^t+1\right) t\right)}{4\, s\, t\, (s+t)}.$$

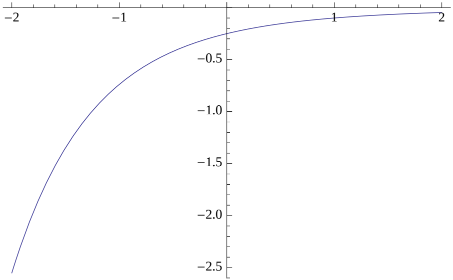
$$k(s) = \frac{1}{2} - \frac{s}{4} + \frac{s^2}{12} - \frac{s^3}{48} + \frac{s^4}{240} - \frac{s^5}{1440} + O\left(s^6\right).$$



$$
\begin{aligned}
H(s,t) \quad = \quad & \left(-\frac{1}{4}+\frac{t}{24}+O\left(t^3\right)\right)+s\left(\frac{5}{24}-\frac{t}{16}+\frac{t^2}{80}+O\left(t^3\right)\right) \\
& +s^2\left(-\frac{1}{12}+\frac{7t}{240}-\frac{t^2}{144}+O\left(t^3\right)\right)+O\left(s^3\right).
\end{aligned}
$$

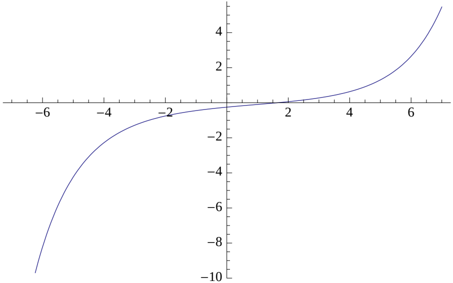


$$
\begin{aligned}
H(s,s) &= -\frac{e^{-2s}\left(e^{s}-1\right)^{2}}{4s^{2}} \\
&= -\frac{1}{4}+\frac{s}{4}-\frac{7s^{2}}{48}+\frac{s^{3}}{16}-\frac{31s^{4}}{1440}+\frac{s^{5}}{160}+O\left(s^{6}\right).
\end{aligned}
$$



$$G(s) := H(s,-s) = \frac{-4s-3e^{-s}+e^{s}+2}{4s^2}$$

$$= -\frac{1}{4}+\frac{s}{6}-\frac{s^2}{48}+\frac{s^3}{120}-\frac{s^4}{1440}+\frac{s^5}{5040}+O\left(s^6\right).$$



Einstein-Hilbert Action for $\mathbb{T}_\theta^4$

Theorem. We have the local expression (up to a factor of π^2)

$$\varphi_0(R) = \frac{1}{2}\sum_{i=1}^{4} \varphi_0\Big(e^{-h}\delta_i^2(h)\Big) + \sum_{i=1}^{4} \varphi_0\Big(G(\nabla)(e^{-h}\delta_i(h))\,\delta_i(h)\Big).$$

Extremum of the Einstein-Hilbert Action

Theorem. For any Weyl factor $e^{-h} \in C^\infty(\mathbb{T}^4_\theta)$

$$\varphi_0(R) \leq 0,$$

and the equality happens if and only if h is a constant.

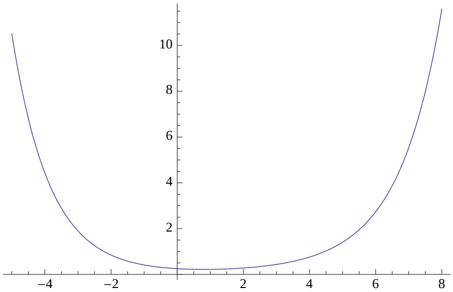
Proof.

$$\varphi_0(R) = \sum_{i=1}^{4} \varphi_0\big(e^{-h} T(\nabla)(\delta_i(h))\, \delta_i(h)\big),$$

where

$$T(s) = \frac{1}{2}\frac{e^{-s}-1}{-s} + G(s) = \frac{-2s + e^{s} - e^{-s}(2s+3) + 2}{4s^2}.$$

$$T(s) = \frac{1}{4} - \frac{s}{12} + \frac{s^2}{16} - \frac{s^3}{80} + \frac{s^4}{288} - \frac{s^5}{2016} + O\left(s^6\right).$$



Weyl's Law for $\mathbb{T}^4_\theta$

Theorem. For the eigenvalue counting function

$$N(\lambda) = \#\{\lambda_j \leq \lambda\}$$

of the Laplacian $\triangle_\varphi$ on $\mathbb{T}^4_\theta$, we have

$$N(\lambda) \sim \frac{\pi^2 \varphi_0(e^{-2h})}{2} \lambda^2 \qquad (\lambda \to \infty).$$

Corollary.

$$\lambda_j \sim \frac{\sqrt{2}}{\pi \varphi_0(e^{-2h})^{1/2}} j^{1/2} \qquad (j \to \infty),$$

$$\mathrm{Tr}_\omega\Big((1 + \triangle_\varphi)^{-2}\Big) = \frac{\pi^2}{2} \varphi_0(e^{-2h}).$$

A Noncommutative Residue

Classical symbols: $\rho : \mathbb{R}^4 \to A_\theta^\infty$

$$\rho(\xi) \sim \sum_{i=0}^{\infty} \rho_{m-i}(\xi) \qquad (\xi \to \infty),$$

$$\rho_{m-i}(t\,\xi) = t^{m-i}\,\rho_{m-i}(\xi), \qquad t > 0, \quad \xi \in \mathbb{R}^4.$$

$$\mathrm{Res}(P_\rho) := \int_{\mathbb{S}^3} \varphi_0\big(\rho_{-4}(\xi)\big)\, d\xi.$$

Theorem. Res is the unique trace on the algebra of classical pseudodifferential operators on $\mathbb{T}_\theta^4$.

Connes' Trace Theorem for $\mathbb{T}^4_\theta$

Theorem. For any classical symbol ρ of order -4 on $\mathbb{T}^4_\theta$, we have

$$P_\rho \in \mathcal{L}^{1,\infty}(\mathcal{H}_0),$$

and

$$\mathrm{Tr}_\omega(P_\rho) = \frac{1}{4}\mathrm{Res}(P_\rho).$$

The geometry in noncommutative geometry

- Geometry starts with metric and curvature. While there are a good number of 'soft' topological tools in NCG, like cyclic cohomology, K and KK-theory, and index theory, a truly noncommutative theory of curvature is still illusive. The situation is better with scalar curvature, but computations are quite tough at the moment.

- Metric aspects of NCG are informed by Spectral Geometry. Spectral invariants are the only means by which we can formulate metric ideas of NCG.