

A One Lecture Introduction to Fredholm Operators

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1. Introduction

This is an elementary introduction to Fredholm operators on a Hilbert space H . Fredholm operators are named after a Swedish mathematician, Ivar Fredholm (1866-1927), who studied integral equations. We will introduce two definitions of a Fredholm operator and prove their equivalence. We will also discuss briefly the index map defined on the set of Fredholm operators. Note that results proved in 4154A Functional Analysis by Prof. Khalkhali may be used without proof.

Definition 1.1: A bounded, linear operator $T : H \rightarrow H$ is said to be Fredholm if

- 1) $\text{range}T \subseteq H$ is closed
- 2) $\dim \ker T < \infty$
- 3) $\dim \ker T^* < \infty$

Notice that for a bounded linear operator T on a Hilbert space H , $\ker T^* = (\text{Image}T)^\perp$.

Indeed:

$$x \in \ker T^*$$

$$\Leftrightarrow T^*x = 0$$

$$\Leftrightarrow \langle T^*x, y \rangle = 0 \text{ for all } y \in H$$

$$\Leftrightarrow \langle x, Ty \rangle = 0$$

$$\Leftrightarrow x \in (\text{Image}T)^\perp$$

A good way to think about these Fredholm operators is as operators that are "almost invertible". This notion of "almost invertible" will be made precise later. For now, notice that that operator is almost injective as it has only a finite dimensional kernel, and almost surjective as $\ker T^* = (\text{Image}T)^\perp$ is also finite dimensional.

We will now introduce some algebraic structure on the set of all bounded linear functionals, $L(H)$ and from here make the above discussion precise.

2. Bounded Linear Operators as a Banach Algebra

Definition 2.1: A \mathbb{C} algebra is a ring A with identity along with a ring homomorphism $f : \mathbb{C} \rightarrow A$ such that $1 \mapsto 1_A$ and $f(\mathbb{C}) \subseteq Z(A)$.

Observe that $L(H)$ is a \mathbb{C} algebra. We can add and multiply operators, and have additive inverses. We also have a zero element, the zero map, and an identity element, the identity map. Also, \mathbb{C} sits inside of $Z(L(H))$ as the maps that are simply multiplication by a complex scalar.

Definition 2.2: A Banach Algebra B is an algebra over \mathbb{C} with identity which has a norm $\|\cdot\|$ making it into a Banach space. Further we require that $\|1\| = 1$ and $\|fg\| \leq \|f\|\|g\|$ for all $f, g \in B$.

Indeed, $L(H)$ is a Banach Algebra under the sup norm. We have seen that the sup norm satisfies the required properties.

Definition 2.3: If U is a Banach Algebra, then an involution on U is a mapping $T \rightarrow T^*$ such that:

- 1) $T^{**} = T$
- 2) $(\alpha S + \beta T)^* = \bar{\alpha}S^* + \bar{\beta}T^*$
- 3) $(ST)^* = T^*S^*$

The adjoint map on $L(H)$ satisfies all of these properties as observed in class.

Definition 2.4: A Banach Algebra with an involution map such that $\|TT^*\| = \|T\|^2$ is called a \mathbb{C}^* - algebra.

We notice that the adjoint map satisfies this final condition. Now we have our final algebraic structure on $L(H)$. We can now say that $L(H)$ is a \mathbb{C}^* - algebra.

Proposition 2.5: The set of all compact operators on H , denoted $K(H)$, is a closed two-sided ideal in $L(H)$.

Proof: $K(H)$ is closed as it is the closure of the finite rank operators. We have also seen that if $S \in L(H)$ and $T \in K(H)$ then $ST \in K(H)$ and $TS \in K(H)$. Furthermore, if T is a compact operator, then so is T^* .

3. The Calkin Algebra

Now that we have an algebra and a closed ideal it is very natural to consider the quotient algebra: $L(H)/K(H)$. This quotient algebra is called the Calkin algebra, and it is again a $(C)^*$ -algebra. We will prove that the invertible elements of the Calkin algebra are exactly the images of the Fredholm operators under the quotient map. This result will show that Fredholm operators are invertible up to a compact operator exactly.

First, we have to prove a few Lemmas.

Lemma 3.1: The unit ball of a Hilbert space H , $(H)_1$, is compact in norm topology if and only if H is finite dimensional.

Proof: Suppose that H is finite dimensional. Then H is isometrically isomorphic to $(C)^n$. Since the unit ball in $(C)^n$ is compact, the unit ball in H is also compact.

Now assume that H is ∞ -dimensional. Then there exists an orthonormal subset $\{e_n\}_{n=1}^{\infty} \subset (H)_1$.

$\|e_n - e_m\| = \sqrt{2}$ for any $n, m \in \mathbb{N}$ with $n \neq m$.

Thus $(H)_1$ cannot be compact.

Lemma 3.2: If H is an ∞ -dimensional Hilbert space and T is a compact operator on H , then the range of T contains no closed ∞ -dimensional subspace.

Proof: Let M be a closed subspace in the range of T .

Let P_M be the projection map onto M . Then $P_M T$ is compact.

Define $A : H \rightarrow M$ by $Af = P_M T f$.

Then A is bounded and onto. Thus by the open mapping theorem, A is an open map.

Therefore, $A((H)_1)$ contains the open ball in M of radius δ for some small $\delta > 0$ centered at 0.

Now, since the closed ball of radius δ is contained in the compact set $P_M T((H)_1)$ we get that M is finite dimensional by *Lemma 3.2*.

Lemma 3.3: Let H be a Hilbert space, M a closed subspace of H , and

N a finite dimensional subspace of H . Then the linear space $M + N$ is a closed subspace of H .

Proof: Omitted. See Lemma 5.16 in 1.

We will now state and prove the main result of this lecture.

Theorem 3.4(Atkinson): If H is a Hilbert space, then $T \in L(H)$ is a Fredholm operator if and only if the image of T in the Calkin algebra is invertible.

Proof: Suppose that the image of T is invertible in the Calkin algebra. Then there exists $A \in L(H)$ and $K \in K(H)$ such that $AT = I + K$.

Let $f \in \ker(I + K)$ then $(I + K)f = 0$.

$$\Rightarrow If = -Kf \Rightarrow Kf = -f$$

So, $f \in \text{range}K$.

Thus, $\ker T \subset \ker AT = \ker(I + K) \subset \text{range}K$

So, by Lemma 3.2 the dimension of $\ker T < \infty$ as $\text{range}K$ contains no closed ∞ dimensional subspace ($\ker T$ is closed as it is a kernel).

Now, since T^* is also invertible in the Calkin algebra, $\dim \ker T^* < \infty$ by symmetry.

Next, we need to show that $\text{range}T$ is closed in H .

We know that the finite rank operators are dense in the compact operators, so there exists F , a finite rank operator, such that $\|K - F\| < \frac{1}{2}$.

Claim: T is bounded below on $\ker F$.

Let $f \in \ker F$.

$$\begin{aligned} \|ATf\| &= \|f + Kf\| = \|f + Kf - Ff\| = \|f - (Kf - Ff)\| \\ &\geq \|f\| - \|Kf - Ff\| \geq \|f\| - \|K - F\|\|f\| \geq \|f\| - \frac{1}{2}\|f\| = \frac{\|f\|}{2} \end{aligned}$$

$$\text{Thus, } \frac{\|f\|}{2} \leq \|ATf\| \leq \|A\|\|Tf\|$$

$$\Rightarrow \frac{\|f\|}{2\|A\|} \leq \|Tf\|$$

So, T is bounded below on $\ker F$ and we have proved the claim.

Claim: $T(\ker F)$ is a closed subspace of H .

Let $\{Tf_n\}_{n=1}^\infty$ be a Cauchy sequence in $T(\ker F)$.

$$\text{Then } \|f_n - f_m\| \leq 2\|A\|\|Tf_n - Tf_m\|$$

$\Rightarrow \{f_n\}$ is Cauchy.

So, if $f = \lim f_n$ then $Tf = \lim Tf_n$ is in $T(\ker F)$.

Therefore, $T(\ker F)$ is closed to complete the proof of the claim.

Now, since F is a finite rank operator, we have that $(\ker(F))^{perp}$ is finite dimensional.

So, $\text{range}T = T(\ker F) + T((\ker F)^{perp})$ is a closed subspace of H by *Lemma 3.3*.

Thus, T is a Fredholm operator.

Now, assume that T is Fredholm operator. We will show that it is invertible up to a finite rank operator, and thus up to a compact operator.

Define $T_0 : (\ker T)^{perp} \rightarrow \text{range}T$ by $T_0f = Tf$.

Claim: T_0 is a bijection.

First we will show that it is surjective.

Let $g \in \text{range}T$ then there exists $f \in H$ such that $Tf = g$.

If $f \in (\ker T)^{perp}$ then we are done.

If $f \notin (\ker T)^{perp}$ then $T(f) = 0$. In this case $g = 0$ and $T(0) = 0$ so T_0 is surjective.

Now we will show that it is injective. If $T_0f = T_0g$ for $f, g \in (\ker T)^{perp}$.

Then, $T_0(f - g) = 0 \Rightarrow f - g = 0$ so T_0 is injective.

Thus, T_0 is bijective and so it has an inverse.

Now, define $S : H \rightarrow H$ by $Sf = T_0^{-1}f$ if $f \in \text{range}T$, and $Sf = 0$ if $f \in (\text{range}T)^{perp}$.

S is bounded since it is bounded on $\text{range}T$ as it is equal to T_0^{-1} there and zero otherwise.

So, $ST = I - P_1$ and $TS = I - P_2$ where P_1 is the projection map onto $\ker T$ and P_2 is the projection map onto $(\text{range}T)^{perp} = \ker T^*$.

Since $\dim \ker T < \infty$ P_1 is a finite rank operator and thus is compact. Similarly as $\dim \ker T^* < \infty$ P_2 is finite rank and thus compact.

So, the image of S is the inverse of the image of T is the Calkin algebra. Therefore, T is invertible in the Calkin algebra as required.

There are a couple interesting things to note in this proof. First, we only used the fact that T had a one sided inverse in the first direction. Then, in the second direction we proved that T was invertible up to a finite rank operator instead of a compact operator.

4. Examples

Example 4.1

If T is an invertible operator, then T is a Fredholm operator.

This is clear, as if T is invertible then certainly the image of T in the Calkin algebra is invertible.

Or, we can see this using the original definition:

$\dimker T = 0$, $\dimker T^* = 0$, and $range T = H$ is closed.

Example 4.2

Let T be the right shift operator. $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. Then T is Fredholm. Indeed, $range T = \{(0, a_1, a_2, \dots)\}$ is closed, $\dimker T = 1$, and T^* is the left shift operator, so $\dimker T^* = 1$. Thus T is Fredholm.

Example 4.3

If T is a bounded linear operator, and K is compact then $T + K$ is a Fredholm operator.

5. Some additional results

We will state a few additional results without proof. The proofs can be found in the referenced text 1. These are the next steps after the given introduction to Fredholm operators.

Definition 5.1: Denote the set of all Fredholm operators on H by $F(H)$. Then we define the map $i : F(H) \rightarrow (\mathbb{Z})$ by $i(T) = \dimker T - \dimker T^*$. We call i the index map. Also define $F_n = \{T \in F(H) | i(T) = n\}$.

Proposition 5.2: F_0 is invariant under compact perturbation.

Proposition 5.3 $i(T + K) = i(T)$ for $T \in F(H)$ and $K \in K(H)$.

This following Theorem is a very famous result with many different formulations. This is the formulation in terms of functional analysis.

Theorem 5.4(Fredholm Alternative): If λ is a nonzero element of $\sigma(K)$, then λ is an eigenvalue of K with finite multiplicity and $\bar{\lambda}$ is an eigenvalue of K^* with the same multiplicity. Moreover, the generalized eigenspace E_λ for λ is finite dimensional and has the same dimension as the generalized eigenspace for K^* with $\bar{\lambda}$.

6. Bibliography

1. Ronald G. Douglas, Banach Algebra Techniques in Operator Theory Second Edition, 1998 Springer-Verlag New York, Inc.