Scalar Curvature and Local Spectral Invariants in Noncommutative Geometry

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From spectral geometry to noncommutative geometry

- One of the backbones of Alain Connes' program of NCG, specially its metric and differential geometric aspects, is Spectral Geometry and the Correspondence Principle which relates QM to CM. Both subjects have their roots in Planck's derivation of his celebrated Radiation Law and in Bohr-Sommerfeld Quantization Rules.

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$$\Omega^0(M) \longrightarrow \Omega^1(M) \longrightarrow \cdots \longrightarrow \Omega^n(M).$$

- Laplacian on forms

$$\triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

has pure point spectrum:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

Laplace spectrum

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- (M, g) and (M', g') are called isospectral if they have the same Laplace spectrum, counting multiplicities.
- Isometric manifolds are isospectral, but the converse is not always true (Milnor, Sunada, $\cdots$).
- Fact: isospectral manifolds have the same Dimension, volume, total scalar curvature, Betti numbers, and hence the same Euler characteristic.

First examples: flat tori and round spheres

- Flat tori: $M = \mathbb{R}^m/\Gamma, \quad \Gamma \subset \mathbb{R}^m$ a cocompact lattice;

$$\mathrm{spec}(\triangle) = \{4\pi^2||\gamma||^2; \ \ \gamma \in \Gamma^*\}$$
$$\varphi_\gamma(x) = e^{2\pi i\langle\gamma,x\rangle} \qquad \gamma \in \Gamma^*$$

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- Round sphere S^n. Eigenvalues

$$\bar{\lambda}_k = k(k+n-1) \qquad k = 0, 1, \cdots ,$$

with multiplicity $\binom{n+k}{k} - \binom{n+k-2}{k-2}$.

- In particular $\bar{\lambda}_1(S^n) = n$ with eigenfunctions

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- Eigenspace of $\bar{\lambda}_k$: Harmonic polynomials of degree k.

- Except for very few cases, no general formulas are known for eigenvalues.

Patterns in eigenvalues

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- But this is far from obviuos, and clues as to why such a statment should be true, and what C should be, first came from from spectroscopy, and in particular attemps to find the black body radiation formula.

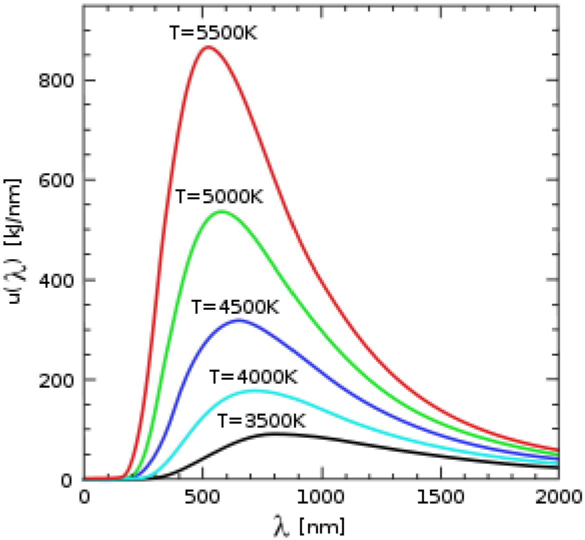

Figure: Black body spectrum

Planck's Radiation Law

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- Kirchhoff predicted: The total energy emitted by a black body is independent of the shape of the cavity and should only depend on its volume.
- Plnack's formula: $\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}$. But nobody was happy with his derivation, untill Bose gave a satisfactory derivation in 1924.

Conjecture of Lorentz and Sommerfeld

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- In 1910 H. A. Lorentz gave a series of lectures in Göttingen under the title "old and new problems of physics". Weyl and Hilbert were in attendance. In particular he mentioned attempts to drive Planck's radiation formula in a mathematically satisfactory way and remarked:

  'It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume. ....There is no doubt that it holds in general even for multiply connected spaces'.

- Hilbert was not very optimistic to see a solution in his lifetime. But Hermann Weyl settled this conjecture of Lorentz and Sommerfeld affirmatively within a year and announced a proof in 1911! All he needed was Hilbert's theory of integral equations and his spectral theorem for compact operators developed by Hilbert and his students in 1900-1910.

Figure: Hermann Weyl in Göttingen

Dirichlet eigenvalues and Weyl law

- Let $\Omega \subset \mathbb{R}^2$ be a compact connected domain with a piecewise smooth boundary.

$$\begin{cases} \Delta u = \lambda u \\ u|\partial\Omega = 0 \end{cases}$$

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

$$\langle u_i, u_j \rangle = \delta_{ij} \quad \text{o.n. basis for } L^2(\Omega)$$

- Eigenvalue Counting Function:

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

- Weyl Law for planar domains $\Omega \subset \mathbb{R}^2$:

$$N(\lambda) \sim \frac{\mathrm{Area}(\Omega)}{4\pi}\lambda, \qquad \lambda \to \infty$$

where $N(\lambda)$ is the eigenvalue counting function.

- In general, for $\Omega \subset \mathbb{R}^n$

$$N(\lambda) \sim \frac{\omega_n \mathrm{Vol}(\Omega)}{(2\pi)^n}\lambda^{\frac{n}{2}} \qquad \lambda \to \infty$$

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- But the ultimate question is: what else one can hear about the shape of a manifold, or the shape of a noncommutative space?

- It is now known that one can hear, among other things, the total scalar curvature, and, in many cases, lengths of closed geodesics (as in Selberg trace formula).

- Years later, in his Gibbs lecture to the American Mathematical Society (1950) Weyl said:

  "I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still very incomplete. I have certain conjectures of what a complete analysis of their asymptotic behaviour should aim at but, since for more than 35 years I have made no serious effort to prove them, I think I had better keep them to myself".

First impacts of Weyl's law: how to quantize

- Consider a classical system (X, h);
  $X =$ symplectic manifold, $h : X \to \mathbb{R}$, Hamiltonian.
  Assume
  $$\{x \in X;\ h(x) \le \lambda\}$$
  are compact for all λ (confined system).

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- Typical example: $X = T^*M$, (M, g) = compact Riemannian manifold, $h = T + V$.
  T= kinetic energy, V= potential energy.

- How to quantize this ?

$$(X, h) \rightsquigarrow (\mathcal{H}, H)$$

No one knows! No functor! But Dirac rules, geometric quantization, (strict) deformation quantization, etc. provide some ideas.

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  No one knows! No functor! But Dirac rules, geometric quantization, (strict) deformation quantization, etc. provide some ideas.

- Weyl's law imposes some constraints that are universally agreed on. This is an aspect of the celebrated correspondence principle:

- $H =$ Hamiltonian, with pure point spectrum

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

s.t.

$$N(\lambda) \sim c\,\text{Volume}\,(h \leq \lambda) \qquad \lambda \to \infty$$

- Thus: quantized energy levels are approximated by phase space volumes.

- Apply this to $X = T^*M$, (M, g) = compact Riemannian manifold, $h(q,p) = ||p||^2$; set

$$\mathcal{H} = L^2(M), \quad H = \Delta \quad \text{Laplacian}$$

obtain Weyl's Law:

$$N(\lambda) \sim c\,\mathrm{Vol}\,(\mathrm{M})\lambda^{m/2} \qquad (\lambda \to \infty)$$

Heat kernel asymptotics

- $(M, g) =$ compact Riemannian manifold

$$\Delta = d^*d : L^2(M) \to L^2(M), \qquad \text{Laplacian}$$

$$\Delta = -g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B$$

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- Weyl's Law:

$$N(\lambda) = \frac{\mathrm{Vol}\ (M)}{(4\pi)^{m/2}\Gamma(1 + m/2)}\lambda^{m/2} + \ \mathrm{o}(\lambda^{m/2})$$

Method of proof: bring in the heat kernel

- Heat equation for functions: $\partial_t + \triangle = 0$
- $k(t, x, y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, \triangle) + a_1(x, \triangle)t + a_2(x, \Delta)t^2 + \cdots).$$

- $a_i(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in curvature tensor R and its covariant derivatives:

$$\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}$$

Heat trace asymptotics

Compute $\mathrm{Trace}(e^{-t\triangle})$ in two ways:

$$\text{Spectral Sum} = \text{Geometric Sum}.$$

$$\sum e^{-t\lambda_i} = \int_M k(t,x,x)d\mathrm{vol}_x \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0).$$

Hence

$$a_j = \int_M a_j(x, \triangle) d\mathrm{vol}_x,$$

are manifestly spectral invariants:

$$\begin{aligned} a_0 &= \int_M d\mathrm{vol}_x = \mathrm{Vol}(M), & \Longrightarrow \text{Weyl's law} \\ a_1 &= \frac{1}{6}\int_M S(x) d\mathrm{vol}_x, & \text{total scalar curvature} \end{aligned}$$

Tauberian theory and $a_0 = 1$, implies Weyl's law:

$$N(\lambda) \sim \frac{\text{Vol }(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} \qquad \lambda \to \infty,$$

where

$$N(\lambda) = \#\{\lambda_i \le \lambda\}$$

is the eigenvalue counting function.

Simplest example: flat tori

- $\Gamma \subset \mathbb{R}^m$ a cocompact lattice; $M = \mathbb{R}^m/\Gamma$

$$\mathrm{spec}(\triangle) = \{4\pi^2 ||\gamma^*||^2;\ \ \gamma^* \in \Gamma^*\}$$

- Then:

$$K(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2 ||x-y+\gamma||^2/4t}$$

- Poisson summation formula $\Longrightarrow$

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 ||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2 ||\gamma||^2/4t}$$

- And from this we obtain the asymptotic expansion of the heat trace near $t = 0$

$$\mathrm{Tr} e^{-t\Delta} \sim \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \qquad (t \to 0)$$

Application 1: heat equation proof of the Atiyah-Singer index theorem

- Dirac operator

$$D : C^{\infty}(S_+) \to C^{\infty}(S_-)$$

McKean-Singer formula:

$$\mathrm{Index}(D) = \mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*}), \qquad \forall t > 0$$

Heat trace asymptotics $\Longrightarrow$

$$\mathrm{Index}(D) = \int_M a_n(x)dx,$$

where $a_n(x) = a_n^+(x) - a_n^-(x)$, $m = 2n$, can be explicitly computed and recovers the A-S integrand (The simplest proof is due to Getzler).

Application 2: meromorphic extension of spectral zeta functions

$$\zeta_\triangle(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

$$\begin{aligned} \zeta_\triangle(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle}) - \mathrm{Dim}\,\mathrm{Ker}\,\triangle) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \{ \int_0^c \cdots + \int_c^\infty \cdots \} \end{aligned}$$

The second term defines an entire function, while the first term has a meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Simplest example: For $M = S^1$ with round metric, we have

$$\zeta_\triangle(s) = 2\zeta(2s) \qquad \text{Riemann zeta function}$$

Scalar curvature

The spectral invariants a_i in the heat asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_{\triangle}(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x) dvol_x$, we obtain a formula for scalar curvature density as follows:

Let $\zeta_f(s) := \mathrm{Tr}\,(f\triangle^{-s}),\ f \in C^\infty(M)$.

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi}\int_M fS(x)dvol_x - \mathrm{Tr}(fP) \qquad m = 2$$

$\log\det(\triangle) = -\zeta'(0),$ Ray-Singer regularized determinant

Works on $\mathbb{T}^2_\theta$ and $\mathbb{T}^4_\theta$

A. Connes, P. Tretkoff, *The Gauss-Bonnet Theorem for the noncommutative two torus* (2009).

F. Fathizadeh, M. Khalkhali, *The Gauss-Bonnet Theorem for Noncommutative Two Tori With a General Conformal Structure* (2010), Journal of Noncommutative Geometry.

F. Fathizadeh, M. Khalkhali, *Scalar Curvature for the Noncommutative Two Torus* (2011), Journal of Noncommutative Geometry.

F. Fathizadeh, M. Khalkhali, *Weyl's Law and Connes' Trace Theorem for Noncommutative Two Tori* (2011), LMP.

A. Connes, H. Moscovici, *Modular Curvature for Noncommutative Two-Tori* (2011).

F. Fathizadeh, M. Wang, *Weyl's Law and Connes' Trace Theorem for Noncommutative Two Tori* (2011), LMP.

F. Fathizadeh, M. Khalkhali, *Scalar Curvature for Noncommutative Four-Tori*, Preprint, 2013.

L. Dabrowski and A. Sitarz, *Curved noncommutative torus and Gauss-Bonnet,* (2012).