From Spectral Geometry to Geometry of Noncommutative Spaces III

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Recall from last lecture

Friedmann-Lemaître-Robertson-Walker metric

- (Euclidean) FLRW metric with the scale factor $a(t)$:

$$ds^2 = dt^2 + a^2(t)\, d\sigma^2.$$

Where $d\sigma^2$ is the round metric on 3-sphere. It describes a homogeneous, isotropic (expanding or contracting) universe with spatially closed universe.

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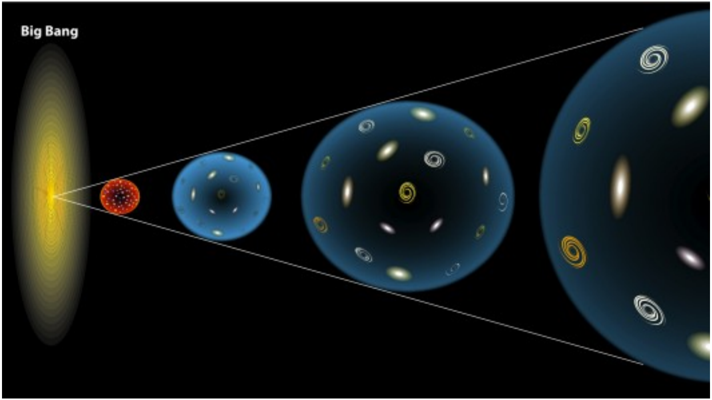
- For $a(t) = \sin(t)$ one obtains the round metric on S^4.

$$ds^2 = dt^2 + a^2(t)\left(d\chi^2 + \sin^2(\chi)\left(d\theta^2 + \sin^2(\theta)\, d\varphi^2\right)\right)$$

FLRW Metric



References

1. Chamseddine and Connes: Spectral Action for Robertson-Walker metrics (JHEP 2012)

2. Fathizadeh, Ghorbanpour, and Khalkhali: Rationality of Spectral Action for Robertson-Walker Metrics (JHEP 2014)

Dirac spectrum

- Spectrum of Dirac for round S^4 :

	eigenvalues	multiplicity
D	$\pm k$	$\frac{2}{3}(k^3 - k)$
D^2	k^2	$\frac{4}{3}(k^3 - k)$

- To find heat kernel coefficients of D^2 we apply the Euler Maclaurin formula for $a = 0$, $b = \infty$ and

$$g(x) = \frac{4}{3}(x^3 - x)f(x) = \frac{4}{3}(x^3 - x)e^{-tx^2}$$

The integral term gives

$$\int_a^b g(x)dx = \frac{4}{3}\int_0^\infty (x^3 - x)e^{-tx^2}dx = \frac{2}{3}(t^{-2} - t^{-1})$$

The term $\frac{g(a)+g(b)}{2}$ is zero since $g(0) = g(\infty) = 0$.
And

$$g^{(2m-1)}(0)/(2m-1)! = (-1)^m \frac{4}{3}\left(\frac{t^{m-2}}{(m-2)!} + \frac{t^{m-1}}{(m-1)!}\right)$$

Putting all these together we get

$$\frac{3}{4}\mathrm{Tr}(e^{-tD^2}) = \frac{1}{2t^2} - \frac{1}{2t} + \frac{11}{120} + \sum_{k=1}^{m}(-1)^k\left(\frac{B_{2k+2}}{2k+2} + \frac{B_{2k+4}}{2k+4}\right)\frac{t^k}{k!} + o(t^m)$$

Euler Maclaurin formula and spectral action for S^4

For general f the Euler Maclaurin formula gives

$$\frac{3}{4}\mathrm{Tr}(f(tD^2)) = \int_0^\infty f(tx^2)(x^3 - x)dx + \frac{11f(0)}{120} - \frac{31f'(0)}{2520}t$$

$$+\frac{41f''(0)}{10080}t^2 - \frac{31f^{(3)}(0)}{15840}t^3 + \frac{10331f^{(4)}(0)}{8648640}t^4 + \ldots + R_m$$

Levi-Civita Connection and the Spin Connection

Fix a frame $\{\theta_\alpha\}$ and coframe $\{\theta^\alpha\}$. Connection 1-forms

$$\nabla\theta^\alpha = \omega^\alpha_\beta \theta^\beta.$$

Metric connection:

$$\omega^\alpha_\beta = -\omega^\beta_\alpha.$$

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Cartan structure equations: curvature and torsion 2-forms:

$$\Omega = d\omega - \omega \wedge \omega$$

$$T^\alpha = d\theta^\alpha - \omega^\alpha_\beta \wedge \theta^\beta$$

For torsion free connections:

$$d\theta^\beta = \omega^\beta_\alpha \wedge \theta^\alpha.$$

Connection one-form for Levi-civita connection

Orthonormal basis for the cotangent space

$$\begin{aligned}
\theta^1 &= dt, \\
\theta^2 &= a(t)\, d\chi, \\
\theta^3 &= a(t)\, \sin\chi\, d\theta, \\
\theta^4 &= a(t)\, \sin\chi\, \sin\theta\, d\varphi.
\end{aligned}$$

The computation by Chamseddin-Connes shows that the connection one-form is given by

$$\omega = \begin{bmatrix} 0 & -\frac{a'(t)}{a(t)}\theta^2 & -\frac{a'(t)}{a(t)}\theta^3 & -\frac{a'(t)}{a(t)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^2 & 0 & -\frac{\cot(\chi)}{a(t)}\theta^3 & -\frac{\cot(\chi)}{a(t)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^3 & \frac{\cot(\chi)}{a(t)}\theta^3 & 0 & -\frac{\cot(\theta)}{a(t)\sin(\chi)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^4 & \frac{\cot(\chi)}{a(t)}\theta^4 & \frac{\cot(\theta)}{a\sin(\chi)}\theta^4 & 0 \end{bmatrix}$$

The Spin Connection

The spin connection is the lift of the Levi-Civita connection defined on T^*M. Now we have the connection one-forms ω, which is a skew symmetric matrix, i.e. $\omega \in \mathfrak{so}(4)$. Using the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \to \mathfrak{spin}(4)$ given by

$$A \mapsto \frac{1}{4} \sum_{\alpha,\beta} \langle A\theta^\alpha, \theta^\beta \rangle c(\theta^\alpha) c(\theta^\beta)$$

Since ω is written in the orthonormal basis θ^α so $\langle \omega\theta^\alpha, \theta^\beta \rangle = \omega^\alpha_\beta$. So the connection one forms for the spinor connection is given by

$$\tilde{\omega} = \frac{1}{2}\omega^1_2\gamma^{12} + \frac{1}{2}\omega^1_3\gamma^{13} + \frac{1}{2}\omega^1_4\gamma^{14} + \frac{1}{2}\omega^2_3\gamma^{23} + + \frac{1}{2}\omega^2_4\gamma^{24} + \frac{1}{2}\omega^3_4\gamma^{34}$$

Chamseddine-Connes Computations

They used Gilkey's local formulae to obtain the heat kernel coefficients

$$a_0 = \frac{a(t)^3}{2}$$

$$a_2 = \frac{1}{4}a(t)\left(a(t)a''(t) + a'(t)^2 - 1\right)$$

$$a_4 = \frac{1}{120}(3a^{(4)}(t)a(t)^2 + 3a(t)a''(t)^2 - 5a''(t) + 9a^{(3)}(t)a(t)a'(t) - 4a'(t)^2a''(t))$$

$$a_6 = \frac{1}{5040a(t)^2}(9a^{(6)}(t)a(t)^4 - 21a^{(4)}(t)a(t)^2 - 3a^{(3)}(t)^2a(t)^3 - 56a(t)^2a''(t)^3 + 42a(t)a''(t)^2 + 36a^{(5)}(t)a(t)^3a'(t) + 6a^{(4)}(t)a(t)^3a''(t) - 42a^{(4)}(t)a(t)^2a'(t)^2 + 60a^{(3)}(t)a(t)a'(t)^3 + 21a^{(3)}(t)a(t)a'(t) + 240a(t)a'(t)^2a''(t)^2 - 60a'(t)^4a''(t) - 21a'(t)^2a''(t) - 252a^{(3)}(t)a(t)^2a'(t)a''(t))$$

Chamseddine-Connes Computations

Using Euler-Maclaurin summation and Feynman-Kac formula they computed up to a_{10}:

$$a_8 = -\frac{1}{10080a(t)^4}(-a^{(8)}(t)a(t)^6 + 3a^{(6)}(t)a(t)^4 + 13a^{(4)}(t)^2a(t)^5 - 24a^{(3)}(t)^2a(t)^3 - 114a(t)^3a''(t)^4 + 43a(t)^2a''(t)^3 - 5a^{(7)}(t)a(t)^5a'(t) + 2a^{(6)}(t)a(t)^5a''(t) + 9a^{(6)}(t)a(t)^4a'(t)^2 + 16a^{(3)}(t)a^{(5)}(t)a(t)^5 - 24a^{(5)}(t)a(t)^3a'(t)^3 - 6a^{(5)}(t)a(t)^3a'(t) + 69a^{(4)}(t)a(t)^4a''(t)^2 - 36a^{(4)}(t)a(t)^3a''(t) + 60a^{(4)}(t)a(t)^2a'(t)^4 + 15a^{(4)}(t)a(t)^2a'(t)^2 + 90a^{(3)}(t)^2a(t)^4a''(t) - 216a^{(3)}(t)^2a(t)^3a'(t)^2 - 108a^{(3)}(t)a(t)a'(t)^5 - 27a^{(3)}(t)a(t)a'(t)^3 + 801a(t)^2a'(t)^2a''(t)^3 - 588a(t)a'(t)^4a''(t)^2 - 87a(t)a'(t)^2a''(t)^2 + 108a'(t)^6a''(t) + 27a'(t)^4a''(t) + 78a^{(5)}(t)a(t)^4a'(t)a''(t) + 132a^{(3)}(t)a^{(4)}(t)a(t)^4a'(t) - 312a^{(4)}(t)a(t)^3a'(t)^2a''(t) - 819a^{(3)}(t)a(t)^3a'(t)a''(t)^2 + 768a^{(3)}(t)a(t)^2a'(t)^3a''(t) + 102a^{(3)}(t)a(t)^2a'(t)a''(t))$$

$$
\begin{aligned}
a_{10} = {} & \frac{1}{665280a(t)^6}(3a^{(10)}(t)a(t)^8 - 222a^{(5)}(t)^2a(t)^7 - 348a^{(4)}(t)a^{(6)}(t)a(t)^7 - 147a^{(3)}(t)a^{(7)}(t)a(t)^7 - 18a''(t)a^{(8)}(t)a(t)^7 + \\
& 18a'(t)a^{(9)}(t)a(t)^7 - 482a''(t)a^{(4)}(t)^2a(t)^6 - 331a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 1110a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - \\
& 1556a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 448a''(t)^2a^{(6)}(t)a(t)^6 - 1074a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 476a'(t)a''(t)a^{(7)}(t)a(t)^6 - \\
& 43a'(t)^2a^{(8)}(t)a(t)^6 - 11a^{(8)}(t)a(t)^6 + 8943a'(t)a^{(3)}(t)^3a(t)^5 + 21846a''(t)^2a^{(3)}(t)^2a(t)^5 + 4092a'(t)^2a^{(4)}(t)^2a(t)^5 + \\
& 396a^{(4)}(t)^2a(t)^5 + 10560a''(t)^3a^{(4)}(t)a(t)^5 + 39402a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 11352a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + \\
& 6336a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 594a^{(3)}(t)a^{(5)}(t)a(t)^5 + 2904a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 264a''(t)a^{(6)}(t)a(t)^5 + \\
& 165a'(t)^3a^{(7)}(t)a(t)^5 + 33a'(t)a^{(7)}(t)a(t)^5 - 10338a''(t)^5a(t)^4 - 95919a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 3729a''(t)a^{(3)}(t)^2a(t)^4 - \\
& 117600a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 68664a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 2772a''(t)^2a^{(4)}(t)a(t)^4 - 23976a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - \\
& 2640a'(t)a^{(3)}(t)a^{(4)}(t)a(t)^4 - 12762a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 1386a'(t)a''(t)a^{(5)}(t)a(t)^4 - 651a'(t)^4a^{(6)}(t)a(t)^4 - \\
& 132a'(t)^2a^{(6)}(t)a(t)^4 + 111378a'(t)^2a''(t)^4a(t)^3 + 2354a''(t)^4a(t)^3 + 31344a'(t)^4a^{(3)}(t)^2a(t)^3 + 3729a'(t)^2a^{(3)}(t)^2a(t)^3 + \\
& 236706a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 13926a'(t)a''(t)^2a^{(3)}(t)a(t)^3 + 43320a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 5214a'(t)^2a''(t)a^{(4)}(t)a(t)^3 + \\
& 2238a'(t)^5a^{(5)}(t)a(t)^3 + 462a'(t)^3a^{(5)}(t)a(t)^3 - 162162a'(t)^4a''(t)^3a(t)^2 - 11880a'(t)^2a''(t)^3a(t)^2 - \\
& 103884a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 13332a'(t)^3a''(t)a^{(3)}(t)a(t)^2 - 6138a'(t)^6a^{(4)}(t)a(t)^2 - 1287a'(t)^4a^{(4)}(t)a(t)^2 + \\
& 76440a'(t)^6a''(t)^2a(t) + 10428a'(t)^4a''(t)^2a(t) + 11700a'(t)^7a^{(3)}(t)a(t) + 2475a'(t)^5a^{(3)}(t)a(t) - 11700a'(t)^8a''(t) - \\
& 2475a'(t)^6a''(t))
\end{aligned}
$$

Conjectures and question about coefficients (CC):

- Check the agreement between the above formulas for a_8 and a_{10} and the universal formulas.
- Show that the term a_{2n} of the asymptotic expansion of the spectral action for Robertson-Walker metric is of the form $P_n(a, \cdots, a^{(2n)})/a^{2n-4}$ where P_n is a polynomial with rational coefficients and compute P_n.

Our approach: spectral analysis via pseudodifferential calculus

$$
\begin{aligned}
D &= \gamma^\alpha \nabla_{\theta_\alpha} = \gamma^\alpha \left(\theta_\alpha + \omega(\theta_\alpha)\right) \\
&= \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{1}{a} \frac{\partial}{\partial \chi} + \gamma^2 \frac{1}{a \sin\chi} \frac{\partial}{\partial \theta} + \gamma^3 \frac{1}{a \sin\chi \sin\theta} \frac{\partial}{\partial \varphi} \\
&+ \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

So the symbol of the Dirac operator would be

$$
\begin{aligned}
\sigma_D(\mathbf{x}, \xi) \quad = i\gamma^0\xi_1 \quad &+ \frac{i}{a}\gamma^1\xi_2 + \frac{i}{a\sin(\chi)}\gamma^2\xi_3 + \frac{i}{a\sin(\chi)\sin(\theta)}\gamma^3\xi_4 \\
&+ \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

Symbol of D^2

Using the symbol multiplication rule one can compute the symbol of the square of the Dirac operator. The symbol of D^2 has following homogeneous parts.

$$p_2 = \xi_1^2 + \frac{1}{a(t)^2}\xi_2^2 + \frac{1}{a(t)^2\sin^2(\chi)}\xi_3^2 + \frac{1}{a(t)^2\sin^2(\theta)\sin^2(\chi)}\xi_4^2,$$

$$\begin{aligned} p_1 = &-\frac{3ia'(t)}{a(t)}\xi_1 - \frac{i}{a(t)^2}\left(\gamma^{12}a'(t) + 2\cot(\chi)\right)\xi_2 \\ &- \frac{i}{a(t)^2}\left(\gamma^{13}\csc(\chi)a'(t) + \cot(\theta)\csc^2(\chi) + \gamma^{23}\cot(\chi)\csc(\chi)\right)\xi_3 \\ &- \frac{i}{a(t)^2}\left(\csc(\theta)\csc(\chi)a'(t)\gamma^{14} + \cot(\theta)\csc(\theta)\csc^2(\chi)\gamma^{34} + \csc(\theta)\cot(\chi)\csc(\chi)\gamma^{24}\right)\xi_4, \end{aligned}$$

$$\begin{aligned} p_0 = &+\frac{1}{8a(t)^2}\Big(-12a(t)a''(t) - 6a'(t)^2 + 3\csc^2(\theta)\csc^2(\chi) - \cot^2(\theta)\csc^2(\chi) \\ &+4i\cot(\theta)\cot(\chi)\csc(\chi) - 4i\cot(\theta)\cot(\chi)\csc(\chi) - 4\cot^2(\chi) + 5\csc^2(\chi) + 4\Big) \\ &- \frac{\left(\cot(\theta)\csc(\chi)a'(t)\right)}{2a(t)^2}\gamma^{13} - \frac{\left(\cot(\chi)a'(t)\right)}{a(t)^2}\gamma^{12} - \frac{(\cot(\theta)\cot(\chi)\csc(\chi))}{2a(t)^2}\gamma^{23} \end{aligned}$$

Symbol of the parametrix

Parametrix: $(P-\lambda)\tilde{R}(\lambda) = I$.

$$\sigma(\tilde{R}(\lambda)) = r_0 + r_1 + r_2 + \cdots$$

Recursive formulas:

$$r_n = -r_0 \sum_{|\alpha|+j+2-k=n} (-i)^{|\alpha|} d_\xi^\alpha p_k \cdot d_x^\alpha r_j/\alpha!,$$

where $r_0 = (p_2-\lambda)^{-1} = (\|\xi\|^2-\lambda)^{-1}$. So the summation, for $n > 1$, will only have the following possible summands.

$k=0, \lvert\alpha\rvert=0, j=n-2$	$-r_0 p_0 r_{n-2}$
$k=1, \lvert\alpha\rvert=0, j=n-1$	$-r_0 p_1 r_{n-1}$
$k=1, \lvert\alpha\rvert=0, j=n-2$	$ir_0 \frac{\partial}{\partial \xi_0} p_1 \cdot \frac{\partial}{\partial t} r_{n-2} + ir_0 \frac{\partial}{\partial \xi_1} p_1 \cdot \frac{\partial}{\partial \chi} r_{n-2} + ir_0 \frac{\partial}{\partial \xi_2} p_1 \cdot \frac{\partial}{\partial \theta} r_{n-2}$
$k=2, \lvert\alpha\rvert=1, j=n-1$	$ir_0 \frac{\partial}{\partial \xi_0} p_2 \cdot \frac{\partial}{\partial t} r_{n-1} + ir_0 \frac{\partial}{\partial \xi_1} p_2 \cdot \frac{\partial}{\partial \chi} r_{n-1} + ir_0 \frac{\partial}{\partial \xi_2} p_2 \cdot \frac{\partial}{\partial \theta} r_{n-1}$
$k=2, \lvert\alpha\rvert=2, j=n-2$	$\frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_0^2} p_2 \cdot \frac{\partial^2}{\partial t^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_1^2} p_2 \cdot \frac{\partial^2}{\partial \chi^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_2^2} p_2 \cdot \frac{\partial^2}{\partial \theta^2} r_{n-2}$

Heat Kernel of D^2 in terms of symbols of the parametrix.

Let

$$e_n = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} r_n(x,\xi,\lambda) d\lambda d\xi$$
$$= \frac{1}{2\pi i (2\pi)^4} \sum r_{n,j,\alpha}(x) \int_{\mathbb{R}^4} \xi^{\alpha} \int_{\gamma} e^{-t\lambda} r_0^j d\lambda d\xi$$
$$= \sum c_\alpha \frac{1}{(j-1)!} r_{n,j,\alpha} a(t)^{\alpha_2+\alpha_3+\alpha_4+3} \sin(\chi)^{\alpha_3+\alpha_4+2} \sin(\theta)^{\alpha_4+1}$$

Where $c_\alpha = \frac{1}{(2\pi)^4} \prod_k \Gamma\left(\frac{\alpha_k+1}{2}\right) \frac{(-1)^{\alpha_k}+1}{2}$.

where

$$a_n = \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} e_n d\chi d\theta d\phi$$

New term a_{12}

$$
\begin{aligned}
a_{12} = {} & \frac{1}{17297280a(t)^8}\left(3a^{(12)}(t)a(t)^{10} - 1057a^{(6)}(t)^2a(t)^9 - 1747a^{(5)}(t)a^{(7)}(t)a(t)^9 - 970a^{(4)}(t)a^{(8)}(t)a(t)^9 - \right. \\
& 317a^{(3)}(t)a^{(9)}(t)a(t)^9 - 34a''(t)a^{(10)}(t)a(t)^9 + 21a'(t)a^{(11)}(t)a(t)^9 + 5001a^{(4)}(t)^3a(t)^8 + 2419a''(t)a^{(5)}(t)^2a(t)^8 + \\
& 19174a^{(3)}(t)a^{(4)}(t)a^{(5)}(t)a(t)^8 + 4086a^{(3)}(t)^2a^{(6)}(t)a(t)^8 + 2970a''(t)a^{(4)}(t)a^{(6)}(t)a(t)^8 - 5520a'(t)a^{(5)}(t)a^{(6)}(t)a(t)^8 - \\
& 511a''(t)a^{(3)}(t)a^{(7)}(t)a(t)^8 - 4175a'(t)a^{(4)}(t)a^{(7)}(t)a(t)^8 - 745a''(t)^2a^{(8)}(t)a(t)^8 - 2289a'(t)a^{(3)}(t)a^{(8)}(t)a(t)^8 - \\
& 828a'(t)a''(t)a^{(9)}(t)a(t)^8 - 62a'(t)^2a^{(10)}(t)a(t)^8 - 13a^{(10)}(t)a(t)^8 + 45480a^{(3)}(t)^4a(t)^7 + 152962a''(t)^2a^{(4)}(t)^2a(t)^7 + \\
& 203971a'(t)a^{(3)}(t)a^{(4)}(t)^2a(t)^7 + 21369a'(t)^2a^{(5)}(t)^2a(t)^7 + 1885a^{(5)}(t)^2a(t)^7 + 410230a''(t)a^{(3)}(t)^2a^{(4)}(t)a(t)^7 + \\
& 163832a'(t)a^{(3)}(t)^2a^{(5)}(t)a(t)^7 + 250584a''(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^7 + 244006a'(t)a''(t)a^{(4)}(t)a^{(5)}(t)a(t)^7 + \\
& 42440a''(t)^3a^{(6)}(t)a(t)^7 + 163390a'(t)a''(t)a^{(3)}(t)a^{(6)}(t)a(t)^7 + 35550a'(t)^2a^{(4)}(t)a^{(6)}(t)a(t)^7 + 3094a^{(4)}(t)a^{(6)}(t)a(t)^7 + \\
& 34351a'(t)a''(t)^2a^{(7)}(t)a(t)^7 + 19733a'(t)^2a^{(3)}(t)a^{(7)}(t)a(t)^7 + 1625a^{(3)}(t)a^{(7)}(t)a(t)^7 + 6784a'(t)^2a''(t)a^{(8)}(t)a(t)^7 + \\
& 520a''(t)a^{(8)}(t)a(t)^7 + 308a'(t)^3a^{(9)}(t)a(t)^7 + 52a'(t)a^{(9)}(t)a(t)^7 - 2056720a'(t)a''(t)a^{(3)}(t)^3a(t)^6 - \\
& 1790580a''(t)^3a^{(3)}(t)^2a(t)^6 - 900272a'(t)^2a''(t)a^{(4)}(t)^2a(t)^6 - 31889a''(t)a^{(4)}(t)^2a(t)^6 - 643407a''(t)^4a^{(4)}(t)a(t)^6 - \\
& 1251548a'(t)^2a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 43758a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 4452042a'(t)a''(t)^2a^{(3)}(t)a^{(4)}(t)a(t)^6 - \\
& 836214a'(t)a''(t)^3a^{(5)}(t)a(t)^6 - 1400104a'(t)^2a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 48620a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - \\
& 181966a'(t)^3a^{(4)}(t)a^{(5)}(t)a(t)^6 - 18018a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 319996a'(t)^2a''(t)^2a^{(6)}(t)a(t)^6 - 11011a''(t)^2a^{(6)}(t)a(t)^6 - \\
& 115062a'(t)^3a^{(3)}(t)a^{(6)}(t)a(t)^6 - 11154a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 42764a'(t)^3a''(t)a^{(7)}(t)a(t)^6 - 4004a'(t)a''(t)a^{(7)}(t)a(t)^6 - \\
& 1649a'(t)^4a^{(8)}(t)a(t)^6 - 286a'(t)^2a^{(8)}(t)a(t)^6 + 460769a''(t)^6a(t)^5 + 1661518a'(t)^3a^{(3)}(t)^3a(t)^5 + 83486a'(t)a^{(3)}(t)^3a(t)^5 + \\
& 13383328a'(t)^2a''(t)^2a^{(3)}(t)^2a(t)^5 + 222092a''(t)^2a^{(3)}(t)^2a(t)^5 + 342883a'(t)^4a^{(4)}(t)^2a(t)^5 + 36218a'(t)^2a^{(4)}(t)^2a(t)^5 + \\
& 7922361a'(t)a''(t)^4a^{(3)}(t)a(t)^5 + 6367314a'(t)^2a''(t)^3a^{(4)}(t)a(t)^5 + 109330a''(t)^3a^{(4)}(t)a(t)^5 +
\end{aligned}
$$

$$+7065862a'(t)^3a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 360386a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 1918386a'(t)^3a''(t)^2a^{(5)}(t)a(t)^5 + 98592a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + 524802a'(t)^4a^{(3)}(t)a^{(5)}(t)a(t)^5 + 55146a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 226014a'(t)^4a''(t)a^{(6)}(t)a(t)^5 + 23712a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 8283a'(t)^5a^{(7)}(t)a(t)^5 + 1482a'(t)^3a^{(7)}(t)a(t)^5 - 7346958a'(t)^2a''(t)^5a(t)^4 - 72761a''(t)^5a(t)^4 - 11745252a'(t)^4a''(t)a^{(3)}(t)^2a(t)^4 - 725712a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 27707028a'(t)^3a''(t)^3a^{(3)}(t)a(t)^4 - 819520a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 8247105a'(t)^4a''(t)^2a^{(4)}(t)a(t)^4 - 520260a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 1848228a'(t)^5a^{(3)}(t)a^{(4)}(t)a(t)^4 - 205296a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - 973482a'(t)^5a''(t)a^{(5)}(t)a(t)^4 - 110136a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 36723a'(t)^6a^{(6)}(t)a(t)^4 - 6747a'(t)^4a^{(6)}(t)a(t)^4 + 17816751a'(t)^4a''(t)^4a(t)^3 + 721058a'(t)^2a''(t)^4a(t)^3 + 2352624a'(t)^6a^{(3)}(t)^2a(t)^3 + 274170a'(t)^4a^{(3)}(t)^2a(t)^3 + 24583191a'(t)^5a''(t)^2a^{(3)}(t)a(t)^3 + 1771146a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 3256248a'(t)^6a''(t)a^{(4)}(t)a(t)^3 + 389376a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 135300a'(t)^7a^{(5)}(t)a(t)^3 + 25350a'(t)^5a^{(5)}(t)a(t)^3 - 15430357a'(t)^6a''(t)^3a(t)^2 - 1252745a'(t)^4a''(t)^3a(t)^2 - 7747848a'(t)^7a''(t)a^{(3)}(t)a(t)^2 - 967590a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 385200a'(t)^8a^{(4)}(t)a(t)^2 - 73125a'(t)^6a^{(4)}(t)a(t)^2 + 5645124a'(t)^8a''(t)^2a(t) + 741195a'(t)^6a''(t)^2a(t) + 749700a'(t)^9a^{(3)}(t)a(t) + 143325a'(t)^7a^{(3)}(t)a(t) - 749700a'(t)^{10}a''(t) - 143325a'(t)^8a''(t))\Big)$$

Check on round sphere S^4

For $a(t) = sin(t)$ we have

$$a_{12}(sphere) = \frac{10331 \sin^3(t)}{8648640}.$$

Hence

$$\int_0^{\pi} a_{12}(spher)dt = \frac{4}{3}\frac{10331}{8648640} = \frac{10331}{6486480}.$$

Which agrees with the direct computation done in Connes-Chamseddine.

Rationality of heat coefficients

Theorem (Fathizadeh, Ghorbanpour, K.) The terms a_{2n} in the expansion of the spectral action for the Robertson-Walker metric with scale factor $a(t)$ is of the form

$$\frac{1}{a(t)^{2n-3}}\, Q_{2n}\left(a(t), a'(t), \ldots, a^{(2n)}(t)\right),$$

where Q_{2n} is a polynomial with *rational* coefficients.

By direct computation in Hopf coordinates, we found the vector fields which respectively form bases for left and right invariant vector fields on $\mathrm{SU}(2)$:

$$
\begin{aligned}
X_1^L &= \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\
X_2^L &= \sin(\phi_1 + \phi_2)\frac{\partial}{\partial \eta} + \cot(\eta)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_1} - \\
&\quad \tan(\eta)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_2}, \\
X_3^L &= \cos(\phi_1 + \phi_2)\frac{\partial}{\partial \eta} - \cot(\eta)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_1} + \\
&\quad \tan(\eta)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_2},
\end{aligned}
$$

and X_1^R, X_2^R, X_3^R. One checks that these vector fields are Killing vector fields for the Robertson-Walker metrics on the four dimensional space.

Quillen's determinant line bundle for NC tori

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- Recall: regularized determinants. Given a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \qquad \text{spec}(\Delta)$$

How one defines $\prod \lambda_i = \det \Delta$?

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How one defines $\prod \lambda_i = \det \Delta$?

- Define the spectral zeta function:

$$\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \qquad Re(s) \gg 0$$

Assume: $\zeta_\Delta(s)$ has meromorphic extension to $\mathbb{C}$ and is regular at 0.

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- Zeta regularized determinant:

$$\prod \lambda_i := e^{-\zeta_\Delta'(0)} = \det\Delta$$

Holomorphic determinants?

- Example: For Riemann zeta function, $\zeta'(0) = -\log\sqrt{2\pi}$. Hence

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- Quillen's approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.

Space of Fredholm operators

- The Space of Fredholm operators is one of the gifts of operator algebra theory to geometry, topology, and physics:

$$F = \mathrm{Fred}(H_0, H_1) = \{\, T : H_0 \to H_1; \;\; T \text{ is Fredholm}\}$$

- Atiyah-Jänich: $K_0(X) = [X, F]$. So F is a classifying space for K-theory.

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The determinant line bundle

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- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.

$$(DET)_T = \lambda(KerT)^* \otimes \lambda(KerT^*)$$

  2) There map $\sigma : F_0 \to DET$

$$\sigma(T) = \begin{cases} 1 & T \quad invertible \\ 0 & otherwise \end{cases}$$

  is a holomorphic section of DET over F_0.

Cauchy-Riemann operators on $\mathcal{A}_\theta$

- Families of spectral triples

$$\mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, \quad \begin{pmatrix} 0 & \bar\partial^* + \alpha^* \\ \bar\partial + \alpha & 0 \end{pmatrix},$$

with $\alpha \in \mathcal{A}_\theta,\ \bar\partial = \delta_1 + \tau\delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar\partial + \alpha$.

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- Let $\mathcal{A} =$ space of elliptic operators $D = \bar\partial + \alpha$.

- Pull back DET to a holomorphic line bundle $\mathcal{L} \to \mathcal{A}$ with

$$\mathcal{L}_D = \lambda(KerD)^* \otimes \lambda(KerD^*).$$

From det section to det function

- If $\mathcal{L}$ admits a canonical global holomorphic frame s, then

$$\sigma(D) = det(D)s$$

defines a holomorphic determinant function $\det(D)$. A canonical frame is defined once we have a canonical flat holomorphic connection.

Quillen's metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $\mathit{Index}(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta, \quad \Delta = D^*D.$$

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

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- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

- A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from

$$\bar{\partial}\partial \log ||s||^2,$$

where s is any local holomorphic frame.

Connes' pseudodifferential calculus

- To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.
- Symbols of order m: smooth maps $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ with

$$||\delta^{(i_1,i_2)}\partial^{(j_1,j_2)}\sigma(\xi)|| \leq c(1+|\xi|)^{m-j_1-j_2}.$$

The space of symbols of order m is denoted by $\mathcal{S}^m(\mathcal{A}_\theta)$.

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  The space of symbols of order m is denoted by $\mathcal{S}^m(\mathcal{A}_\theta)$.

- To a symbol σ of order m, one associates an operator

$$P_\sigma(a) = \int\int e^{-is\cdot\xi}\sigma(\xi)\alpha_s(a)\,ds\,d\xi.$$

  The operator $P_\sigma : \mathcal{A}_\theta \to \mathcal{A}_\theta$ is said to be a pseudodifferential operator of order m.

Classical symbols

- Classical symbol of order $\alpha \in \mathbb{C}$:

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{\alpha-j} \qquad \text{ord}\, \sigma_{\alpha-j} = \alpha - j.$$

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.$$

- We denote the set of classical symbols of order α by $\mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$.

A cutoff integral

- Any pseudo P_σ of order < -2 is trace-class with

$$\mathrm{Tr}(P_\sigma) = \varphi_0\left(\int_{\mathbb{R}^2} \sigma(\xi) d\xi\right).$$

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- For $\mathrm{ord}(P) \geq -2$ the integral is divergent, but, assuming P is classical, and of non-integral order, one has an asymptotic expansion as $R \to \infty$

$$\int_{B(R)} \sigma(\xi) d\xi \sim \sum_{j=0, \alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma) \log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi =$ Wodzicki residue of P (Fathizadeh).

The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}_{cl}^{\alpha}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi) d\xi$.

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- The canonical trace of a classical pseudo $P \in \Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$ of non-integral order α is defined as

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- NC residue in terms of TR:

$$\mathrm{Res}_{z=0} \mathrm{TR}(AQ^{-z}) = \frac{1}{q} \mathrm{Res}(A).$$

Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j\geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^{l} |\xi| \quad |\xi| > 0,$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

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- Example: $\log Q$ where $Q \in \Psi^q_{cl}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$.

- Wodzicki residue: $\mathrm{Res}(A) = \varphi_0(\mathrm{res}(A))$,

$$\mathrm{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.$$

Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section σ,

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}$$

Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section σ,

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}$$

- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$. Want to compute

$$\bar{\partial}\partial \log \|\sigma\|^2 = \delta_{\bar{w}}\delta_w \zeta'_\Delta(0) = \delta_{\bar{w}}\delta_w \frac{d}{dz}\mathrm{TR}(\Delta^{-z})|_{z=0}.$$

The second variation of logDet

- Prop 1: For a holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ is given by :

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{2}\varphi_0\left(\delta_w D\delta_{\bar{w}}\mathrm{res}(\log\Delta\, D^{-1})\right).$$

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- Prop 2: The residue density of $\log\Delta\, D^{-1}$:

$$\sigma_{-2,0}(\log\Delta\, D^{-1}) = \frac{(\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2}{(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)(\xi_1+\tau\xi_2)}$$

$$-\log\left(\frac{\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2}{|\xi|^2}\right)\frac{\alpha}{\xi_1+\tau\xi_2},$$

and

$$\delta_{\bar{w}}\mathrm{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Curvature of the determinant line bundle

- Theorem (A. Fathi, A. Ghorbanpour, MK.): The curvature of the determinant line bundle for the noncommutative two torus is given by

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right).$$

- Remark: For $\theta = 0$ this reduces to Quillen's theorem (for elliptic curves).

A holomorphic determinant à la Quillen

- Modify the metric to get a flat connection:

$$||s||_f^2 = e^{||D-D_0||^2}||s||^2$$

A holomorphic determinant à la Quillen

- Modify the metric to get a flat connection:

$$||s||_f^2 = e^{||D-D_0||^2}||s||^2$$

- Get a flat holomorphic global section. This gives a holomorphic determinant function

$$det(D, D_0) : \mathcal{A} \to \mathbb{C}$$

It satisfies

$$|det(D, D_0)|^2 = e^{||D-D_0||^2} det_\zeta(D^* D)$$

Summary of my 3 lectures

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