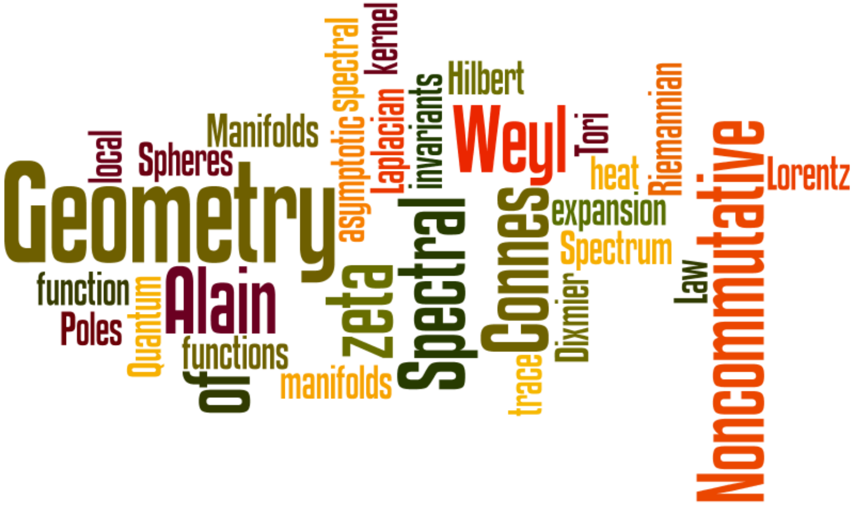
From Spectral Geometry to Geometry of Noncommutative Spaces II

Masoud Khalkhali

University of Western Ontario
(Joint work with Farzad Fathizadeh)

Workshop on the Geometry of Noncommutative Manifolds
Fields Inst. Toronto, March 2015

Recall from last lecture



Example: heat trace for flat tori

- $\Gamma \subset \mathbb{R}^m$ a cocompact lattice; $M = \mathbb{R}^m/\Gamma$

$$\mathrm{spec}(\triangle) = \{4\pi^2 ||\gamma^*||^2; \ \ \gamma^* \in \Gamma^*\}$$

- Then:

$$K(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2 ||x-y+\gamma||^2/4t}$$

- Poisson summation formula $\Longrightarrow$

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 ||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2 ||\gamma||^2 / 4t}$$

- And from this we obtain the asymptotic expansion of the heat trace near $t = 0$

$$\mathrm{Tr} e^{-t\Delta} \sim \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \qquad (t \to 0)$$

Quantum sphere ζ-function

- The spectral ζ-function of the quantum sphere is (Eckstein-Iochum-Sitarz)

$$\zeta_q(s) = 4(1-q^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{k!\Gamma(s)} \frac{q^{2k}}{(1-q^{s+2k})^2}$$

- All poles of $\zeta_q(s)$ are complex of the second order:

$$-2k + i\frac{2\pi}{\log q} m$$

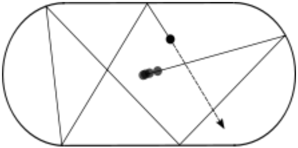
where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Heat trace and wave trace

- The heat trace $Z(t)$ is a smooth function. The wave trace

$$W(t) = \sum e^{it\sqrt{\lambda_k}},$$

is divergent for all t, but is a well defined distribution. In fact it contains more information than the heat trace. Its singular support contains the length spectrum of closed geodesics.



Closed geodesics are critical points of the energy functional and contribute most in path integral expression for wave trace.

Algebra of differential operators

- Given (M, E, ∇), let

$$D(M, E) \subset \mathrm{End}_{\mathbb{C}}(\Gamma(M, E)$$

be the subalgebra generated by $\Gamma(M, \mathrm{End}(E))$ and ∇_X for all vector fields X.

Algebra of differential operators

- Given (M, E, ∇), let

$$D(M, E) \subset \mathrm{End}_{\mathbb{C}}(\Gamma(M, E)$$

  be the subalgebra generated by $\Gamma(M, \mathrm{End}(E))$ and ∇_X for all vector fields X.

- $D(M, E)$ is indendent of the choice of connection ∇
- $D(M, E)$ is a filtered algebra-degree filteration.
- The associated graded algebra is
- This construction is purely algebraic and can be done for commutative algebras
- In local coordinates:

$$P = \sum f_I \partial_I, \qquad \partial_I = \partial_1^{i_1} \cdots \partial_m^{i_m}$$

Laplace type operators

- Fix a Riemannian metric g on M and a vector bundle V on M. An operator $P : \Gamma(M, V) \to \Gamma(M, V)$ is a Laplace type operator if in local coordinates it looks like

$$P = -g^{ij}\partial_i\partial_j + \text{lower orders}$$

- Lemma: P is Laplace type iff for all smooth sections f,

$$[[P, f], f] = -2|df|^2,$$

Examples of Laplace type operators

- Laplacian on forms

$$\Delta = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

- Dirac Laplacians $\Delta = D^* D$, where

$$D : \Gamma(S) \to \Gamma(S)$$

is a generalized Dirac operator.

Lemma: Let P be a Laplace type operator. Then there exists a unique connection ∇ on the vector bundle V and an endomorphism $E \in \mathrm{End}(V)$ such that

$$P = \nabla^* \nabla - E.$$

Here $\nabla^* \nabla$ is the connection Laplacian which is locally given by $-g^{ij} \nabla_i \nabla_j$.

Heat kernel asymtotics

- e^{-tP} is a smoothing operator with a smooth kernel $k(t,x,y)$ with

$$k(t,x,y) \in V_x \otimes V_y^*$$

$$e^{-tP} f(x) = \int_M k(t,x,y) f(y) dy.$$

- There is an asymptotic expansion near $t = 0$

$$k(t,x,x) \sim \frac{1}{(4\pi t)^{m/2}} (a_0(x,P) + a_1(x,P)t + a_2(x,P)t^2 + \cdots).$$

- $a_i(x,P)$, Seeley-De Witt-Gilkey coefficients.

$$
\begin{aligned}
a_0(x,P) &= \mathrm{tr}(\mathrm{Id}),\\
a_2(x,P) &= \mathrm{tr}\big(E - \frac{1}{6}R\,\mathrm{Id}\big),\\
a_4(x,P) &= \frac{1}{360}\mathrm{tr}\Big(\big(-12R_{;kk} + 5R^2 - 2R_{jk}R_{jk} + 2R_{ijkl}R_{ijkl}\big)\mathrm{Id}\\
&\qquad -60RE + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\Big).
\end{aligned}
$$

$$
\begin{aligned}
a_6(x,P) = \mathrm{tr}\Big\{ \frac{1}{7!}\Big(& -18R_{;kkll} + 17R_{;k}R_{;k} - 2R_{jk;l}R_{jk;l} - 4R_{jk;l}R_{jl;k} \\
& + 9R_{ijku;l}R_{ijku;l} + 28RR_{;ll} - 8R_{jk}R_{jk;ll} + 24R_{jk}R_{jl;kl} \\
& + 12R_{ijkl}R_{ijkl;uu}\Big)\mathrm{Id} \\
+ \frac{1}{9\cdot 7!}\Big(& -35R^3 + 42RR_{lp}R_{lp} - 42RR_{klpq}R_{klpq} + 208R_{jk}R_{jl}R_{kl} \\
& -192R_{jk}R_{ul}R_{jukl} + 48R_{jk}R_{julp}R_{kulp} - 44R_{ijku}R_{ijlp}R_{kulp} \\
& -80R_{ijku}R_{ilkp}R_{jlup}\Big)\mathrm{Id} \\
+ \frac{1}{360}\Big(& 8\Omega_{ij;k}\Omega_{ij;k} + 2\Omega_{ij;j}\Omega_{ik;k} + 12\Omega_{ij}\Omega_{ij;kk} - 12\Omega_{ij}\Omega_{jk}\Omega_{ki} \\
& -6R_{ijkl}\Omega_{ij}\Omega_{kl} + 4R_{jk}\Omega_{jl}\Omega_{kl} - 5R\Omega_{kl}\Omega_{kl}\Big) \\
+ \frac{1}{360}\Big(& 6E_{;iijj} + 60EE_{;ii} + 30E_{;i}E_{;i} + 60E^3 + 30E\Omega_{ij}\Omega_{ij} - 10RE_{;kk} \\
& -4R_{jk}E_{;jk} - 12R_{;k}E_{;k} - 30RE^2 - 12R_{;kk}E + 5R^2E \\
& -2R_{jk}R_{jk}E + 2R_{ijkl}R_{ijkl}E\Big)\Big\}.
\end{aligned}
$$

Note: all tensors are written in normal coordinates passing through the base point x, and Ω is the curvature two form of the connection.

Scalar curvature

The spectral invariants a_i in the heat asymptotic expansion

$$\text{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\text{Res}_{s=\alpha}\zeta_{\triangle}(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x) dvol_x$, we obtain a formula for scalar curvature density as follows:

Let $\zeta_f(s) := \mathrm{Tr}\,(f\triangle^{-s}),\ f \in C^\infty(M)$.

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)} \int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi} \int_M fS(x)dvol_x - \mathrm{Tr}(fP) \qquad m = 2$$

$$\log\det(\triangle) = -\zeta'(0), \qquad \text{Ray-Singer regularized determinant}$$

Curved noncommutative tori A_θ

$A_\theta = C(\mathbb{T}^2_\theta) =$ universal C^*-algebra generated by unitaries U and V

$$VU = e^{2\pi i\theta} UV.$$

$$A_\theta^\infty = C^\infty(\mathbb{T}^2_\theta) = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n : \; (a_{m,n}) \, \text{Schwartz class} \Big\}.$$

- Differential operators $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V$$

- Integration $\varphi_0 : A_\theta \to \mathbb{C}$ on smooth elements:

$$\varphi_0(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = a_{0,0}.$$

- Complex structures: Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0$. Dolbeault operators

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

Conformal perturbation of the metric (Connes-Tretkoff)

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form φ_0 by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \varphi_0(ae^{-h}).$$

- It is a twisted trace (KMS state):

$$\varphi(ab) = \varphi(b\Delta(a)),$$

where

$$\Delta(x) = e^{-h}xe^{h}.$$

Perturbed Dolbeault operator

- Hilbert space $\mathcal{H}_\varphi = L^2(A_\theta, \varphi), \quad GNS$ construction.

- Let $\partial_\varphi = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$,

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi.$$

and $\triangle = \partial_\varphi^* \partial_\varphi$, perturbed non-flat Laplacian.

Scalar curvature for A_θ

- Gilkey-De Witt-Seeley formulae in spectral geometry motivates the following definition:

  The scalar curvature of the curved nc torus (A_θ, τ, h) is the unique element $R \in A_\theta^\infty$ satisfying

  $$\mathrm{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \mathrm{Trace}\,(aP) = \varphi_0\,(aR), \qquad \forall a \in A_\theta^\infty,$$

  where P is the projection onto the kernel of $\triangle$.

Scalar curvature for A_θ

- Gilkey-De Witt-Seeley formulae in spectral geometry motivates the following definition:

  The scalar curvature of the curved nc torus (A_θ, τ, h) is the unique element $R \in A_\theta^\infty$ satisfying

  $$\mathrm{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \mathrm{Trace}\,(aP) = \varphi_0\,(aR), \qquad \forall a \in A_\theta^\infty,$$

  where P is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansion for the kernel of the operator $e^{-t\triangle}$, using Connes' pseudodifferential calculus for nc tori.

Final formula for the scalar curvature (Connes-Moscovici; Fathizadeh-K

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\Big(\triangle_0(\log k)\Big)+$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2+|\tau|^2\delta_2(\log k)^2+\tau_1\big\{\delta_1(\log k), \delta_2(\log k)\big\}\Big)+$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2\big[\delta_1(\log k), \delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1+\cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s+\sinh t-\sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

What remains to be done

- Define new curved NC spaces and extend these spectral computations to them. Compare algebraic and analytic approaches. They match in the commutative case (Gilkey's theorem), but this is far from obvious in the NC case.

- Other curvature related work: Rosenberg, Marcolli-Buhyan, Dabrowski-Sitarz, Fathi, Ghorbanpour, Moatadelro, and Arnlind-Wilson.

- Recently Fathizadeh has simplified the four dimensional calculations and its Einstein-Hilbert action. He will report on his work in this conference.

Rationality of Spectral Action for Robertson-Walker Metrics

Masoud Khalkhali
(Joint work with F. Fathizadeh and A. Ghorbanpour)

The spectral action principle of Connes and Chamseddine

Some relevant references:

- A. Connes, *Noncommutative Geometry*, 1994.
- A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Comm. Math. Phys. 182 (1996) 155-176.
- A. H. Chamseddine, A. Connes, *The spectral action principle,* Comm. Math. Phys. 186 (1997), no. 3, 731–750.
- A. H. Chamseddine, A. Connes, *Spectral action for Robertson-Walker metrics,* J. High Energy Phys. 2012, no. 10, 101.

Classical action

- Classical Lagrangian action

$$S = \int_M \mathcal{L}(\varphi, \partial_\mu \varphi) d^n x$$

Classical action

- Classical Lagrangian action

$$S = \int_M \mathcal{L}(\varphi, \partial_\mu \varphi) d^n x$$

- Classical equations of motion

$$\delta S = 0,$$

Classical action

- Classical Lagrangian action

$$S = \int_M \mathcal{L}(\varphi, \partial_\mu \varphi) d^n x$$

- Classical equations of motion

$$\delta S = 0,$$

- Quantum expectation values

$$\langle \mathcal{O} \rangle = \int D[\varphi] \, \mathcal{O}(\varphi) \, e^{\frac{i}{\hbar} S}$$

Spectral action of Connes-Chamseddine

- Replace the classical action $S = \int_M \mathcal{L}(\varphi, \partial_\mu \varphi) d^n x$ by the spectral action
$$S = \text{Trace}(f(D/\Lambda)),$$
where D is a Dirac operator, f is a positive even function, and the cutoff Λ is the mass scale.

- *S only depends on the spectrum of D and moments of the cutoff,* $f_k = \int_0^\infty f(v) v^{k-1} dv$.

- Results from spectral geometry (Gilkey's formulae for heat trace asymptotics) can be used to show that one indeed recovers the classical action from the spectral action (more on this later).

- Results from spectral geometry (Gilkey's formulae for heat trace asymptotics) can be used to show that one indeed recovers the classical action from the spectral action (more on this later).
- Spectral action is manifestly quantum mechanical and one does not need a geometric background to write it down.

- Results from spectral geometry (Gilkey's formulae for heat trace asymptotics) can be used to show that one indeed recovers the classical action from the spectral action (more on this later).
- Spectral action is manifestly quantum mechanical and one does not need a geometric background to write it down.
- Spectral action makes perfect sense for spectral triples.

Hard calculations made easy

- Compute

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}, \qquad t = 0.01.$$

You have to add 21 terms to get it to one decimal digit.

Hard calculations made easy

- Compute
$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \qquad t = 0.01.$$
You have to add 21 terms to get it to one decimal digit.

- But: you can compute it to 130 decimal digits without any calculation!

Modular equation is the key

- Modular equation (Jacobi, 1828)

$$\theta(t) = \frac{1}{\sqrt{t}}\,\theta(\frac{1}{t}).$$

Modular equation is the key

- Modular equation (Jacobi, 1828)

$$\theta(t) = \frac{1}{\sqrt{t}}\,\theta(\frac{1}{t}).$$

- In particular

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} + O(e^{-\frac{1}{t}}) \qquad (t \to 0)$$

$$\theta(0.01) = 10.000000000000000000\cdots$$

Modular equation is the key

- Modular equation (Jacobi, 1828)

$$\theta(t) = \frac{1}{\sqrt{t}}\,\theta(\frac{1}{t}).$$

- In particular

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} + O(e^{-\frac{1}{t}}) \qquad (t \to 0)$$

$$\theta(0.01) = 10.000000000000000000\cdots$$

- So: Jacobi computed the first heat trace asymptotic expansion and in fact the first trace formula.

Poisson summation formula

- Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} \delta_n(x) = \sum_{n\in\mathbb{Z}} e^{2\pi inx}$$

Poisson summation formula

- Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} \delta_n(x) = \sum_{n\in\mathbb{Z}} e^{2\pi inx}$$

- Application: for any lattice $\Gamma \subset \mathbb{R}^n$:

$$\sum_{\gamma^*\in\Gamma^*} e^{-4\pi^2||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{d/2}} \sum_{\gamma\in\Gamma} e^{-||\gamma||^2/4t}$$

Zeta Values

- To evaluate
$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
to 6 decimal places you need to add 1000,000 terms!

Zeta Values

- To evaluate
$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
to 6 decimal places you need to add 1000,000 terms!

- Euler: $\zeta(2) = 1.644944$

Zeta Values

- To evaluate
$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
to 6 decimal places you need to add 1000,000 terms!
- Euler: $\zeta(2) = 1.644944$
$$\zeta(2) = (\log 2)^2 + 2\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

Zeta Values

- To evaluate
$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
to 6 decimal places you need to add 1000,000 terms!

- Euler: $\zeta(2) = 1.644944$
$$\zeta(2) = (\log 2)^2 + 2\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

- Dilogarithm function $Li_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}$
$$Li_2(x) + Li_2(1-x) + \log x \log(1-x) = Li_2(1)$$

Zeta values

- Euler computed zeta values $\zeta(2), \zeta(3), \cdots \zeta(23)$ with at least 15 decimal digits! How? Dilogarithm identities are not useful for finding

$$\zeta(3) = 1,202056903159594285399 7\cdots$$

Zeta values

- Euler computed zeta values $\zeta(2), \zeta(3), \cdots \zeta(23)$ with at least 15 decimal digits! How? Dilogarithm identities are not useful for finding

$$\zeta(3) = 1,202056903159594285399 7 \cdots$$

- Euler-Maclaurin Summation Formula:

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \sum_{j=2}^{m} \frac{B_j}{j!}(g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m$$

Friedmann-Lemaître-Robertson-Walker metric

- (Euclidean) FLRW metric with the scale factor $a(t)$:

$$ds^2 = dt^2 + a^2\,(t)\,d\sigma^2.$$

Where $d\sigma^2$ is the round metric on 3-sphere. It describes a homogeneous, isotropic (expanding or contracting) universe with spatially closed universe.

Friedmann-Lemaître-Robertson-Walker metric

- (Euclidean) FLRW metric with the scale factor $a(t)$:

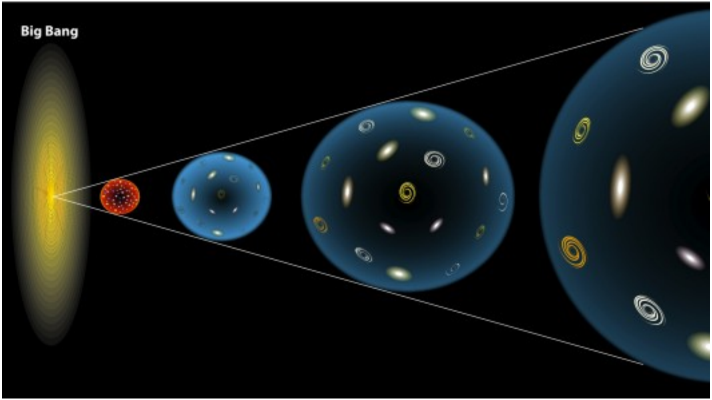
$$ds^2 = dt^2 + a^2(t)\,d\sigma^2.$$

Where $d\sigma^2$ is the round metric on 3-sphere. It describes a homogeneous, isotropic (expanding or contracting) universe with spatially closed universe.

- For $a(t) = \sin(t)$ one obtains the round metric on S^4.

$$ds^2 = dt^2 + a^2(t)\left(d\chi^2 + \sin^2(\chi)\left(d\theta^2 + \sin^2(\theta)\,d\varphi^2\right)\right)$$

FLRW Metric



References

1. Chamseddine and Connes: Spectral Action for Robertson-Walker metrics (2012)

2. Fathizadeh, Ghorbanpour, and Khalkhali: Rationality of Spectral Action for Robertson-Walker Metrics (2014)

Euler Maclaurin formula and Heat kernel for S^4

Euler Maclaurin formula

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \sum_{j=2}^{m} \frac{B_j}{j!}(g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m$$

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

Dirac spectrum

- Spectrum of Dirac for round S^4 :

	eigenvalues	multiplicity
D	$\pm k$	$\frac{2}{3}(k^3 - k)$
D^2	k^2	$\frac{4}{3}(k^3 - k)$

- To find heat kernel coefficients of D^2 we apply the Euler Maclaurin formula for $a = 0$, $b = \infty$ and

$$g(x) = \frac{4}{3}(x^3 - x)f(x) = \frac{4}{3}(x^3 - x)e^{-tx^2}$$

The integral term gives

$$\int_a^b g(x)dx = \frac{4}{3}\int_0^\infty (x^3 - x)e^{-tx^2}dx = \frac{2}{3}(t^{-2} - t^{-1})$$

The term $\frac{g(a)+g(b)}{2}$ is zero since $g(0) = g(\infty) = 0$.
And

$$g^{(2m-1)}(0)/(2m-1)! = (-1)^m \frac{4}{3}\left(\frac{t^{m-2}}{(m-2)!} + \frac{t^{m-1}}{(m-1)!}\right)$$

Putting all these together we get

$$\frac{3}{4}\mathrm{Tr}(e^{-tD^2}) = \frac{1}{2t^2} - \frac{1}{2t} + \frac{11}{120} + \sum_{k=1}^{m}(-1)^k\left(\frac{B_{2k+2}}{2k+2} + \frac{B_{2k+4}}{2k+4}\right)\frac{t^k}{k!} + o(t^m)$$

Euler Maclaurin formula and spectral action for S^4

For general f the Euler Maclaurin formula gives

$$\frac{3}{4}\mathrm{Tr}(f(tD^2)) = \int_0^\infty f(tx^2)(x^3 - x)dx + \frac{11f(0)}{120} - \frac{31f'(0)}{2520}t$$

$$+\frac{41f''(0)}{10080}t^2 - \frac{31f^{(3)}(0)}{15840}t^3 + \frac{10331f^{(4)}(0)}{8648640}t^4 + \ldots + R_m$$

Levi-Civita Connection and the Spin Connection

Fix a frame $\{\theta_\alpha\}$ and coframe $\{\theta^\alpha\}$. Connection 1-forms

$$\nabla\theta^\alpha = \omega^\alpha_\beta \theta^\beta.$$

Metric connection:

$$\omega^\alpha_\beta = -\omega^\beta_\alpha.$$

Cartan equations: torsion and curvature 2-forms

$$T^\alpha = d\theta^\alpha - \omega^\alpha_\beta \wedge \theta^\beta$$

For torsion free connections:

$$d\theta^\beta = \omega^\beta_\alpha \wedge \theta^\alpha.$$

Connection one-form for Levi-civita connection

Orthonormal basis for the cotangent space

$$\begin{aligned}
\theta^1 &= dt, \\
\theta^2 &= a(t)\, d\chi, \\
\theta^3 &= a(t)\, \sin\chi\, d\theta, \\
\theta^4 &= a(t)\, \sin\chi\, \sin\theta\, d\varphi.
\end{aligned}$$

The computation by Chamseddin-Connes shows that the connection one-form is given by

$$\omega = \begin{bmatrix}
0 & -\frac{a'(t)}{a(t)}\theta^2 & -\frac{a'(t)}{a(t)}\theta^3 & -\frac{a'(t)}{a(t)}\theta^4 \\
\frac{a'(t)}{a(t)}\theta^2 & 0 & -\frac{\cot(\chi)}{a(t)}\theta^3 & -\frac{\cot(\chi)}{a(t)}\theta^4 \\
\frac{a'(t)}{a(t)}\theta^3 & \frac{\cot(\chi)}{a(t)}\theta^3 & 0 & -\frac{\cot(\theta)}{a(t)\sin(\chi)}\theta^4 \\
\frac{a'(t)}{a(t)}\theta^4 & \frac{\cot(\chi)}{a(t)}\theta^4 & \frac{\cot(\theta)}{a\sin(\chi)}\theta^4 & 0
\end{bmatrix}$$

The Spin Connection

The spin connection is the lift of the Levi-Civita connection defined on T^*M. Now we have the connection one-forms ω, which is a skew symmetric matrix, i.e. $\omega \in \mathfrak{so}(4)$. Using the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \to \mathfrak{spin}(4)$ given by

$$A \mapsto \frac{1}{4} \sum_{\alpha,\beta} \langle A\theta^\alpha, \theta^\beta \rangle c(\theta^\alpha) c(\theta^\beta)$$

Since ω is written in the orthonormal basis θ^α so $\langle \omega\theta^\alpha, \theta^\beta \rangle = \omega^\alpha_\beta$. So the connection one forms for the spinor connection is given by

$$\tilde{\omega} = \frac{1}{2}\omega^1_2\gamma^{12} + \frac{1}{2}\omega^1_3\gamma^{13} + \frac{1}{2}\omega^1_4\gamma^{14} + \frac{1}{2}\omega^2_3\gamma^{23} + + \frac{1}{2}\omega^2_4\gamma^{24} + \frac{1}{2}\omega^3_4\gamma^{34}$$

Gilkey's local formulae

For an operator of Laplace type $P = \nabla^*\nabla - E$,

$$a_0 = (4\pi)^{-m/2}\,\mathrm{Tr}(1).$$

$$a_2 = (4\pi)^{-m/2}\,\mathrm{Tr}\big(E - \frac{1}{6}R\big).$$

$$\begin{aligned} a_4 \quad = \quad & \frac{(4\pi)^{-m/2}}{360}\mathrm{Tr}\big(-12R_{ijij;kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} \\ & +2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\big). \end{aligned}$$

$$
\begin{aligned}
a_6 \quad = \quad & (4\pi)^{-m/2}\, \mathrm{Tr}\Big(\frac{1}{7!} \big(-18R_{ijij;kkll} + 17R_{ijij;k}R_{ulul;k} - 2R_{ijik;l}R_{ujuk;l} \\
& \qquad -4R_{ijik;l}R_{ujul;k} + 9R_{ijku;l}R_{ijku;l} + 28R_{ijij}R_{kuku;ll} \\
& \qquad -8R_{ijijk}R_{ujuk;ll} + 24R_{ijik}R_{ujul;kl} + 12R_{ijkl}R_{ijkl;uu} \big) \\
& + \frac{1}{9\cdot 7!} \big(-35R_{ijij}R_{klkl}R_{pqpq} + 42R_{ijij}R_{klkp}R_{qlqp} \\
& -42R_{ijij}R_{klpq}R_{klpq} + 208R_{ijik}R_{julu}R_{kplp} - 192R_{ijik}R_{uplp}R_{jukl} \\
& +48R_{ijik}R_{julp}R_{kulp} - 44R_{ijku}R_{ijlp}R_{kulp} - 80R_{ijku}R_{ilkp}R_{jlup} \big) \\
& \quad + \frac{1}{360} \big(8\Omega_{ij;k}\Omega_{ij;k} + 2\Omega_{ij;j}\Omega_{ik;k} + 12\Omega_{ij}\Omega_{ij;kk} - 12\Omega_{ij}\Omega_{jk}\Omega_{ki} \\
& \quad -6R_{ijkl}\Omega_{ij}\Omega_{kl} + 4R_{ijik}\Omega_{jl}\Omega_{kl} - 5R_{ijij}\Omega_{kl}\Omega_{kl} \big) \\
& + \frac{1}{360} \big(6E_{;iijj} + 60EE_{;ii} + 30E_{;i}E_{;i} + 60E^3 + 30E\Omega_{ij}\Omega_{ij} \\
& -10R_{ijij}E_{;kk} - 4R_{ijik}E_{;jk} - 12R_{ijij;k}E_{;k} - 30R_{ijij}E^2 \\
& -12R_{ijij;kk}E + 5R_{ijij}R_{klkl}E - 2RijikR_{ijkl}E + 2R_{ijkl}R_{ijkl}E \big) \Big).
\end{aligned}
$$

For the Dirac operator $D^2 = \nabla^* \nabla - \frac{1}{4} R$, so

$$
E = \frac{1}{4} R.
$$

Chamseddine-Connes Computations

They used Gilkey's local formulae to obtain

$$a_0 = \frac{a(t)^3}{2}$$

$$a_2 = \frac{1}{4}a(t)\left(a(t)a''(t) + a'(t)^2 - 1\right)$$

$$a_4 = \frac{1}{120}(3a^{(4)}(t)a(t)^2 + 3a(t)a''(t)^2 - 5a''(t) + 9a^{(3)}(t)a(t)a'(t) - 4a'(t)^2a''(t))$$

$$a_6 = \frac{1}{5040a(t)^2}(9a^{(6)}(t)a(t)^4 - 21a^{(4)}(t)a(t)^2 - 3a^{(3)}(t)^2a(t)^3 - 56a(t)^2a''(t)^3 + 42a(t)a''(t)^2 + 36a^{(5)}(t)a(t)^3a'(t) + 6a^{(4)}(t)a(t)^3a''(t) - 42a^{(4)}(t)a(t)^2a'(t)^2 + 60a^{(3)}(t)a(t)a'(t)^3 + 21a^{(3)}(t)a(t)a'(t) + 240a(t)a'(t)^2a''(t)^2 - 60a'(t)^4a''(t) - 21a'(t)^2a''(t) - 252a^{(3)}(t)a(t)^2a'(t)a''(t))$$

Chamseddine-Connes Computations

Using Euler-Maclaurin summation and Feynman-Kac formula they computed up to a_{10}:

$$a_8 = -\frac{1}{10080a(t)^4}(-a^{(8)}(t)a(t)^6 + 3a^{(6)}(t)a(t)^4 + 13a^{(4)}(t)^2a(t)^5 - 24a^{(3)}(t)^2a(t)^3 - 114a(t)^3a''(t)^4 + 43a(t)^2a''(t)^3 - 5a^{(7)}(t)a(t)^5a'(t) + 2a^{(6)}(t)a(t)^5a''(t) + 9a^{(6)}(t)a(t)^4a'(t)^2 + 16a^{(3)}(t)a^{(5)}(t)a(t)^5 - 24a^{(5)}(t)a(t)^3a'(t)^3 - 6a^{(5)}(t)a(t)^3a'(t) + 69a^{(4)}(t)a(t)^4a''(t)^2 - 36a^{(4)}(t)a(t)^3a''(t) + 60a^{(4)}(t)a(t)^2a'(t)^4 + 15a^{(4)}(t)a(t)^2a'(t)^2 + 90a^{(3)}(t)^2a(t)^4a''(t) - 216a^{(3)}(t)^2a(t)^3a'(t)^2 - 108a^{(3)}(t)a(t)a'(t)^5 - 27a^{(3)}(t)a(t)a'(t)^3 + 801a(t)^2a'(t)^2a''(t)^3 - 588a(t)a'(t)^4a''(t)^2 - 87a(t)a'(t)^2a''(t)^2 + 108a'(t)^6a''(t) + 27a'(t)^4a''(t) + 78a^{(5)}(t)a(t)^4a'(t)a''(t) + 132a^{(3)}(t)a^{(4)}(t)a(t)^4a'(t) - 312a^{(4)}(t)a(t)^3a'(t)^2a''(t) - 819a^{(3)}(t)a(t)^3a'(t)a''(t)^2 + 768a^{(3)}(t)a(t)^2a'(t)^3a''(t) + 102a^{(3)}(t)a(t)^2a'(t)a''(t))$$

$$
\begin{aligned}
a_{10} = {} & \frac{1}{665280a(t)^6}\left(3a^{(10)}(t)a(t)^8 - 222a^{(5)}(t)^2a(t)^7 - 348a^{(4)}(t)a^{(6)}(t)a(t)^7 - 147a^{(3)}(t)a^{(7)}(t)a(t)^7 - 18a''(t)a^{(8)}(t)a(t)^7 + \right. \\
& 18a'(t)a^{(9)}(t)a(t)^7 - 482a''(t)a^{(4)}(t)^2a(t)^6 - 331a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 1110a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - \\
& 1556a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 448a''(t)^2a^{(6)}(t)a(t)^6 - 1074a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 476a'(t)a''(t)a^{(7)}(t)a(t)^6 - \\
& 43a'(t)^2a^{(8)}(t)a(t)^6 - 11a^{(8)}(t)a(t)^6 + 8943a'(t)a^{(3)}(t)^3a(t)^5 + 21846a''(t)^2a^{(3)}(t)^2a(t)^5 + 4092a'(t)^2a^{(4)}(t)^2a(t)^5 + \\
& 396a^{(4)}(t)^2a(t)^5 + 10560a''(t)^3a^{(4)}(t)a(t)^5 + 39402a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 11352a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + \\
& 6336a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 594a^{(3)}(t)a^{(5)}(t)a(t)^5 + 2904a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 264a''(t)a^{(6)}(t)a(t)^5 + \\
& 165a'(t)^3a^{(7)}(t)a(t)^5 + 33a'(t)a^{(7)}(t)a(t)^5 - 10338a''(t)^5a(t)^4 - 95919a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 3729a''(t)a^{(3)}(t)^2a(t)^4 - \\
& 117600a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 68664a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 2772a''(t)^2a^{(4)}(t)a(t)^4 - 23976a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - \\
& 2640a'(t)a^{(3)}(t)a^{(4)}(t)a(t)^4 - 12762a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 1386a'(t)a''(t)a^{(5)}(t)a(t)^4 - 651a'(t)^4a^{(6)}(t)a(t)^4 - \\
& 132a'(t)^2a^{(6)}(t)a(t)^4 + 111378a'(t)^2a''(t)^4a(t)^3 + 2354a''(t)^4a(t)^3 + 31344a'(t)^4a^{(3)}(t)^2a(t)^3 + 3729a'(t)^2a^{(3)}(t)^2a(t)^3 + \\
& 236706a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 13926a'(t)a''(t)^2a^{(3)}(t)a(t)^3 + 43320a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 5214a'(t)^2a''(t)a^{(4)}(t)a(t)^3 + \\
& 2238a'(t)^5a^{(5)}(t)a(t)^3 + 462a'(t)^3a^{(5)}(t)a(t)^3 - 162162a'(t)^4a''(t)^3a(t)^2 - 11880a'(t)^2a''(t)^3a(t)^2 - \\
& 103884a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 13332a'(t)^3a''(t)a^{(3)}(t)a(t)^2 - 6138a'(t)^6a^{(4)}(t)a(t)^2 - 1287a'(t)^4a^{(4)}(t)a(t)^2 + \\
& 76440a'(t)^6a''(t)^2a(t) + 10428a'(t)^4a''(t)^2a(t) + 11700a'(t)^7a^{(3)}(t)a(t) + 2475a'(t)^5a^{(3)}(t)a(t) - 11700a'(t)^8a''(t) - \\
& \left. 2475a'(t)^6a''(t)\right)
\end{aligned}
$$

Conjectures and question about coefficients (CC):

- Check the agreement between the above formulas for a_8 and a_{10} and the universal formulas.
- Show that the term a_{2n} of the asymptotic expansion of the spectral action for Robertson-Walker metric is of the form $P_n(a, \cdots, a^{(2n)})/a^{2n-4}$ where P_n is a polynomial with rational coefficients and compute P_n.

Our approach: spectral analysis via pseudodifferential calculus

$$
\begin{aligned}
D &= \gamma^\alpha \nabla_{\theta_\alpha} = \gamma^\alpha \left(\theta_\alpha + \omega(\theta_\alpha)\right) \\
&= \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{1}{a}\frac{\partial}{\partial \chi} + \gamma^2 \frac{1}{a\,\sin\chi}\frac{\partial}{\partial \theta} + \gamma^3 \frac{1}{a\,\sin\chi\,\sin\theta}\frac{\partial}{\partial \varphi} \\
&+ \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

So the symbol of the Dirac operator would be

$$
\begin{aligned}
\sigma_D(\mathbf{x}, \xi) \quad = i\gamma^0\xi_1 \quad & + \frac{i}{a}\gamma^1\xi_2 + \frac{i}{a\sin(\chi)}\gamma^2\xi_3 + \frac{i}{a\sin(\chi)\sin(\theta)}\gamma^3\xi_4 \\
& + \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

Symbol of D^2

Using the symbol multiplication rule one can compute the symbol of the square of the Dirac operator. The symbol of D^2 has following homogeneous parts.

$$p_2 = \xi_1^2 + \frac{1}{a(t)^2}\xi_2^2 + \frac{1}{a(t)^2 \sin^2(\chi)}\xi_3^2 + \frac{1}{a(t)^2 \sin^2(\theta)\sin^2(\chi)}\xi_4^2,$$

$$\begin{aligned} p_1 = &-\frac{3ia'(t)}{a(t)}\xi_1 - \frac{i}{a(t)^2}\left(\gamma^{12}a'(t) + 2\cot(\chi)\right)\xi_2 \\ &- \frac{i}{a(t)^2}\left(\gamma^{13}\csc(\chi)a'(t) + \cot(\theta)\csc^2(\chi) + \gamma^{23}\cot(\chi)\csc(\chi)\right)\xi_3 \\ &- \frac{i}{a(t)^2}\left(\csc(\theta)\csc(\chi)a'(t)\gamma^{14} + \cot(\theta)\csc(\theta)\csc^2(\chi)\gamma^{34} + \csc(\theta)\cot(\chi)\csc(\chi)\gamma^{24}\right)\xi_4, \end{aligned}$$

$$\begin{aligned} p_0 = &+\frac{1}{8a(t)^2}\Big(-12a(t)a''(t) - 6a'(t)^2 + 3\csc^2(\theta)\csc^2(\chi) - \cot^2(\theta)\csc^2(\chi) \\ &+4i\cot(\theta)\cot(\chi)\csc(\chi) - 4i\cot(\theta)\cot(\chi)\csc(\chi) - 4\cot^2(\chi) + 5\csc^2(\chi) + 4\Big) \\ &- \frac{\left(\cot(\theta)\csc(\chi)a'(t)\right)}{2a(t)^2}\gamma^{13} - \frac{\left(\cot(\chi)a'(t)\right)}{a(t)^2}\gamma^{12} - \frac{\left(\cot(\theta)\cot(\chi)\csc(\chi)\right)}{2a(t)^2}\gamma^{23} \end{aligned}$$

Symbol of the parametrix

Parametrix: $(P-\lambda)\tilde{R}(\lambda) = I$.

$$\sigma(\tilde{R}(\lambda)) = r_0 + r_1 + r_2 + \cdots$$

Recursive formulas:

$$r_n = -r_0 \sum_{|\alpha|+j+2-k=n} (-i)^{|\alpha|} d_\xi^\alpha p_k \cdot d_x^\alpha r_j / \alpha!,$$

where $r_0 = (p_2-\lambda)^{-1} = (\|\xi\|^2-\lambda)^{-1}$. So the summation, for $n>1$, will only have the following possible summands.

$k=0, \lvert\alpha\rvert=0, j=n-2$	$-r_0 p_0 r_{n-2}$
$k=1, \lvert\alpha\rvert=0, j=n-1$	$-r_0 p_1 r_{n-1}$
$k=1, \lvert\alpha\rvert=0, j=n-2$	$ir_0 \frac{\partial}{\partial\xi_0} p_1 \cdot \frac{\partial}{\partial t} r_{n-2} + ir_0 \frac{\partial}{\partial\xi_1} p_1 \cdot \frac{\partial}{\partial\chi} r_{n-2} + ir_0 \frac{\partial}{\partial\xi_2} p_1 \cdot \frac{\partial}{\partial\theta} r_{n-2}$
$k=2, \lvert\alpha\rvert=1, j=n-1$	$ir_0 \frac{\partial}{\partial\xi_0} p_2 \cdot \frac{\partial}{\partial t} r_{n-1} + ir_0 \frac{\partial}{\partial\xi_1} p_2 \cdot \frac{\partial}{\partial\chi} r_{n-1} + ir_0 \frac{\partial}{\partial\xi_2} p_2 \cdot \frac{\partial}{\partial\theta} r_{n-1}$
$k=2, \lvert\alpha\rvert=2, j=n-2$	$\frac{1}{2} r_0 \frac{\partial^2}{\partial\xi_0^2} p_2 \cdot \frac{\partial^2}{\partial t^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial\xi_1^2} p_2 \cdot \frac{\partial^2}{\partial\chi^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial\xi_2^2} p_2 \cdot \frac{\partial^2}{\partial\theta^2} r_{n-2}$

Heat Kernel of D^2 in terms of symbols of the parametrix.

Let

$$\begin{aligned} e_n &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)d\lambda d\xi \\ &= \frac{1}{2\pi i(2\pi)^4}\sum r_{n,j,\alpha}(x)\int_{\mathbb{R}^4}\xi^\alpha\int_\gamma e^{-t\lambda}r_0^j d\lambda d\xi \\ &= \sum c_\alpha \frac{1}{(j-1)!} r_{n,j,\alpha} a(t)^{\alpha_2+\alpha_3+\alpha_4+3}\sin(\chi)^{\alpha_3+\alpha_4+2}\sin(\theta)^{\alpha_4+1} \end{aligned}$$

Where $c_\alpha = \frac{1}{(2\pi)^4}\prod_k \Gamma\left(\frac{\alpha_k+1}{2}\right)\frac{(-1)^{\alpha_k}+1}{2}$.

where

$$a_n = \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} e_n d\chi d\theta d\phi$$

new term a_{12}

$$a_{12} = \frac{1}{17297280a(t)^8}\Big(3a^{(12)}(t)a(t)^{10} - 1057a^{(6)}(t)^2a(t)^9 - 1747a^{(5)}(t)a^{(7)}(t)a(t)^9 - 970a^{(4)}(t)a^{(8)}(t)a(t)^9 - 317a^{(3)}(t)a^{(9)}(t)a(t)^9 - 34a''(t)a^{(10)}(t)a(t)^9 + 21a'(t)a^{(11)}(t)a(t)^9 + 5001a^{(4)}(t)^3a(t)^8 + 2419a''(t)a^{(5)}(t)^2a(t)^8 + 19174a^{(3)}(t)a^{(4)}(t)a^{(5)}(t)a(t)^8 + 4086a^{(3)}(t)^2a^{(6)}(t)a(t)^8 + 2970a''(t)a^{(4)}(t)a^{(6)}(t)a(t)^8 - 5520a'(t)a^{(5)}(t)a^{(6)}(t)a(t)^8 - 511a''(t)a^{(3)}(t)a^{(7)}(t)a(t)^8 - 4175a'(t)a^{(4)}(t)a^{(7)}(t)a(t)^8 - 745a''(t)^2a^{(8)}(t)a(t)^8 - 2289a'(t)a^{(3)}(t)a^{(8)}(t)a(t)^8 - 828a'(t)a''(t)a^{(9)}(t)a(t)^8 - 62a'(t)^2a^{(10)}(t)a(t)^8 - 13a^{(10)}(t)a(t)^8 + 45480a^{(3)}(t)^4a(t)^7 + 152962a''(t)^2a^{(4)}(t)^2a(t)^7 + 203971a'(t)a^{(3)}(t)a^{(4)}(t)^2a(t)^7 + 21369a'(t)^2a^{(5)}(t)^2a(t)^7 + 1885a^{(5)}(t)^2a(t)^7 + 410230a''(t)a^{(3)}(t)^2a^{(4)}(t)a(t)^7 + 163832a'(t)a^{(3)}(t)^2a^{(5)}(t)a(t)^7 + 250584a''(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^7 + 244006a'(t)a''(t)a^{(4)}(t)a^{(5)}(t)a(t)^7 + 42440a''(t)^3a^{(6)}(t)a(t)^7 + 163390a'(t)a''(t)a^{(3)}(t)a^{(6)}(t)a(t)^7 + 35550a'(t)^2a^{(4)}(t)a^{(6)}(t)a(t)^7 + 3094a^{(4)}(t)a^{(6)}(t)a(t)^7 + 34351a'(t)a''(t)^2a^{(7)}(t)a(t)^7 + 19733a'(t)^2a^{(3)}(t)a^{(7)}(t)a(t)^7 + 1625a^{(3)}(t)a^{(7)}(t)a(t)^7 + 6784a'(t)^2a''(t)a^{(8)}(t)a(t)^7 + 520a''(t)a^{(8)}(t)a(t)^7 + 308a'(t)^3a^{(9)}(t)a(t)^7 + 52a'(t)a^{(9)}(t)a(t)^7 - 2056720a'(t)a''(t)a^{(3)}(t)^3a(t)^6 - 1790580a''(t)^3a^{(3)}(t)^2a(t)^6 - 900272a'(t)^2a''(t)a^{(4)}(t)^2a(t)^6 - 31889a''(t)a^{(4)}(t)^2a(t)^6 - 643407a''(t)^4a^{(4)}(t)a(t)^6 - 1251548a'(t)^2a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 43758a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 4452042a'(t)a''(t)^2a^{(3)}(t)a^{(4)}(t)a(t)^6 - 836214a'(t)a''(t)^3a^{(5)}(t)a(t)^6 - 1400104a'(t)^2a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 48620a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 181966a'(t)^3a^{(4)}(t)a^{(5)}(t)a(t)^6 - 18018a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 319996a'(t)^2a''(t)^2a^{(6)}(t)a(t)^6 - 11011a''(t)^2a^{(6)}(t)a(t)^6 - 115062a'(t)^3a^{(3)}(t)a^{(6)}(t)a(t)^6 - 11154a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 42764a'(t)^3a''(t)a^{(7)}(t)a(t)^6 - 4004a'(t)a''(t)a^{(7)}(t)a(t)^6 - 1649a'(t)^4a^{(8)}(t)a(t)^6 - 286a'(t)^2a^{(8)}(t)a(t)^6 + 460769a''(t)^6a(t)^5 + 1661518a'(t)^3a^{(3)}(t)^3a(t)^5 + 83486a'(t)a^{(3)}(t)^3a(t)^5 + 13383328a'(t)^2a''(t)^2a^{(3)}(t)^2a(t)^5 + 222092a''(t)^2a^{(3)}(t)^2a(t)^5 + 342883a'(t)^4a^{(4)}(t)^2a(t)^5 + 36218a'(t)^2a^{(4)}(t)^2a(t)^5 + 7922361a'(t)a''(t)^4a^{(3)}(t)a(t)^5 + 6367314a'(t)^2a''(t)^3a^{(4)}(t)a(t)^5 + 109330a''(t)^3a^{(4)}(t)a(t)^5 +$$

$$+7065862a'(t)^3a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 360386a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 1918386a'(t)^3a''(t)^2a^{(5)}(t)a(t)^5 + 98592a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + 524802a'(t)^4a^{(3)}(t)a^{(5)}(t)a(t)^5 + 55146a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 226014a'(t)^4a''(t)a^{(6)}(t)a(t)^5 + 23712a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 8283a'(t)^5a^{(7)}(t)a(t)^5 + 1482a'(t)^3a^{(7)}(t)a(t)^5 - 7346958a'(t)^2a''(t)^5a(t)^4 - 72761a''(t)^5a(t)^4 - 11745252a'(t)^4a''(t)a^{(3)}(t)^2a(t)^4 - 725712a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 27707028a'(t)^3a''(t)^3a^{(3)}(t)a(t)^4 - 819520a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 8247105a'(t)^4a''(t)^2a^{(4)}(t)a(t)^4 - 520260a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 1848228a'(t)^5a^{(3)}(t)a^{(4)}(t)a(t)^4 - 205296a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - 973482a'(t)^5a''(t)a^{(5)}(t)a(t)^4 - 110136a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 36723a'(t)^6a^{(6)}(t)a(t)^4 - 6747a'(t)^4a^{(6)}(t)a(t)^4 + 17816751a'(t)^4a''(t)^4a(t)^3 + 721058a'(t)^2a''(t)^4a(t)^3 + 2352624a'(t)^6a^{(3)}(t)^2a(t)^3 + 274170a'(t)^4a^{(3)}(t)^2a(t)^3 + 24583191a'(t)^5a''(t)^2a^{(3)}(t)a(t)^3 + 1771146a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 3256248a'(t)^6a''(t)a^{(4)}(t)a(t)^3 + 389376a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 135300a'(t)^7a^{(5)}(t)a(t)^3 + 25350a'(t)^5a^{(5)}(t)a(t)^3 - 15430357a'(t)^6a''(t)^3a(t)^2 - 1252745a'(t)^4a''(t)^3a(t)^2 - 7747848a'(t)^7a''(t)a^{(3)}(t)a(t)^2 - 967590a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 385200a'(t)^8a^{(4)}(t)a(t)^2 - 73125a'(t)^6a^{(4)}(t)a(t)^2 + 5645124a'(t)^8a''(t)^2a(t) + 741195a'(t)^6a''(t)^2a(t) + 749700a'(t)^9a^{(3)}(t)a(t) + 143325a'(t)^7a^{(3)}(t)a(t) - 749700a'(t)^{10}a''(t) - 143325a'(t)^8a''(t))\Big)$$

Check on round sphere S^4

For $a(t) = sin(t)$ we have

$$a_{12}(sphere) = \frac{10331\sin^3(t)}{8648640}.$$

Hence

$$\int_0^{\pi} a_{12}(spher)dt = \frac{4}{3}\frac{10331}{8648640} = \frac{10331}{6486480}.$$

Which agrees with the direct computation done in Connes-Chamseddine.

Rationality of heat coefficients

Theorem: The terms a_{2n} in the expansion of the spectral action for the Robertson-Walker metric with scale factor $a(t)$ is of the form

$$\frac{1}{a(t)^{2n-3}}\, Q_{2n}\left(a(t), a'(t), \ldots, a^{(2n)}(t)\right),$$

where Q_{2n} is a polynomial with *rational* coefficients.

By direct computation in Hopf coordinates, we found the vector fields which respectively form bases for left and right invariant vector fields on $\mathrm{SU}(2)$:

$$
\begin{aligned}
X_1^L &= \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\
X_2^L &= \sin(\phi_1 + \phi_2)\frac{\partial}{\partial \eta} + \cot(\eta)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_1} - \\
&\quad \tan(\eta)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_2}, \\
X_3^L &= \cos(\phi_1 + \phi_2)\frac{\partial}{\partial \eta} - \cot(\eta)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_1} + \\
&\quad \tan(\eta)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial \phi_2},
\end{aligned}
$$

and X_1^R, X_2^R, X_3^R. One checks that these vector fields are Killing vector fields for the Robertson-Walker metrics on the four dimensional space.

Warm up: zeta regularized determinants

Warm up: zeta regularized determinants

- Given a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \qquad \mathsf{spec}(\Delta)$$

How one defines $\prod \lambda_i = \mathsf{det}\Delta$?

Warm up: zeta regularized determinants

- Given a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \qquad \text{spec}(\Delta)$$

How one defines $\prod \lambda_i = \det\Delta$?

- Define the spectral zeta function:

$$\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \qquad Re(s) \gg 0$$

Assume: $\zeta_\Delta(s)$ has meromorphic extension to $\mathbb{C}$ and is regular at 0.

Warm up: zeta regularized determinants

- Given a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \qquad \text{spec}(\Delta)$$

How one defines $\prod \lambda_i = \det\Delta$?

- Define the spectral zeta function:

$$\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \qquad Re(s) \gg 0$$

Assume: $\zeta_\Delta(s)$ has meromorphic extension to $\mathbb{C}$ and is regular at 0.

- Zeta regularized determinant:

$$\prod \lambda_i := e^{-\zeta'_\Delta(0)} = \det\Delta$$

Holomorphic determinants

- Example: For Riemann zeta function, $\zeta'(0) = -\log\sqrt{2\pi}$. Hence
$$1 \cdot 2 \cdot 3 \cdots = \sqrt{2\pi}.$$

Holomorphic determinants

- Example: For Riemann zeta function, $\zeta'(0) = -\log\sqrt{2\pi}$. Hence

$$1 \cdot 2 \cdot 3 \cdots = \sqrt{2\pi}.$$

- Usually $\Delta = D^*D$. The determinant map $D \rightsquigarrow \sqrt{\det D^*D}$ is not holomorphic. How to define a holomorphic regularized determinant? This is hard.

Holomorphic determinants

- Example: For Riemann zeta function, $\zeta'(0) = -\log\sqrt{2\pi}$. Hence
$$1 \cdot 2 \cdot 3 \cdots = \sqrt{2\pi}.$$

- Usually $\Delta = D^*D$. The determinant map $D \rightsquigarrow \sqrt{\det D^*D}$ is not holomorphic. How to define a holomorphic regularized determinant? This is hard.

- Quillen's approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.

Holomorphic determinants

- Logdet is not a holomorphic function. How to define a holomorphic determinant det : $\mathcal{A} \to \mathbb{C}$.

Holomorphic determinants

- Logdet is not a holomorphic function. How to define a holomorphic determinant $\det : \mathcal{A} \to \mathbb{C}$.

- Quillen's approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.

Holomorphic determinants

- Logdet is not a holomorphic function. How to define a holomorphic determinant $\det : \mathcal{A} \to \mathbb{C}$.

- Quillen's approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.

- Recall: Space of Fredholm operators:

$$F = \text{Fred}(H_0, H_1) = \{T : H_0 \to H_1; \;\; T \text{ is Fredholm}\}$$

$$K_0(X) = [X, F], \quad \text{classifying space for K-theory}$$

The determinant line bundle

- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.

The determinant line bundle

- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.
- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.
$$(DET)_T = \lambda(KerT)^* \otimes \lambda(KerT^*)$$

The determinant line bundle

- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.

- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.

$$(DET)_T = \lambda(KerT)^* \otimes \lambda(KerT^*)$$

2) There map $\sigma : F_0 \to DET$

$$\sigma(T) = \begin{cases} 1 & T \quad invertible \\ 0 & otherwise \end{cases}$$

is a holomorphic section of DET over F_0.

Cauchy-Riemann operators on $\mathcal{A}_\theta$

- Families of spectral triples

$$\mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, \quad \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix},$$

with $\alpha \in \mathcal{A}_\theta$, $\bar{\partial} = \delta_1 + \tau\delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar{\partial} + \alpha$.

Cauchy-Riemann operators on $\mathcal{A}_\theta$

- Families of spectral triples

$$\mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, \quad \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix},$$

with $\alpha \in \mathcal{A}_\theta,\ \bar{\partial} = \delta_1 + \tau\delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar{\partial} + \alpha$.

- Pull back DET to a holomorphic line bundle $\mathcal{L} \to \mathcal{A}$ with

$$\mathcal{L}_D = \lambda(KerD)^* \otimes \lambda(KerD^*).$$

From det section to det function

- If $\mathcal{L}$ admits a canonical global holomorphic frame s, then

$$\sigma(D) = det(D)s$$

  defines a holomorphic determinant function $\det(D)$. A canonical frame is defined once we have a canonical flat holomorphic connection.

Quillen's metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $Index(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta, \quad \Delta = D^* D.$$

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

Quillen's metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $Index(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta, \quad \Delta = D^* D.$$

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

- A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from

$$\bar{\partial}\partial \log ||s||^2,$$

where s is any local holomorphic frame.

Connes' pseudodifferential calculus

- To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.
- Symbols of order m: smooth maps $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ with

$$||\delta^{(i_1,i_2)}\partial^{(j_1,j_2)}\sigma(\xi)|| \leq c(1+|\xi|)^{m-j_1-j_2}.$$

The space of symbols of order m is denoted by $\mathcal{S}^m(\mathcal{A}_\theta)$.

Connes' pseudodifferential calculus

- To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.
- Symbols of order m: smooth maps $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ with

$$||\delta^{(i_1,i_2)}\partial^{(j_1,j_2)}\sigma(\xi)|| \leq c(1+|\xi|)^{m-j_1-j_2}.$$

  The space of symbols of order m is denoted by $\mathcal{S}^m(\mathcal{A}_\theta)$.

- To a symbol σ of order m, one associates an operator

$$P_\sigma(a) = \int\int e^{-is\cdot\xi}\sigma(\xi)\alpha_s(a)\,ds\,d\xi.$$

  The operator $P_\sigma : \mathcal{A}_\theta \to \mathcal{A}_\theta$ is said to be a pseudodifferential operator of order m.

Classical symbols

- Classical symbol of order $\alpha \in \mathbb{C}$:

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{\alpha-j} \qquad \text{ord}\, \sigma_{\alpha-j} = \alpha - j.$$

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.$$

- We denote the set of classical symbols of order α by $\mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$.

A cutoff integral

- Any pseudo P_σ of order < -2 is trace-class with

$$\mathrm{Tr}(P_\sigma) = \varphi_0 \left(\int_{\mathbb{R}^2} \sigma(\xi) d\xi \right).$$

A cutoff integral

- Any pseudo P_σ of order < -2 is trace-class with

$$\mathrm{Tr}(P_\sigma) = \varphi_0 \left(\int_{\mathbb{R}^2} \sigma(\xi) d\xi \right).$$

- For $\mathrm{ord}(P) \geq -2$ the integral is divergent, but, assuming P is classical, and of non-integral order, one has an asymptotic expansion as $R \to \infty$

$$\int_{B(R)} \sigma(\xi) d\xi \sim \sum_{j=0, \alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma) \log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi =$ Wodzicki residue of P (Fathizadeh).

The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}_{cl}^{\alpha}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi) d\xi$.

The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi) d\xi$.

- The canonical trace of a classical pseudo $P \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ of non-integral order α is defined as

$$\mathrm{TR}(P) := \varphi_0 \left(⨍ \sigma_P(\xi) d\xi \right).$$

The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}_{cl}^{\alpha}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi) d\xi$.

- The canonical trace of a classical pseudo $P \in \Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$ of non-integral order α is defined as

$$\mathrm{TR}(P) := \varphi_0 \left(⨍ \sigma_P(\xi) d\xi \right).$$

- NC residue in terms of TR:

$$\mathrm{Res}_{z=0} \mathrm{TR}(AQ^{-z}) = \frac{1}{q} \mathrm{Res}(A).$$

Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j\geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,$$

  with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

- Example: $\log Q$ where $Q \in \Psi^q_{cl}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$.

Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^{l} |\xi| \quad |\xi| > 0,$$

  with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

- Example: $\log Q$ where $Q \in \Psi^{q}_{cl}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$.

- Wodzicki residue: $\mathrm{Res}(A) = \varphi_0(\mathrm{res}(A))$,

$$\mathrm{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.$$

Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section σ,

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}$$

Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section σ,

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}$$

- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$. Want to compute

$$\bar{\partial}\partial \log \|\sigma\|^2 = \delta_{\bar{w}}\delta_w \zeta'_\Delta(0) = \delta_{\bar{w}}\delta_w \frac{d}{dz}\mathrm{TR}(\Delta^{-z})|_{z=0}.$$

The second variation of logDet

- Prop 1: For a holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ is given by :

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{2}\varphi_0\left(\delta_w D\delta_{\bar{w}}\mathrm{res}(\log\Delta\, D^{-1})\right).$$

The second variation of logDet

- Prop 1: For a holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ is given by :

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{2}\varphi_0\left(\delta_w D\delta_{\bar{w}}\mathrm{res}(\log\Delta\, D^{-1})\right).$$

- Prop 2: The residue density of $\log\Delta\, D^{-1}$:

$$\sigma_{-2,0}(\log\Delta\, D^{-1}) = \frac{(\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2}{(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)(\xi_1+\tau\xi_2)}$$

$$-\log\left(\frac{\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2}{|\xi|^2}\right)\frac{\alpha}{\xi_1+\tau\xi_2},$$

and

$$\delta_{\bar{w}}\mathrm{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Curvature of the determinant line bundle

- Theorem (A. Fathi, A. Ghorbanpour, MK.): The curvature of the determinant line bundle for the noncommutative two torus is given by
$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right).$$

- Remark: For $\theta = 0$ this reduces to Quillen's theorem (for elliptic curves).

A holomorphic determinant a la Quillen

- Modify the metric to get a flat connection:

$$||s||_f^2 = e^{||D-D_0||^2}||s||^2$$

A holomorphic determinant a la Quillen

- Modify the metric to get a flat connection:

$$||s||_f^2 = e^{||D-D_0||^2}||s||^2$$

- Get a flat holomorphic global section. This gives a holomorphic determinant function

$$det(D, D_0) : \mathcal{A} \to \mathbb{C}$$

It satisfies

$$|det(D, D_0)|^2 = e^{||D-D_0||^2} det_\zeta(D^* D)$$