From Spectral Geometry to Geometry of Noncommutative Spaces I

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- Spectral geometry, starting with Weyl's law on the asymptotic distribution of eigenvalues of Laplacians, was in part motivated by theories of sound and heat, but above all by qunatum mechanics and a desire to see laws of classical physics as a limit of laws of quantum physics. (Correspondence principle, Bohr-Sommerfeld quantization rules, semiclassical approximations.)

Toward Weyl's Law: conjecture of Lorentz and Sommerfeld

- In 1910 H. A. Lorentz gave a series of lectures in Göttingen under the title "old and new problems of physics". Weyl and Hilbert were in attendance. In particular he mentioned attempts to drive Planck's 1900 radiation formula in a mathematically satisfactory way and remarked:

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  'It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume. ....There is no doubt that it holds in general even for multiply connected spaces'.

A brief history of Weyl's asymptotic law

- Hilbert was not very optimistic to see a solution in his lifetime. But Hermann Weyl, his bright student, settled this conjecture of Lorentz and Sommerfeld affirmatively within a year and announced a proof in 1911! All he needed was Hilbert's theory of integral equations and his spectral theorem for compact selfadjoint operators developed by Hilbert and his students in 1906-1910, as well as some original ideas of his own.

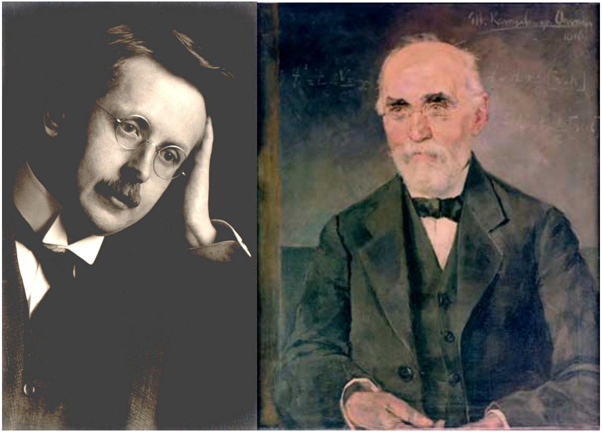

Figure : Weyl and Lorentz

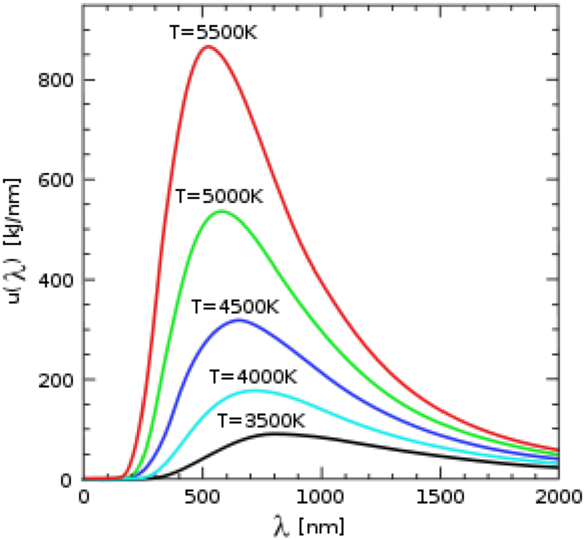

Figure : Black body spectrum. Nobody was happy with Planck's derivation, untill Bose gave a satisfactory derivation in 1924.

Eigenvalues of the Laplacian and Weyl's law

- Let $\Omega \subset \mathbb{R}^n$ be a compact connected domain with piecewise smooth boundary, and Δ = Laplacian. Consider the Helmholtz equation

$$\begin{cases} \Delta u = \lambda u \\ u|\partial\Omega = 0 \quad \text{Dirichlet boundary condition} \end{cases}$$

- Spectral theory shows that Δ has pure point spectrum

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

$$\langle u_i, u_j \rangle = \delta_{ij} \quad \text{o.n. basis for } L^2(\Omega)$$

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- Eigenvalue Counting Function: How fast eigenvalues grow?

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

Patterns in eigenvalues?

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- Except for few cases (Boxes, Balls, ellipses, and equilateral triangles), no explicit formulas are known for eigenvalues.
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$$\lambda_k \sim C k^{\frac{2}{m}} \qquad k \to \infty$$

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$$\lambda_k \sim C k^{\frac{2}{m}} \qquad k \to \infty$$

- But this is far from obvious, and clues as to why such a statment should be true, and what C should be, first came from spectroscopy, and in particular attemps to prove the black body radiation formula, as Lorentz emphasized.

Examples: flat tori and round spheres

- Flat tori: $M = \mathbb{R}^m/\Gamma$, $\quad \Gamma \subset \mathbb{R}^m$ a cocompact lattice;

$$\mathrm{spec}(\triangle) = \{4\pi^2||\gamma||^2; \ \gamma \in \Gamma^*\}$$
$$u_\gamma(x) = e^{2\pi i\langle\gamma,x\rangle} \qquad \gamma \in \Gamma^*$$

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- Round sphere S^n. Distinct eigenvalues

$$\lambda_k = k(k+n-1) \qquad k = 0, 1, \cdots,$$

multiplicity: $\binom{n+k}{k} - \binom{n+k-2}{k-2}$.

Weyl's Asymptotic Law

- Weyl's Law for domains $\Omega \subset \mathbb{R}^n$:

$$N(\lambda) \sim \frac{\omega_n \mathrm{Vol}(\Omega)}{(2\pi)^n} \lambda^{\frac{n}{2}} \qquad \lambda \to \infty$$

where $N(\lambda)$ is the eigenvalue counting function, and ω_n= volume of unit ball in $\mathbb{R}^n$.

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- Equivalently

$$\lambda_k \sim \frac{(2\pi)^2}{(\omega_n \mathrm{Vol}(\Omega))^{2/n}} \, k^{\frac{2}{n}} \qquad k \to \infty$$

Weyl's law and the theory of sound

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- The idea is very old of course. In late 18 century E. Darwin in his book Zoonomia reports: *The late blind Justice Fielding walked for the first time into my room, when he once visited me, and after speaking a few words said, This room is about 22 feet long, 18 wide, and 12 high; all which he guessed by the ear with great accuracy.* (quoted by Lord Rayleigh in his book *The Theory of Sound* (1877)).

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- But the ultimate question is: what else one can hear about the shape of a manifold, or the shape of a noncommutative space?

- A modern version of Weyl's law: One can hear the volume and dimension of a Riemannian manifold. We shall see one can hear the volume of curved noncommutative tori too.

- But the ultimate question is: what else one can hear about the shape of a manifold, or the shape of a noncommutative space?

- It is now known that one can hear, among other things, the total scalar curvature, and, in many cases, lengths of closed geodesics (as in Selberg trace formula).

The limits $n \to 0$ and $n \to \infty$ of Weyl's law?

For zero dimensional spaces we expect the eigenvalues grow exponentially fast, while for infinite dimensional spaces they are expected to grow logarithmically-perhaps.

Quantum sphere S_q^2

Relations for Podles quantum sphere S_q^2, $0 < q < 1$.

$$AB = q^2 BA$$
$$AB^* = q^{-2} B^* A$$
$$BB^* = q^{-2} A(1 - A)$$
$$B^* B = A(1 - q^2 A)$$

Eigenvalues of the Dirac operator for quantum spheres

Eigenvalues of the Dirac operator for S_q^2:

$$\lambda_k = \frac{q^{k+1/2} - q^{-(k+1/2)}}{q - q^{-1}}$$

where

$$k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

Note: This grows exponentially fast. As $q \to 1$, the above numbers approach the spectrum of the Dirac operator D for the round sphere S^2.

First impact of Weyl's law: how to quantize

- Consider a classical system (X, ω, h);
  $(X, \omega) =$ symplectic manifold, $h : X \to \mathbb{R}$, Hamiltonian. Assume

  $$\{x \in X;\ h(x) \leq \lambda\}$$

  are compact for all λ (confined system).

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  are compact for all λ (confined system).

- Typical example: $X = T^*M$, $(M, g) =$ compact Riemannian manifold, $h = T + V$.
  $T=$ kinetic energy, $V=$ potential energy.

- How to quantize this ?

$$(X, h) \rightsquigarrow (\mathcal{H}, H)$$

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  No one knows! No functor! But Dirac rules, geometric quantization, (strict) deformation quantization, etc. provide some ideas.

- Weyl's law imposes some constraints that are universally agreed on. This is an aspect of the celebrated correspondence principle:

- $H =$ Hamiltonian, with pure point spectrum

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

  s.t.

$$N(\lambda) \sim c \, \text{Volume} \, (h \leq \lambda) \qquad \lambda \to \infty$$

- Thus: quantized energy levels are approximated by phase space volumes.

- Apply this to $X = T^*M$, (M, g) = compact Riemannian manifold, $h(q,p) = ||p||^2$; set

$$\mathcal{H} = L^2(M), \quad H = \Delta \quad \text{Laplacian}$$

obtain Weyl's Law:

$$N(\lambda) \sim c \, \text{Vol (M)} \lambda^{m/2} \qquad (\lambda \to \infty)$$

- Years later, in his Gibbs lecture to the American Mathematical Society (1950) Weyl said:

  "I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still very incomplete. I have certain conjectures of what a complete analysis of their asymptotic behaviour should aim at but, since for more than 35 years I have made no serious effort to prove them, I think I had better keep them to myself".

Weyl's law and the Dixmier trace

- The Dixmier ideal:

$$\mathcal{L}^{1,\infty}(\mathcal{H}) := \{T \in \mathcal{K}(\mathcal{H});\, \sum_{1}^{N} \mu_n(T) = O\left(\log N\right)\}.$$

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- Example: $\mu_n(T) \sim \frac{1}{n}$ implies $T \in \mathcal{L}^{1,\infty}(\mathcal{H})$. By Weyl's law $\Delta^{-\frac{n}{2}}$ satisfies this estimate. So

$$\Delta^{-\frac{n}{2}} \in \mathcal{L}^{1,\infty}(\mathcal{H})$$

The Dixmier trace of an operator $T \in \mathcal{L}^{1,\infty}(\mathcal{H})$ measures the *logarithmic divergence* of its ordinary trace. More precisely, we are interested in taking some kind of limit of the bounded sequence

$$\sigma_N(T) = \frac{\sum_1^N \mu_n(T)}{\log N}$$

as $N \to \infty$. The problem is that, while by our assumption the sequence is bounded, the usual limit may not exists and must be replaced by a carefully chosen 'generalized limit' Lim_ω. Then we can define

$$\mathrm{Tr}_\omega(T) := \mathrm{Lim}_\omega(\sigma_N(T)).$$

Weyl's law is equivalent to: $\Delta^{-\frac{n}{2}} \in \mathcal{L}^{1,\infty}(\mathcal{H})$ and

$$\mathrm{Tr}_\omega(\Delta^{-\frac{n}{2}}) = \frac{\mathrm{Vol}(\Omega)}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2}+1)}$$

Weyl's law and zeta functions

- Spectral zeta function

$$\zeta(s) = \mathrm{Trace}(\Delta^{-s}), \qquad \mathrm{Re}\, s > n/2.$$

Weyl's law implies $\zeta(s)$ is holomorphic in $\mathrm{Re}\, s > n/2$, but certainly is not enough to show that $\zeta(s)$ has a meromorphic extension to $\mathbb{C}$.

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- To study the analytic continuation of zeta, one needs much deeper information on eigenvalues than afforded by Weyl's law. But what kind of information? How to formulate it? Analytic number theory provided the right ideas!

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- In fact, Riemann gave two proofs for anayltic continuation (and functional equation) for his zeta function. His second proof is a blue print for all later proofs of anaytic continuation in number theory and spectral geometry.

Heat kernel asymptotics: Minakshisundaram-Pleijel

- Let (M, g) be a Riemannian manifold, and $\triangle = d^*d$ the Laplacian on functions, and $k(t, x, y) =$ kernel of $e^{-t\triangle}$.

- Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}} \sum_{k=0}^{\infty} a_k(x, \triangle) t^{\frac{k}{2}} \quad t \to 0.$$

  $a_k(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.

- $a_0(x, \triangle) = 1$ is already a deep fact, equivalent to Weyl's law.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in curvature tensor R and its covariant derivatives:

$$\begin{aligned} a_0(x, \triangle) &= 1 \\ a_1(x, \triangle) &= \frac{1}{6} S(x) && \text{scalar curvature} \\ a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\ a_3(x, \triangle) &= \cdots\cdots \end{aligned}$$

Heat trace asymptotics

Compute Trace$(e^{-t\triangle})$ in two ways:

$$\text{Spectral Sum} = \text{Geometric Sum}.$$

$$\sum e^{-t\lambda_i} = \int_M k(t,x,x)d\mathrm{vol}_x$$

But

$$\int_M k(t,x,x)d\mathrm{vol}_x \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0).$$

where

$$a_j = \int_M a_j(x,\triangle)dx.$$

Hence a_j are manifestly spectral invariants:

$$a_0 \;=\; \int_M d\mathrm{vol}_x = \mathrm{Vol}(M), \qquad \Longrightarrow \text{Weyl's law}$$

Tauberian theory and $a_0 = 1$, implies Weyl's asymptotic law for closed Riemannian manifolds:

$$N(\lambda) \sim \frac{\mathrm{Vol}\,(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} \qquad \lambda \to \infty,$$

where

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

is the eigenvalue counting function.

Analytic continuation of zeta functions

$$\zeta_\triangle(s) := \sum \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

$$\begin{aligned} \zeta_\triangle(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Trace}(e^{-t\triangle}) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \{ \int_0^c \cdots + \int_c^\infty \cdots \} \end{aligned}$$

The second term defines an entire function, while the first term has a meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Simplest example: For $M = S^1$ with round metric, we have

$$\zeta_\triangle(s) = 2\zeta(2s) \qquad \text{Riemann zeta function}$$

Application: heat equation proof of the Atiyah-Singer index theorem

- Dirac operator

$$D : C^\infty(S_+) \to C^\infty(S_-)$$

McKean-Singer formula:

$$\mathrm{Index}(D) = \mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*}), \qquad \forall t > 0$$

Heat trace asymptotics $\Longrightarrow$

$$\mathrm{Index}(D) = \int_M a_n(x)dx,$$

where $a_n(x) = a_n^+(x) - a_n^-(x), m = 2n$, can be explicitly computed and recovers the A-S integrand (The simplest proof is due to Getzler).

Example: heat trace for round sphere S^2

- The heat trace for S^2

$$Z(t) = \sum_{k=0}^{\infty} (2k+1) e^{-(k^2+k)t}.$$

- We can use the *Euler-Maclaurin Summation Formula* to find the short time asymptotics for the heat trace.

- Euler-Maclaurin Summation Formula:

$$\sum_{k=a}^{b} f(k) \sim \int\limits_{a}^{b} f(x)dx + \frac{f(a)+f(b)}{2} + \sum_{k=2}^{m} \frac{B_k}{k!}(f^{(k-1)}(b) - f^{(k-1)}(a))$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}, \cdots$ are the *Bernoulli numbers*

- Euler-Maclaurin Summation Formula:

$$\sum_{k=a}^{b} f(k) \sim \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \sum_{k=2}^{m} \frac{B_k}{k!}\left(f^{(k-1)}(b) - f^{(k-1)}(a)\right)$$

where $B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6}, \cdots$ are the *Bernoulli numbers*

- In our case, $f(k) = (2k+1)e^{-(k^2+k)t}$, $a = 0,\ b = \infty$. So

$$Z(t) = \int_0^\infty (2x+1)e^{-(x^2+x)t}\, dx + \frac{1}{2} +$$

$$+ \frac{1}{6}\left(2e^{-(x^2+x)t} - t(2x+1)^2 e^{-(x^2+x)t}\right)\Big|_0^\infty + \ldots$$

The first integral is:

$$\int_0^{\infty} (2x+1)e^{-\left(x^2+x\right)t}\, dx = -\left.\frac{1}{t}e^{-\left(x^2+x\right)t}\right|_0^{\infty} = \frac{1}{t}.$$

Then we have

$$\begin{aligned} Z(t) &= \frac{1}{t} + \frac{1}{2} + \frac{1}{12}(-2+t) + \ldots = \\ &= \frac{1}{4\pi t}\left(4\pi + \frac{4}{3}\pi t + \ldots\right) \end{aligned}$$

At the same time, we know that

$$a_0 = Area\left(S^2\right) = 4\pi$$

and

$$a_1 = \frac{1}{3}\int_{S^2} K\, dx = \frac{4}{3}\pi$$

where K is Gaussian curvature and scalar curvature $S(x) = 2K$.
In this way, we can compute all terms in the expansion.

Example: heat trace for quantum spheres

- The heat trace for quantum sphere is:

$$Z(t) = \sum_{k=1}^{\infty} e^{\frac{q^{-k} - q^{k}}{q - q^{-1}} t}$$

However, we don't know the heat trace expansion for quantum sphere.

Example: heat trace for flat tori

- $\Gamma \subset \mathbb{R}^m$ a cocompact lattice; $M = \mathbb{R}^m/\Gamma$

$$\mathrm{spec}(\triangle) = \{4\pi^2||\gamma^*||^2;\ \ \gamma^* \in \Gamma^*\}$$

- Then:

$$K(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \sum_{\gamma\in\Gamma} e^{-4\pi^2||x-y+\gamma||^2/4t}$$

- Poisson summation formula $\Longrightarrow$

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 ||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \sum_{\gamma \in \Gamma} e^{-4\pi^2 ||\gamma||^2 / 4t}$$

- And from this we obtain the asymptotic expansion of the heat trace near $t = 0$

$$\mathrm{Tr} e^{-t\Delta} \sim \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \qquad (t \to 0)$$

Quantum sphere ζ-function

- The spectral ζ-function of the quantum sphere is (Eckstein-Iochum-Sitarz)

$$\zeta_q(s) = 4(1-q^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{k!\Gamma(s)} \frac{q^{2k}}{(1-q^{s+2k})^2}$$

- All poles of $\zeta_q(s)$ are complex of the second order:

$$-2k + i\frac{2\pi}{\log q} m$$

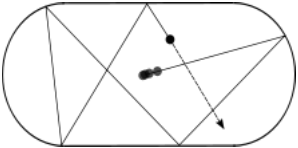
where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Heat trace and wave trace

- The heat trace $Z(t)$ is a smooth function. The wave trace

$$W(t) = \sum e^{it\sqrt{\lambda_k}},$$

is divergent for all t, but is a well defined distribution. In fact it contains more information than the heat trace. Its singular support contains the length spectrum of closed geodesics.



Algebra of differential operators

- Given (M, E, ∇), let

  $$D(M, E) \subset \mathrm{End}_{\mathbb{C}}(\Gamma(M, E)$$

  be the subalgebra generated by $\Gamma(M, \mathrm{End}(E))$ and ∇_X for all vector fields X.

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be the subalgebra generated by $\Gamma(M, \mathrm{End}(E))$ and ∇_X for all vector fields X.

- $D(M,E)$ is indendent of the choice of connection ∇
- $D(M,E)$ is a filtered algebra-degree filteration.
- The associated graded algebra is
- This construction is purely algebraic and can be done for commutative algebras
- In local coordinates:
$$P = \sum f_I \partial_I, \qquad \partial_I = \partial_1^{i_1} \cdots \partial_m^{i_m}$$

Laplace type operators

- Fix a Riemannian metric g on M and a vector bundle V on M. An operator $P : \Gamma(M, V) \to \Gamma(M, V)$ is a Laplace type operator if in local coordinates it looks like

$$P = -g^{ij}\partial_i\partial_j + \text{lower orders}$$

- Lemma: P is Laplace type iff for all smooth sections f,

$$[[P, f], f] = -2|df|^2,$$

Examples of Laplace type operators

- Laplacian on forms

$$\Delta = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

- Dirac Laplacians $\Delta = D^* D$, where

$$D : \Gamma(S) \to \Gamma(S)$$

  is a generalized Dirac operator.

Lemma: Let P be a Laplace type operator. Then there exists a unique connection ∇ on the vector bundle V and an endomorphism $E \in \mathrm{End}(V)$ such that

$$P = \nabla^* \nabla - E.$$

Here $\nabla^* \nabla$ is the connection Laplacian which is locally given by $-g^{ij} \nabla_i \nabla_j$.

Heat kernel asymtotics

- e^{-tP} is a smoothing operator with a smooth kernel $k(t,x,y)$ with

$$k(t,x,y) \in V_x \otimes V_y^*$$

$$e^{-tP} f(x) = \int_M k(t,x,y) f(y) dy.$$

- There is an asymptotic expansion near $t = 0$

$$k(t,x,x) \sim \frac{1}{(4\pi t)^{m/2}} (a_0(x,P) + a_1(x,P)t + a_2(x,P)t^2 + \cdots).$$

- $a_i(x,P)$, Seeley-De Witt-Gilkey coefficients.

$$
\begin{aligned}
a_0(x,P) &= \mathrm{tr}(\mathrm{Id}),\\
a_2(x,P) &= \mathrm{tr}\big(E - \frac{1}{6}R\,\mathrm{Id}\big),\\
a_4(x,P) &= \frac{1}{360}\mathrm{tr}\Big(\big(-12R_{;kk} + 5R^2 - 2R_{jk}R_{jk} + 2R_{ijkl}R_{ijkl}\big)\mathrm{Id}\\
&\qquad -60RE + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\Big).
\end{aligned}
$$

$$
\begin{aligned}
a_6(x,P) = \mathrm{tr}\Big\{ & \frac{1}{7!}\Big(-18R_{;kkll} + 17R_{;k}R_{;k} - 2R_{jk;l}R_{jk;l} - 4R_{jk;l}R_{jl;k} \\
&\qquad + 9R_{ijku;l}R_{ijku;l} + 28RR_{;ll} - 8R_{jk}R_{jk;ll} + 24R_{jk}R_{jl;kl} \\
&\qquad + 12R_{ijkl}R_{ijkl;uu}\Big)\mathrm{Id} \\
&+ \frac{1}{9\cdot 7!}\Big(-35R^3 + 42RR_{lp}R_{lp} - 42RR_{klpq}R_{klpq} + 208R_{jk}R_{jl}R_{kl} \\
&\qquad - 192R_{jk}R_{ul}R_{jukl} + 48R_{jk}R_{julp}R_{kulp} - 44R_{ijku}R_{ijlp}R_{kulp} \\
&\qquad - 80R_{ijku}R_{ilkp}R_{jlup}\Big)\mathrm{Id} \\
&+ \frac{1}{360}\Big(8\Omega_{ij;k}\Omega_{ij;k} + 2\Omega_{ij;j}\Omega_{ik;k} + 12\Omega_{ij}\Omega_{ij;kk} - 12\Omega_{ij}\Omega_{jk}\Omega_{ki} \\
&\qquad - 6R_{ijkl}\Omega_{ij}\Omega_{kl} + 4R_{jk}\Omega_{jl}\Omega_{kl} - 5R\Omega_{kl}\Omega_{kl}\Big) \\
&+ \frac{1}{360}\Big(6E_{;iijj} + 60EE_{;ii} + 30E_{;i}E_{;i} + 60E^3 + 30E\Omega_{ij}\Omega_{ij} - 10RE_{;kk} \\
&\qquad - 4R_{jk}E_{;jk} - 12R_{;k}E_{;k} - 30RE^2 - 12R_{;kk}E + 5R^2E \\
&\qquad - 2R_{jk}R_{jk}E + 2R_{ijkl}R_{ijkl}E\Big)\Big\}.
\end{aligned}
$$

Note: all tensors are written in normal coordinates passing through the base point x, and Ω is the curvature two form of the connection.

Scalar curvature

The spectral invariants a_i in the heat asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_{\triangle}(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x)dvol_x$, we obtain a formula for scalar curvature density as follows:

Let $\zeta_f(s) := \mathrm{Tr}\,(f\triangle^{-s}),\ f \in C^\infty(M)$.

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi}\int_M fS(x)dvol_x - \mathrm{Tr}(fP) \qquad m = 2$$

$\log\det(\triangle) = -\zeta'(0)$, Ray-Singer regularized determinant

Curved noncommutative tori A_θ

$A_\theta = C(\mathbb{T}^2_\theta) =$ universal C^*-algebra generated by unitaries U and V

$$VU = e^{2\pi i\theta} UV.$$

$$A_\theta^\infty = C^\infty(\mathbb{T}^2_\theta) = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n : (a_{m,n}) \, \text{Schwartz class} \Big\}.$$

- Differential operators $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V$$

- Integration $\varphi_0 : A_\theta \to \mathbb{C}$ on smooth elements:

$$\varphi_0\Big(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\Big) = a_{0,0}.$$

- Complex structures: Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0$. Dolbeault operators

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

Conformal perturbation of the metric (Connes-Tretkoff)

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form φ_0 by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \varphi_0(ae^{-h}).$$

- It is a twisted trace (KMS state):

$$\varphi(ab) = \varphi(b\Delta(a)),$$

where

$$\Delta(x) = e^{-h}xe^{h}.$$

Perturbed Dolbeault operator

- Hilbert space $\mathcal{H}_\varphi = L^2(A_\theta, \varphi)$, $\quad$ *GNS* construction.

- Let $\partial_\varphi = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$,

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi.$$

  and $\triangle = \partial_\varphi^* \partial_\varphi$, perturbed non-flat Laplacian.

Scalar curvature for A_θ

- Gilkey-De Witt-Seeley formulae in spectral geometry motivates the following definition:

  The scalar curvature of the curved nc torus (A_θ, τ, h) is the unique element $R \in A_\theta^\infty$ satisfying

  $$\mathrm{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \mathrm{Trace}\,(aP) = \varphi_0\,(aR), \qquad \forall a \in A_\theta^\infty,$$

  where P is the projection onto the kernel of $\triangle$.

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  where P is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansion for the kernel of the operator $e^{-t\triangle}$, using Connes' pseudodifferential calculus for nc tori.

Final formula for the scalar curvature (Connes-Moscovici; Fathizadeh-K

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\big(\triangle_0(\log k)\big)+$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2+|\tau|^2\delta_2(\log k)^2+\tau_1\big\{\delta_1(\log k), \delta_2(\log k)\big\}\Big)+$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2\big[\delta_1(\log k), \delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1 + \cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s-t+t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

What remains to be done

- Define new curved NC spaces and extend these spectral computations to them.

- Other curvature related work: Marcolli-Buhyan, Dabrowski-Sitarz, Lesch, Rosenberg, and Arnlind. Recently Fathizadeh has simplified the four dimensional calculations and its Einstein-Hilbert action.

Rationality of Spectral Action for Robertson-Walker Metrics

Masoud Khalkhali
(Joint work with F. Fathizadeh and A. Ghorbanpour)

The spectral action principle of Connes and Chamseddine

Some relevant references:

- A. Connes, *Noncommutative Geometry*, 1994.
- A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Comm. Math. Phys. 182 (1996) 155-176.
- A. H. Chamseddine, A. Connes, *The spectral action principle,* Comm. Math. Phys. 186 (1997), no. 3, 731–750.
- A. H. Chamseddine, A. Connes, *Spectral action for Robertson-Walker metrics,* J. High Energy Phys. 2012, no. 10, 101.

Classical action

- Classical Lagrangian action

$$S = \int_M \mathcal{L}(\varphi, \partial_\mu \varphi) d^n x$$

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$$\delta S = 0,$$

- Quantum expectation values

$$\langle \mathcal{O} \rangle = \int D[\varphi] \, \mathcal{O}(\varphi) \, e^{\frac{i}{\hbar} S}$$

Spectral action of Connes-Chamseddine

- Replace the classical action $S = \int_M \mathcal{L}(\varphi, \partial_\mu \varphi) d^n x$ by the spectral action
$$S = \mathrm{Trace}(f(D/\Lambda)),$$
where D is a Dirac operator, f is a positive even function, and the cutoff Λ is the mass scale.

- *S only depends on the spectrum of D and moments of the cutoff,*
$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- Results from spectral geometry (Gilkey's formulae for heat trace asymptotics) can be used to show that one indeed recovers the classical action from the spectral action (more on this later).

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- Results from spectral geometry (Gilkey's formulae for heat trace asymptotics) can be used to show that one indeed recovers the classical action from the spectral action (more on this later).
- Spectral action is manifestly quantum mechanical and one does not need a geometric background to write it down.
- Spectral action makes perfect sense for spectral triples.

Hard calculations made easy

- Compute

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}, \qquad t = 0.01.$$

You have to add 21 terms to get it to one decimal digit.

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- But: you can compute it to 130 decimal digits without any calculation!

Modular equation is the key

- Modular equation (Jacobi, 1828)

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$$\theta(0.01) = 10.00000000000000000 \cdots$$

- So: Jacobi computed the first heat trace asymptotic expansion and in fact the first trace formula.

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- Application: for any lattice $\Gamma \subset \mathbb{R}^n$:

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 ||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{d/2}} \sum_{\gamma \in \Gamma} e^{-||\gamma||^2/4t}$$

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- Dilogarithm function $Li_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}$
$$Li_2(x) + Li_2(1-x) + \log x \log(1-x) = Li_2(1)$$

Zeta values

- Euler computed zeta values $\zeta(2), \zeta(3), \cdots \zeta(23)$ with at least 15 decimal digits! How? Dilogarithm identities are not useful for finding

$$\zeta(3) = 1,202056903159594285399 7 \cdots$$

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$$\zeta(3) = 1,202056903159594285399 7 \cdots$$

- Euler-Maclaurin Summation Formula:

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \sum_{j=2}^{m} \frac{B_j}{j!}(g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m$$

Friedmann-Lemaître-Robertson-Walker metric

- (Euclidean) FLRW metric with the scale factor $a(t)$:

$$ds^2 = dt^2 + a^2(t)\, d\sigma^2.$$

Where $d\sigma^2$ is the round metric on 3-sphere. It describes a homogeneous, isotropic (expanding or contracting) universe with spatially closed universe.

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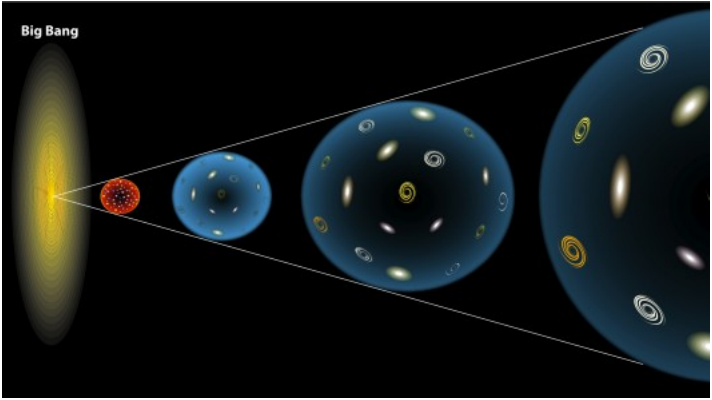
- For $a(t) = \sin(t)$ one obtains the round metric on S^4.

$$ds^2 = dt^2 + a^2\,(t)\left(d\chi^2 + \sin^2(\chi)\left(d\theta^2 + \sin^2(\theta)\,d\varphi^2\right)\right)$$

FLRW Metric



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2. Fathizadeh, Ghorbanpour, and Khalkhali: Rationality of Spectral Action for Robertson-Walker Metrics (2014)

Euler Maclaurin formula and Heat kernel for S^4

Euler Maclaurin formula

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \sum_{j=2}^{m} \frac{B_j}{j!}(g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m$$

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

Dirac spectrum

- Spectrum of Dirac for round S^4 :

	eigenvalues	multiplicity
D	$\pm k$	$\frac{2}{3}(k^3 - k)$
D^2	k^2	$\frac{4}{3}(k^3 - k)$

- To find heat kernel coefficients of D^2 we apply the Euler Maclaurin formula for $a = 0$, $b = \infty$ and

$$g(x) = \frac{4}{3}(x^3 - x)f(x) = \frac{4}{3}(x^3 - x)e^{-tx^2}$$

The integral term gives

$$\int_a^b g(x)dx = \frac{4}{3}\int_0^\infty (x^3 - x)e^{-tx^2}dx = \frac{2}{3}(t^{-2} - t^{-1})$$

The term $\frac{g(a)+g(b)}{2}$ is zero since $g(0) = g(\infty) = 0$.
And

$$g^{(2m-1)}(0)/(2m-1)! = (-1)^m \frac{4}{3}\left(\frac{t^{m-2}}{(m-2)!} + \frac{t^{m-1}}{(m-1)!}\right)$$

Putting all these together we get

$$\frac{3}{4}\mathrm{Tr}(e^{-tD^2}) = \frac{1}{2t^2} - \frac{1}{2t} + \frac{11}{120} + \sum_{k=1}^{m}(-1)^k\left(\frac{B_{2k+2}}{2k+2} + \frac{B_{2k+4}}{2k+4}\right)\frac{t^k}{k!} + o(t^m)$$

Euler Maclaurin formula and spectral action for S^4

For general f the Euler Maclaurin formula gives

$$\frac{3}{4}\mathrm{Tr}(f(tD^2)) = \int_0^\infty f(tx^2)(x^3 - x)dx + \frac{11f(0)}{120} - \frac{31f'(0)}{2520}t$$

$$+\frac{41f''(0)}{10080}t^2 - \frac{31f^{(3)}(0)}{15840}t^3 + \frac{10331f^{(4)}(0)}{8648640}t^4 + \ldots + R_m$$

Levi-Civita Connection and the Spin Connection

Fix a frame $\{\theta_\alpha\}$ and coframe $\{\theta^\alpha\}$. Connection 1-forms

$$\nabla\theta^\alpha = \omega^\alpha_\beta \theta^\beta.$$

Metric connection:

$$\omega^\alpha_\beta = -\omega^\beta_\alpha.$$

Cartan equations: torsion and curvature 2-forms

$$T^\alpha = d\theta^\alpha - \omega^\alpha_\beta \wedge \theta^\beta$$

For torsion free connections:

$$d\theta^\beta = \omega^\beta_\alpha \wedge \theta^\alpha.$$

Connection one-form for Levi-civita connection

Orthonormal basis for the cotangent space

$$\begin{aligned}
\theta^1 &= dt,\\
\theta^2 &= a(t)\, d\chi,\\
\theta^3 &= a(t)\, \sin\chi\, d\theta,\\
\theta^4 &= a(t)\, \sin\chi\, \sin\theta\, d\varphi.
\end{aligned}$$

The computation by Chamseddin-Connes shows that the connection one-form is given by

$$\omega = \begin{bmatrix}
0 & -\frac{a'(t)}{a(t)}\theta^2 & -\frac{a'(t)}{a(t)}\theta^3 & -\frac{a'(t)}{a(t)}\theta^4 \\
\frac{a'(t)}{a(t)}\theta^2 & 0 & -\frac{\cot(\chi)}{a(t)}\theta^3 & -\frac{\cot(\chi)}{a(t)}\theta^4 \\
\frac{a'(t)}{a(t)}\theta^3 & \frac{\cot(\chi)}{a(t)}\theta^3 & 0 & -\frac{\cot(\theta)}{a(t)\sin(\chi)}\theta^4 \\
\frac{a'(t)}{a(t)}\theta^4 & \frac{\cot(\chi)}{a(t)}\theta^4 & \frac{\cot(\theta)}{a\sin(\chi)}\theta^4 & 0
\end{bmatrix}$$

The Spin Connection

The spin connection is the lift of the Levi-Civita connection defined on T^*M. Now we have the connection one-forms ω, which is a skew symmetric matrix, i.e. $\omega \in \mathfrak{so}(4)$. Using the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \to \mathfrak{spin}(4)$ given by

$$A \mapsto \frac{1}{4} \sum_{\alpha,\beta} \langle A\theta^\alpha, \theta^\beta \rangle c(\theta^\alpha) c(\theta^\beta)$$

Since ω is written in the orthonormal basis θ^α so $\langle \omega\theta^\alpha, \theta^\beta \rangle = \omega^\alpha_\beta$. So the connection one forms for the spinor connection is given by

$$\tilde{\omega} = \frac{1}{2}\omega^1_2\gamma^{12} + \frac{1}{2}\omega^1_3\gamma^{13} + \frac{1}{2}\omega^1_4\gamma^{14} + \frac{1}{2}\omega^2_3\gamma^{23} + + \frac{1}{2}\omega^2_4\gamma^{24} + \frac{1}{2}\omega^3_4\gamma^{34}$$

Gilkey's local formulae

For an operator of Laplace type $P = \nabla^*\nabla - E$,

$$a_0 = (4\pi)^{-m/2}\,\mathrm{Tr}(1).$$

$$a_2 = (4\pi)^{-m/2}\,\mathrm{Tr}\big(E - \frac{1}{6}R\big).$$

$$\begin{aligned} a_4 \quad = \quad & \frac{(4\pi)^{-m/2}}{360}\mathrm{Tr}\big(-12R_{ijij;kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} \\ & +2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\big). \end{aligned}$$

$$
\begin{aligned}
a_6 \quad = \quad & (4\pi)^{-m/2}\,\mathrm{Tr}\Big(\frac{1}{7!}\big(-18R_{ijij;kkll}+17R_{ijij;k}R_{ulul;k}-2R_{ijik;l}R_{ujuk;l}\\
&\qquad -4R_{ijik;l}R_{ujul;k}+9R_{ijku;l}R_{ijku;l}+28R_{ijij}R_{kuku;ll}\\
&\qquad -8R_{ijijk}R_{ujuk;ll}+24R_{ijik}R_{ujul;kl}+12R_{ijkl}R_{ijkl;uu}\big)\\
&+\frac{1}{9\cdot 7!}\big(-35R_{ijij}R_{klkl}R_{pqpq}+42R_{ijij}R_{klkp}R_{qlqp}\\
&-42R_{ijij}R_{klpq}R_{klpq}+208R_{ijik}R_{julu}R_{kplp}-192R_{ijik}R_{uplp}R_{jukl}\\
&+48R_{ijik}R_{julp}R_{kulp}-44R_{ijku}R_{ijlp}R_{kulp}-80R_{ijku}R_{ilkp}R_{jlup}\big)\\
&\quad +\frac{1}{360}\big(8\Omega_{ij;k}\Omega_{ij;k}+2\Omega_{ij;j}\Omega_{ik;k}+12\Omega_{ij}\Omega_{ij;kk}-12\Omega_{ij}\Omega_{jk}\Omega_{ki}\\
&\quad -6R_{ijkl}\Omega_{ij}\Omega_{kl}+4R_{ijik}\Omega_{jl}\Omega_{kl}-5R_{ijij}\Omega_{kl}\Omega_{kl}\big)\\
&+\frac{1}{360}\big(6E_{;iijj}+60EE_{;ii}+30E_{;i}E_{;i}+60E^3+30E\Omega_{ij}\Omega_{ij}\\
&-10R_{ijij}E_{;kk}-4R_{ijik}E_{;jk}-12R_{ijij;k}E_{;k}-30R_{ijij}E^2\\
&-12R_{ijij;kk}E+5R_{ijij}R_{klkl}E-2RijikR_{ijkl}E+2R_{ijkl}R_{ijkl}E\big)\Big).
\end{aligned}
$$

For the Dirac operator $D^2 = \nabla^*\nabla - \frac{1}{4}R$, so

$$E = \frac{1}{4}R.$$

Chamseddine-Connes Computations

They used Gilkey's local formulae to obtain

$$a_0 = \frac{a(t)^3}{2}$$

$$a_2 = \frac{1}{4}a(t)\left(a(t)a''(t) + a'(t)^2 - 1\right)$$

$$a_4 = \frac{1}{120}(3a^{(4)}(t)a(t)^2 + 3a(t)a''(t)^2 - 5a''(t) + 9a^{(3)}(t)a(t)a'(t) - 4a'(t)^2a''(t))$$

$$a_6 = \frac{1}{5040a(t)^2}(9a^{(6)}(t)a(t)^4 - 21a^{(4)}(t)a(t)^2 - 3a^{(3)}(t)^2a(t)^3 - 56a(t)^2a''(t)^3 + 42a(t)a''(t)^2 + 36a^{(5)}(t)a(t)^3a'(t) + 6a^{(4)}(t)a(t)^3a''(t) - 42a^{(4)}(t)a(t)^2a'(t)^2 + 60a^{(3)}(t)a(t)a'(t)^3 + 21a^{(3)}(t)a(t)a'(t) + 240a(t)a'(t)^2a''(t)^2 - 60a'(t)^4a''(t) - 21a'(t)^2a''(t) - 252a^{(3)}(t)a(t)^2a'(t)a''(t))$$

Chamseddine-Connes Computations

Using Euler-Maclaurin summation and Feynman-Kac formula they computed up to a_{10}:

$$a_8 = -\frac{1}{10080a(t)^4}(-a^{(8)}(t)a(t)^6 + 3a^{(6)}(t)a(t)^4 + 13a^{(4)}(t)^2a(t)^5 - 24a^{(3)}(t)^2a(t)^3 - 114a(t)^3a''(t)^4 + 43a(t)^2a''(t)^3 - 5a^{(7)}(t)a(t)^5a'(t) + 2a^{(6)}(t)a(t)^5a''(t) + 9a^{(6)}(t)a(t)^4a'(t)^2 + 16a^{(3)}(t)a^{(5)}(t)a(t)^5 - 24a^{(5)}(t)a(t)^3a'(t)^3 - 6a^{(5)}(t)a(t)^3a'(t) + 69a^{(4)}(t)a(t)^4a''(t)^2 - 36a^{(4)}(t)a(t)^3a''(t) + 60a^{(4)}(t)a(t)^2a'(t)^4 + 15a^{(4)}(t)a(t)^2a'(t)^2 + 90a^{(3)}(t)^2a(t)^4a''(t) - 216a^{(3)}(t)^2a(t)^3a'(t)^2 - 108a^{(3)}(t)a(t)a'(t)^5 - 27a^{(3)}(t)a(t)a'(t)^3 + 801a(t)^2a'(t)^2a''(t)^3 - 588a(t)a'(t)^4a''(t)^2 - 87a(t)a'(t)^2a''(t)^2 + 108a'(t)^6a''(t) + 27a'(t)^4a''(t) + 78a^{(5)}(t)a(t)^4a'(t)a''(t) + 132a^{(3)}(t)a^{(4)}(t)a(t)^4a'(t) - 312a^{(4)}(t)a(t)^3a'(t)^2a''(t) - 819a^{(3)}(t)a(t)^3a'(t)a''(t)^2 + 768a^{(3)}(t)a(t)^2a'(t)^3a''(t) + 102a^{(3)}(t)a(t)^2a'(t)a''(t))$$

$$
\begin{aligned}
a_{10} = {} & \frac{1}{665280a(t)^6}(3a^{(10)}(t)a(t)^8 - 222a^{(5)}(t)^2a(t)^7 - 348a^{(4)}(t)a^{(6)}(t)a(t)^7 - 147a^{(3)}(t)a^{(7)}(t)a(t)^7 - 18a''(t)a^{(8)}(t)a(t)^7 + \\
& 18a'(t)a^{(9)}(t)a(t)^7 - 482a''(t)a^{(4)}(t)^2a(t)^6 - 331a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 1110a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - \\
& 1556a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 448a''(t)^2a^{(6)}(t)a(t)^6 - 1074a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 476a'(t)a''(t)a^{(7)}(t)a(t)^6 - \\
& 43a'(t)^2a^{(8)}(t)a(t)^6 - 11a^{(8)}(t)a(t)^6 + 8943a'(t)a^{(3)}(t)^3a(t)^5 + 21846a''(t)^2a^{(3)}(t)^2a(t)^5 + 4092a'(t)^2a^{(4)}(t)^2a(t)^5 + \\
& 396a^{(4)}(t)^2a(t)^5 + 10560a''(t)^3a^{(4)}(t)a(t)^5 + 39402a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 11352a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + \\
& 6336a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 594a^{(3)}(t)a^{(5)}(t)a(t)^5 + 2904a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 264a''(t)a^{(6)}(t)a(t)^5 + \\
& 165a'(t)^3a^{(7)}(t)a(t)^5 + 33a'(t)a^{(7)}(t)a(t)^5 - 10338a''(t)^5a(t)^4 - 95919a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 3729a''(t)a^{(3)}(t)^2a(t)^4 - \\
& 117600a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 68664a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 2772a''(t)^2a^{(4)}(t)a(t)^4 - 23976a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - \\
& 2640a'(t)a^{(3)}(t)a^{(4)}(t)a(t)^4 - 12762a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 1386a'(t)a''(t)a^{(5)}(t)a(t)^4 - 651a'(t)^4a^{(6)}(t)a(t)^4 - \\
& 132a'(t)^2a^{(6)}(t)a(t)^4 + 111378a'(t)^2a''(t)^4a(t)^3 + 2354a''(t)^4a(t)^3 + 31344a'(t)^4a^{(3)}(t)^2a(t)^3 + 3729a'(t)^2a^{(3)}(t)^2a(t)^3 + \\
& 236706a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 13926a'(t)a''(t)^2a^{(3)}(t)a(t)^3 + 43320a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 5214a'(t)^2a''(t)a^{(4)}(t)a(t)^3 + \\
& 2238a'(t)^5a^{(5)}(t)a(t)^3 + 462a'(t)^3a^{(5)}(t)a(t)^3 - 162162a'(t)^4a''(t)^3a(t)^2 - 11880a'(t)^2a''(t)^3a(t)^2 - \\
& 103884a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 13332a'(t)^3a''(t)a^{(3)}(t)a(t)^2 - 6138a'(t)^6a^{(4)}(t)a(t)^2 - 1287a'(t)^4a^{(4)}(t)a(t)^2 + \\
& 76440a'(t)^6a''(t)^2a(t) + 10428a'(t)^4a''(t)^2a(t) + 11700a'(t)^7a^{(3)}(t)a(t) + 2475a'(t)^5a^{(3)}(t)a(t) - 11700a'(t)^8a''(t) - \\
& 2475a'(t)^6a''(t))
\end{aligned}
$$

Conjectures and question about coefficients (CC):

- Check the agreement between the above formulas for a_8 and a_{10} and the universal formulas.

- Show that the term a_{2n} of the asymptotic expansion of the spectral action for Robertson-Walker metric is of the form $P_n(a, \cdots, a^{(2n)})/a^{2n-4}$ where P_n is a polynomial with rational coefficients and compute P_n.

Our approach: spectral analysis via pseudodifferential calculus

$$
\begin{aligned}
D &= \gamma^\alpha \nabla_{\theta_\alpha} = \gamma^\alpha \left(\theta_\alpha + \omega(\theta_\alpha)\right) \\
&= \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{1}{a}\frac{\partial}{\partial \chi} + \gamma^2 \frac{1}{a\,\sin\chi}\frac{\partial}{\partial \theta} + \gamma^3 \frac{1}{a\,\sin\chi\,\sin\theta}\frac{\partial}{\partial \varphi} \\
&+ \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

So the symbol of the Dirac operator would be

$$
\begin{aligned}
\sigma_D(\mathbf{x},\xi) \quad &= i\gamma^0\xi_1 \quad + \frac{i}{a}\gamma^1\xi_2 + \frac{i}{a\sin(\chi)}\gamma^2\xi_3 + \frac{i}{a\sin(\chi)\sin(\theta)}\gamma^3\xi_4 \\
&\qquad\qquad + \frac{3a'}{2a}\gamma^0 + \frac{\cot(\chi)}{a}\gamma^1 + \frac{\cot(\theta)}{2a\sin(\chi)}\gamma^2
\end{aligned}
$$

Symbol of D^2

Using the symbol multiplication rule one can compute the symbol of the square of the Dirac operator. The symbol of D^2 has following homogeneous parts.

$$p_2 = \xi_1^2 + \frac{1}{a(t)^2}\xi_2^2 + \frac{1}{a(t)^2\sin^2(\chi)}\xi_3^2 + \frac{1}{a(t)^2\sin^2(\theta)\sin^2(\chi)}\xi_4^2,$$

$$\begin{aligned} p_1 = &-\frac{3ia'(t)}{a(t)}\xi_1 - \frac{i}{a(t)^2}\left(\gamma^{12}a'(t) + 2\cot(\chi)\right)\xi_2 \\ &- \frac{i}{a(t)^2}\left(\gamma^{13}\csc(\chi)a'(t) + \cot(\theta)\csc^2(\chi) + \gamma^{23}\cot(\chi)\csc(\chi)\right)\xi_3 \\ &- \frac{i}{a(t)^2}\left(\csc(\theta)\csc(\chi)a'(t)\gamma^{14} + \cot(\theta)\csc(\theta)\csc^2(\chi)\gamma^{34} + \csc(\theta)\cot(\chi)\csc(\chi)\gamma^{24}\right)\xi_4, \end{aligned}$$

$$\begin{aligned} p_0 = &+\frac{1}{8a(t)^2}\Big(-12a(t)a''(t) - 6a'(t)^2 + 3\csc^2(\theta)\csc^2(\chi) - \cot^2(\theta)\csc^2(\chi) \\ &+4i\cot(\theta)\cot(\chi)\csc(\chi) - 4i\cot(\theta)\cot(\chi)\csc(\chi) - 4\cot^2(\chi) + 5\csc^2(\chi) + 4\Big) \\ &- \frac{\left(\cot(\theta)\csc(\chi)a'(t)\right)}{2a(t)^2}\gamma^{13} - \frac{\left(\cot(\chi)a'(t)\right)}{a(t)^2}\gamma^{12} - \frac{(\cot(\theta)\cot(\chi)\csc(\chi))}{2a(t)^2}\gamma^{23} \end{aligned}$$

Symbol of the parametrix

Parametrix: $(P-\lambda)\tilde{R}(\lambda) = I$.

$$\sigma(\tilde{R}(\lambda)) = r_0 + r_1 + r_2 + \cdots$$

Recursive formulas:

$$r_n = -r_0 \sum_{|\alpha|+j+2-k=n} (-i)^{|\alpha|} d_\xi^\alpha p_k \cdot d_x^\alpha r_j/\alpha!,$$

where $r_0 = (p_2-\lambda)^{-1} = (\|\xi\|^2-\lambda)^{-1}$. So the summation, for $n>1$, will only have the following possible summands.

$k=0, \lvert\alpha\rvert=0, j=n-2$	$-r_0 p_0 r_{n-2}$
$k=1, \lvert\alpha\rvert=0, j=n-1$	$-r_0 p_1 r_{n-1}$
$k=1, \lvert\alpha\rvert=0, j=n-2$	$ir_0 \frac{\partial}{\partial \xi_0} p_1 \cdot \frac{\partial}{\partial t} r_{n-2} + ir_0 \frac{\partial}{\partial \xi_1} p_1 \cdot \frac{\partial}{\partial \chi} r_{n-2} + ir_0 \frac{\partial}{\partial \xi_2} p_1 \cdot \frac{\partial}{\partial \theta} r_{n-2}$
$k=2, \lvert\alpha\rvert=1, j=n-1$	$ir_0 \frac{\partial}{\partial \xi_0} p_2 \cdot \frac{\partial}{\partial t} r_{n-1} + ir_0 \frac{\partial}{\partial \xi_1} p_2 \cdot \frac{\partial}{\partial \chi} r_{n-1} + ir_0 \frac{\partial}{\partial \xi_2} p_2 \cdot \frac{\partial}{\partial \theta} r_{n-1}$
$k=2, \lvert\alpha\rvert=2, j=n-2$	$\frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_0^2} p_2 \cdot \frac{\partial^2}{\partial t^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_1^2} p_2 \cdot \frac{\partial^2}{\partial \chi^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_2^2} p_2 \cdot \frac{\partial^2}{\partial \theta^2} r_{n-2}$

Heat Kernel of D^2 in terms of symbols of the parametrix.

Let

$$\begin{aligned}
e_n &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)d\lambda d\xi \\
&= \frac{1}{2\pi i(2\pi)^4}\sum r_{n,j,\alpha}(x)\int_{\mathbb{R}^4}\xi^\alpha\int_\gamma e^{-t\lambda} r_0^j d\lambda d\xi \\
&= \sum c_\alpha \frac{1}{(j-1)!} r_{n,j,\alpha} a(t)^{\alpha_2+\alpha_3+\alpha_4+3}\sin(\chi)^{\alpha_3+\alpha_4+2}\sin(\theta)^{\alpha_4+1}
\end{aligned}$$

Where $c_\alpha = \frac{1}{(2\pi)^4}\prod_k \Gamma\left(\frac{\alpha_k+1}{2}\right)\frac{(-1)^{\alpha_k}+1}{2}$.

where

$$a_n = \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} e_n d\chi d\theta d\phi$$

new term a_{12}

$$
\begin{aligned}
a_{12} = {} & \frac{1}{17297280a(t)^8}\Big(3a^{(12)}(t)a(t)^{10} - 1057a^{(6)}(t)^2a(t)^9 - 1747a^{(5)}(t)a^{(7)}(t)a(t)^9 - 970a^{(4)}(t)a^{(8)}(t)a(t)^9 - {} \\
& 317a^{(3)}(t)a^{(9)}(t)a(t)^9 - 34a''(t)a^{(10)}(t)a(t)^9 + 21a'(t)a^{(11)}(t)a(t)^9 + 5001a^{(4)}(t)^3a(t)^8 + 2419a''(t)a^{(5)}(t)^2a(t)^8 + {} \\
& 19174a^{(3)}(t)a^{(4)}(t)a^{(5)}(t)a(t)^8 + 4086a^{(3)}(t)^2a^{(6)}(t)a(t)^8 + 2970a''(t)a^{(4)}(t)a^{(6)}(t)a(t)^8 - 5520a'(t)a^{(5)}(t)a^{(6)}(t)a(t)^8 - {} \\
& 511a''(t)a^{(3)}(t)a^{(7)}(t)a(t)^8 - 4175a'(t)a^{(4)}(t)a^{(7)}(t)a(t)^8 - 745a''(t)^2a^{(8)}(t)a(t)^8 - 2289a'(t)a^{(3)}(t)a^{(8)}(t)a(t)^8 - {} \\
& 828a'(t)a''(t)a^{(9)}(t)a(t)^8 - 62a'(t)^2a^{(10)}(t)a(t)^8 - 13a^{(10)}(t)a(t)^8 + 45480a^{(3)}(t)^4a(t)^7 + 152962a''(t)^2a^{(4)}(t)^2a(t)^7 + {} \\
& 203971a'(t)a^{(3)}(t)a^{(4)}(t)^2a(t)^7 + 21369a'(t)^2a^{(5)}(t)^2a(t)^7 + 1885a^{(5)}(t)^2a(t)^7 + 410230a''(t)a^{(3)}(t)^2a^{(4)}(t)a(t)^7 + {} \\
& 163832a'(t)a^{(3)}(t)^2a^{(5)}(t)a(t)^7 + 250584a''(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^7 + 244006a'(t)a''(t)a^{(4)}(t)a^{(5)}(t)a(t)^7 + {} \\
& 42440a''(t)^3a^{(6)}(t)a(t)^7 + 163390a'(t)a''(t)a^{(3)}(t)a^{(6)}(t)a(t)^7 + 35550a'(t)^2a^{(4)}(t)a^{(6)}(t)a(t)^7 + 3094a^{(4)}(t)a^{(6)}(t)a(t)^7 + {} \\
& 34351a'(t)a''(t)^2a^{(7)}(t)a(t)^7 + 19733a'(t)^2a^{(3)}(t)a^{(7)}(t)a(t)^7 + 1625a^{(3)}(t)a^{(7)}(t)a(t)^7 + 6784a'(t)^2a''(t)a^{(8)}(t)a(t)^7 + {} \\
& 520a''(t)a^{(8)}(t)a(t)^7 + 308a'(t)^3a^{(9)}(t)a(t)^7 + 52a'(t)a^{(9)}(t)a(t)^7 - 2056720a'(t)a''(t)a^{(3)}(t)^3a(t)^6 - {} \\
& 1790580a''(t)^3a^{(3)}(t)^2a(t)^6 - 900272a'(t)^2a''(t)a^{(4)}(t)^2a(t)^6 - 31889a''(t)a^{(4)}(t)^2a(t)^6 - 643407a''(t)^4a^{(4)}(t)a(t)^6 - {} \\
& 1251548a'(t)^2a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 43758a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 4452042a'(t)a''(t)^2a^{(3)}(t)a^{(4)}(t)a(t)^6 - {} \\
& 836214a'(t)a''(t)^3a^{(5)}(t)a(t)^6 - 1400104a'(t)^2a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 48620a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - {} \\
& 181966a'(t)^3a^{(4)}(t)a^{(5)}(t)a(t)^6 - 18018a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 319996a'(t)^2a''(t)^2a^{(6)}(t)a(t)^6 - 11011a''(t)^2a^{(6)}(t)a(t)^6 - {} \\
& 115062a'(t)^3a^{(3)}(t)a^{(6)}(t)a(t)^6 - 11154a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 42764a'(t)^3a''(t)a^{(7)}(t)a(t)^6 - 4004a'(t)a''(t)a^{(7)}(t)a(t)^6 - {} \\
& 1649a'(t)^4a^{(8)}(t)a(t)^6 - 286a'(t)^2a^{(8)}(t)a(t)^6 + 460769a''(t)^6a(t)^5 + 1661518a'(t)^3a^{(3)}(t)^3a(t)^5 + 83486a'(t)a^{(3)}(t)^3a(t)^5 + {} \\
& 13383328a'(t)^2a''(t)^2a^{(3)}(t)^2a(t)^5 + 222092a''(t)^2a^{(3)}(t)^2a(t)^5 + 342883a'(t)^4a^{(4)}(t)^2a(t)^5 + 36218a'(t)^2a^{(4)}(t)^2a(t)^5 + {} \\
& 7922361a'(t)a''(t)^4a^{(3)}(t)a(t)^5 + 6367314a'(t)^2a''(t)^3a^{(4)}(t)a(t)^5 + 109330a''(t)^3a^{(4)}(t)a(t)^5 +
\end{aligned}
$$

$$+7065862a'(t)^3a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 360386a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 1918386a'(t)^3a''(t)^2a^{(5)}(t)a(t)^5 + 98592a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + 524802a'(t)^4a^{(3)}(t)a^{(5)}(t)a(t)^5 + 55146a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 226014a'(t)^4a''(t)a^{(6)}(t)a(t)^5 + 23712a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 8283a'(t)^5a^{(7)}(t)a(t)^5 + 1482a'(t)^3a^{(7)}(t)a(t)^5 - 7346958a'(t)^2a''(t)^5a(t)^4 - 72761a''(t)^5a(t)^4 - 11745252a'(t)^4a''(t)a^{(3)}(t)^2a(t)^4 - 725712a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 27707028a'(t)^3a''(t)^3a^{(3)}(t)a(t)^4 - 819520a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 8247105a'(t)^4a''(t)^2a^{(4)}(t)a(t)^4 - 520260a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 1848228a'(t)^5a^{(3)}(t)a^{(4)}(t)a(t)^4 - 205296a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - 973482a'(t)^5a''(t)a^{(5)}(t)a(t)^4 - 110136a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 36723a'(t)^6a^{(6)}(t)a(t)^4 - 6747a'(t)^4a^{(6)}(t)a(t)^4 + 17816751a'(t)^4a''(t)^4a(t)^3 + 721058a'(t)^2a''(t)^4a(t)^3 + 2352624a'(t)^6a^{(3)}(t)^2a(t)^3 + 274170a'(t)^4a^{(3)}(t)^2a(t)^3 + 24583191a'(t)^5a''(t)^2a^{(3)}(t)a(t)^3 + 1771146a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 3256248a'(t)^6a''(t)a^{(4)}(t)a(t)^3 + 389376a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 135300a'(t)^7a^{(5)}(t)a(t)^3 + 25350a'(t)^5a^{(5)}(t)a(t)^3 - 15430357a'(t)^6a''(t)^3a(t)^2 - 1252745a'(t)^4a''(t)^3a(t)^2 - 7747848a'(t)^7a''(t)a^{(3)}(t)a(t)^2 - 967590a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 385200a'(t)^8a^{(4)}(t)a(t)^2 - 73125a'(t)^6a^{(4)}(t)a(t)^2 + 5645124a'(t)^8a''(t)^2a(t) + 741195a'(t)^6a''(t)^2a(t) + 749700a'(t)^9a^{(3)}(t)a(t) + 143325a'(t)^7a^{(3)}(t)a(t) - 749700a'(t)^{10}a''(t) - 143325a'(t)^8a''(t))\Big)$$

Check on round sphere S^4

For $a(t) = sin(t)$ we have

$$a_{12}(sphere) = \frac{10331 \sin^3(t)}{8648640}.$$

Hence

$$\int_0^\pi a_{12}(spher)dt = \frac{4}{3}\frac{10331}{8648640} = \frac{10331}{6486480}.$$

Which agrees with the direct computation done in Connes-Chamseddine.

Rationality of heat coefficients

Theorem: The terms a_{2n} in the expansion of the spectral action for the Robertson-Walker metric with scale factor $a(t)$ is of the form

$$\frac{1}{a(t)^{2n-3}}\, Q_{2n}\left(a(t), a'(t), \ldots, a^{(2n)}(t)\right),$$

where Q_{2n} is a polynomial with *rational* coefficients.

By direct computation in Hopf coordinates, we found the vector fields which respectively form bases for left and right invariant vector fields on $\mathrm{SU}(2)$:

$$
\begin{aligned}
X_1^L &= \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\
X_2^L &= \sin(\phi_1+\phi_2)\frac{\partial}{\partial \eta} + \cot(\eta)\cos(\phi_1+\phi_2)\frac{\partial}{\partial \phi_1} - \\
&\quad \tan(\eta)\cos(\phi_1+\phi_2)\frac{\partial}{\partial \phi_2}, \\
X_3^L &= \cos(\phi_1+\phi_2)\frac{\partial}{\partial \eta} - \cot(\eta)\sin(\phi_1+\phi_2)\frac{\partial}{\partial \phi_1} + \\
&\quad \tan(\eta)\sin(\phi_1+\phi_2)\frac{\partial}{\partial \phi_2},
\end{aligned}
$$

and X_1^R, X_2^R, X_3^R. One checks that these vector fields are Killing vector fields for the Robertson-Walker metrics on the four dimensional space.

Warm up: zeta regularized determinants

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- Given a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \qquad \mathsf{spec}(\Delta)$$

  How one defines $\prod \lambda_i = \mathsf{det}\Delta$?

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- Define the spectral zeta function:

$$\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \qquad Re(s) \gg 0$$

Assume: $\zeta_\Delta(s)$ has meromorphic extension to $\mathbb{C}$ and is regular at 0.

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- Zeta regularized determinant:

$$\prod \lambda_i := e^{-\zeta'_\Delta(0)} = \det\Delta$$

Holomorphic determinants

- Example: For Riemann zeta function, $\zeta'(0) = -\log\sqrt{2\pi}$. Hence

$$1 \cdot 2 \cdot 3 \cdots = \sqrt{2\pi}.$$

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- Quillen's approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.

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- Recall: Space of Fredholm operators:

$$F = \mathrm{Fred}(H_0, H_1) = \{T : H_0 \to H_1; \ \ T \text{ is Fredholm}\}$$

$$K_0(X) = [X, F], \quad \text{classifying space for K-theory}$$

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- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.
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The determinant line bundle

- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.

- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.
$$(DET)_T = \lambda(KerT)^* \otimes \lambda(KerT^*)$$

  2) There map $\sigma : F_0 \to DET$
$$\sigma(T) = \begin{cases} 1 & T \quad invertible \\ 0 & otherwise \end{cases}$$

  is a holomorphic section of DET over F_0.

Cauchy-Riemann operators on $\mathcal{A}_\theta$

- Families of spectral triples

$$\mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, \quad \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix},$$

with $\alpha \in \mathcal{A}_\theta,\ \bar{\partial} = \delta_1 + \tau\delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar{\partial} + \alpha$.

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- Let $\mathcal{A} =$ space of elliptic operators $D = \bar{\partial} + \alpha$.

- Pull back DET to a holomorphic line bundle $\mathcal{L} \to \mathcal{A}$ with

$$\mathcal{L}_D = \lambda(KerD)^* \otimes \lambda(KerD^*).$$

From det section to det function

- If $\mathcal{L}$ admits a canonical global holomorphic frame s, then

$$\sigma(D) = det(D)s$$

  defines a holomorphic determinant function $\det(D)$. A canonical frame is defined once we have a canonical flat holomorphic connection.

Quillen's metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $Index(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta, \quad \Delta = D^* D.$$

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

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- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

- A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from

$$\bar{\partial}\partial \log ||s||^2,$$

where s is any local holomorphic frame.

Connes' pseudodifferential calculus

- To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.
- Symbols of order m: smooth maps $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ with

$$||\delta^{(i_1,i_2)}\partial^{(j_1,j_2)}\sigma(\xi)|| \leq c(1+|\xi|)^{m-j_1-j_2}.$$

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- To a symbol σ of order m, one associates an operator

$$P_\sigma(a) = \int\int e^{-is\cdot\xi}\sigma(\xi)\alpha_s(a)\, ds\, d\xi.$$

  The operator $P_\sigma : \mathcal{A}_\theta \to \mathcal{A}_\theta$ is said to be a pseudodifferential operator of order m.

Classical symbols

- Classical symbol of order $\alpha \in \mathbb{C}$:

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{\alpha-j} \qquad \operatorname{ord} \sigma_{\alpha-j} = \alpha - j.$$

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.$$

- We denote the set of classical symbols of order α by $\mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$.

A cutoff integral

- Any pseudo P_σ of order < -2 is trace-class with
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- For $\mathrm{ord}(P) \geq -2$ the integral is divergent, but, assuming P is classical, and of non-integral order, one has an asymptotic expansion as $R \to \infty$

$$\int_{B(R)} \sigma(\xi) d\xi \sim \sum_{j=0, \alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma) \log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi =$ Wodzicki residue of P (Fathizadeh).

The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi) d\xi$.

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- The canonical trace of a classical pseudo $P \in \Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$ of non-integral order α is defined as

$$\mathrm{TR}(P) := \varphi_0 \left(⨍ \sigma_P(\xi) d\xi \right).$$

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- NC residue in terms of TR:

$$\mathrm{Res}_{z=0} \mathrm{TR}(AQ^{-z}) = \frac{1}{q} \mathrm{Res}(A).$$

Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

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- Example: $\log Q$ where $Q \in \Psi^q_{cl}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$.

- Wodzicki residue: $\mathrm{Res}(A) = \varphi_0(\mathrm{res}(A))$,

$$\mathrm{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.$$

Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section σ,

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}$$

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- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$. Want to compute

$$\bar{\partial}\partial \log \|\sigma\|^2 = \delta_{\bar{w}}\delta_w \zeta'_\Delta(0) = \delta_{\bar{w}}\delta_w \frac{d}{dz}\mathrm{TR}(\Delta^{-z})|_{z=0}.$$

The second variation of logDet

- Prop 1: For a holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ is given by :

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{2}\varphi_0\left(\delta_w D\delta_{\bar{w}}\mathrm{res}(\log\Delta\, D^{-1})\right).$$

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- Prop 2: The residue density of $\log\Delta\, D^{-1}$:

$$\sigma_{-2,0}(\log\Delta\, D^{-1}) = \frac{(\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2}{(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)(\xi_1+\tau\xi_2)}$$

$$-\log\left(\frac{\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2}{|\xi|^2}\right)\frac{\alpha}{\xi_1+\tau\xi_2},$$

and

$$\delta_{\bar{w}}\mathrm{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Curvature of the determinant line bundle

- Theorem (A. Fathi, A. Ghorbanpour, MK.): The curvature of the determinant line bundle for the noncommutative two torus is given by
$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right).$$

- Remark: For $\theta = 0$ this reduces to Quillen's theorem (for elliptic curves).

A holomorphic determinant a la Quillen

- Modify the metric to get a flat connection:

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- Modify the metric to get a flat connection:

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- Get a flat holomorphic global section. This gives a holomorphic determinant function

$$det(D, D_0) : \mathcal{A} \to \mathbb{C}$$

It satisfies

$$|det(D, D_0)|^2 = e^{||D-D_0||^2} det_\zeta(D^*D)$$

Figure : Hermann Weyl in Göttingen