Spectral Zeta Functions and Scalar Curvature for Noncommutative Tori

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- A. Connes and H. Moscovici, *Modular curvature for noncommutative two-tori,* Oct. 2011.

- F. Fathizadeh and M. Khalkhali, *The Gauss-Bonnet Theorem for noncommutative two tori with a general conformal structure*, May 2010.

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From heat trace and zeta functions to spectral invariants

(M, g) = closed Riemannian manifold. Laplacian on functions

$$\triangle = d^*d : C^\infty(M) \to C^\infty(M)$$

It is positive with pure point spectrum. Let $k(t, x, y)$ = kernel of $e^{-t\triangle}$:

$$k(t, x, y) \sim \frac{1}{(4\pi t)^{m/2}} e^{-d(x,y)^2/4t}, \quad (t \to 0)$$

Restrict to the diagonal: as $t \to 0$,

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}} (a_0(x, \triangle) + a_1(x, \triangle)t + a_2(x, \Delta)t^2 + \cdots)$$

- Functions $a_i(x, \triangle)$: expressed by universal polynomials in curvature tensor R and its covariant derivatives:

$$
\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}
$$

Short time asymptotics of the heat trace

$$\begin{aligned}\mathrm{Trace}(e^{-t\triangle}) &= \sum e^{-t\lambda_i} = \int_M k(t,x,x)d\mathrm{vol}_x \\ &\sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)\end{aligned}$$

So

$$a_j = \int_M a_j(x, \triangle) d\mathrm{vol}_x,$$

are manifestly spectral invariants.

$$\begin{aligned}a_0 &= \int_M d\mathrm{vol}_x = \mathrm{Vol}(M), && \Longrightarrow \text{Weyl's law} \\ a_1 &= \frac{1}{6}\int_M S(x) d\mathrm{vol}_x, && \text{total scalar curvature}\end{aligned}$$

Spectral zeta functions

$$\zeta_\triangle(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion $\Longrightarrow$

$$\zeta_\triangle(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle}) - \mathrm{Dim}\,\mathrm{Ker}\,\triangle) t^{s-1}\, dt$$

has meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Let $\zeta_f(s) := \operatorname{Tr}(f\triangle^{-s}),\ f \in C^\infty(M)$.

$$\operatorname{Res}\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi}\int_M fS(x)dvol_x - \operatorname{Tr}(fP) \qquad m = 2$$

$$\log\det(\triangle) = -\zeta'(0), \qquad \text{regularized determinant}$$

Spectral Triples: $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A}$= involutive unital algebra, $\mathcal{H}$ = Hilbert space,

  $$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \qquad D : \mathcal{H} \to \mathcal{H}$$

  D has compact resolvent and all commutators $[D, \pi(a)]$ are bounded.

- Assume: an asymptotic expansion of the form

  $$\mathrm{Trace}\,(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

  holds.

- Let $\Delta = D^2$. Spectral zeta function

  $$\zeta_D(s) = \mathrm{Tr}\,(|D|^{-s}) = \mathrm{Tr}\,(\Delta^{-s/2}), \quad \mathrm{Re}(s) \gg 0.$$

- Using the Mellin transform and the asymptotic expansion, easy to show that: ζ_D has a meromorphic extension to all of $\mathbb{C}$ and non-zero terms a_α, $\alpha < 0$, give a pole of ζ_D at -2α with

$$\mathrm{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2a_\alpha}{\Gamma(-\alpha)}.$$

Also, $\zeta_D(s)$ is holomorphic at $s = 0$ and

$$\zeta_D(0) + \dim \ker D = a_0$$

Curved noncommutative tori A_θ, $\theta \in \mathbb{R}$

A_θ = universal C^*-algebra generated by unitaries U and V

$$VU = e^{2\pi i\theta} UV.$$

$$A_\theta^\infty = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n \,:\, (a_{m,n}) \,\text{Schwartz class}\Big\}.$$

- Differential operators on A_θ

$$\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty,$$

Infinitesimal generators of the action

$$\alpha_s(U^m V^n) = e^{is.(m,n)} U^m V^n \quad s \in \mathbb{R}^2.$$

Analogues of $\frac{1}{i}\frac{\partial}{\partial x}$, $\frac{1}{i}\frac{\partial}{\partial y}$ on 2-torus.

- Canonical trace $\mathfrak{t} : A_\theta \to \mathbb{C}$ on smooth elements:

$$\mathfrak{t}\Big(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\Big) = a_{0,0}.$$

Complex structures on A_θ

- $\tau = \tau_1 + i\tau_2, \qquad \tau_2 = \Im(\tau) > 0,$

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

- Hilbert space of $(1,0)$-forms:

  $\mathcal{H}^{(1,0)}$:= completion of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, w.r.t.

$$\langle a\partial b, a'\partial b'\rangle := \mathfrak{t}((a'\partial b')^* a\partial b).$$

- Flat Dolbeault Laplacian: $\partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$.

Conformal perturbation of a metric

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form $\mathfrak{t}$ by $\varphi : A_\theta \to \mathbb{C}$,
$$\varphi(a) := \mathfrak{t}(ae^{-h}), \quad a \in A_\theta.$$
- It is a KMS state with modular group
$$\sigma_t(x) = e^{ith} x e^{-ith},$$
and modular automorphism (Tomita-Takesaki theory)
$$\sigma_i(x) = \Delta(x) = e^{-h} x e^{h}.$$
$$\varphi(ab) = \varphi(b\Delta(a)), \quad \forall a, b \in A_\theta.$$
- Warning: $\triangle$ and Δ are very different operators!

Connes-Tretkoff spectral triple

- Hilbert space $\mathcal{H}_\varphi :=$ completion of A_θ w.r.t. $\langle , \rangle_\varphi$,

$$\langle a, b\rangle_\varphi := \varphi(b^* a), \quad a, b \in A_\theta$$

  (GNS construction).

- View

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Full perturbed Laplacian:

$$\triangle := D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Lemma: $\partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$, and $\partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}$ are anti-unitarily equivalent to

$$k\partial^* \partial k : \mathcal{H}_0 \to \mathcal{H}_0,$$

$$\partial^* k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)},$$

where $k = e^{h/2}$.

The Tomita anti-unitary map J is used.

Scalar curvature for A_θ

- The scalar curvature of the curved nc torus $(\mathbb{T}^2_\theta, \tau, k)$ is the unique element $R \in A^\infty_\theta$ satisfying

$$\mathrm{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \mathrm{Trace}\,(aP) = \mathfrak{t}\,(aR), \qquad \forall a \in A^\infty_\theta,$$

  where P is the projection onto the kernel of $\triangle$.

- The chiral scalar curvature R^γ is the unique element $R^\gamma \in A^\infty_\theta$ which satisfies the equation

$$\mathrm{Trace}\,(\gamma a\triangle^{-s})_{|_{s=0}} + \mathfrak{t}(ae^{-h})/\mathfrak{t}(e^{-h}) - \mathfrak{t}(a) = \mathfrak{t}\,(aR^\gamma), \qquad \forall a \in A^\infty_\theta.$$

- We find a formula for R using Connes' pseudodifferential calculus.

Connes' pseudodifferential calculus

- Symbols: $\rho : \mathbb{R}^2 \to A_\theta^\infty$.
- ΨDO's: $P_\rho : A_\theta^\infty \to A_\theta^\infty$,

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi.$$

- For example:

$$\rho(\xi_1, \xi_2) = \sum a_{ij} \xi_1^i \xi_2^j, \quad a_{ij} \in A_\theta^\infty \quad \Rightarrow \quad P_\rho = \sum a_{ij} \delta_1^i \delta_2^j.$$

- Multiplication of symbol.

$$\sigma(PQ) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1! \ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).$$

Local expression for the scalar curvature

- Cauchy integral formula:

$$e^{-t\triangle} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\triangle - \lambda)^{-1} \, d\lambda.$$

- $B_\lambda \approx (\triangle - \lambda)^{-1}$:

$$\sigma(B_\lambda) \sim b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots ,$$

  each $b_j(\xi, \lambda)$ is a symbol of order $-2 - j$, and

$$\sigma(B_\lambda(\triangle - \lambda)) \sim 1.$$

  (Note: λ is considered of order 2.)

Proposition: The scalar curvature of the spectral triple attached to (A_θ, τ, k) is equal to

$$\frac{1}{2\pi i}\int_{\mathbb{R}^2}\int_C e^{-\lambda} b_2(\xi,\lambda)\, d\lambda\, d\xi,$$

where b_2 is defined as above.

The computations for $k\partial^*\partial k$

- The symbol of $k\partial^*\partial k$ is equal to

$$a_2(\xi) + a_1(\xi) + a_0(\xi)$$

where

$$a_2(\xi) = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,$$

$$a_1(\xi) = 2\xi_1 k \delta_1(k) + 2|\tau|^2 \xi_2 k \delta_2(k) + 2\tau_1 \xi_1 k \delta_2(k) + 2\tau_1 \xi_2 k \delta_1(k),$$

$$a_0(\xi) = k\delta_1^2(k) + |\tau|^2 k \delta_2^2(k) + 2\tau_1 k \delta_1 \delta_2(k).$$

- The equation

$$(b_0 + b_1 + b_2 + \cdots)((a_2 + 1) + a_1 + a_0) \sim 1,$$

has a solution with each b_j a symbol of order $-2 - j$.

$$b_0 = (a_2+1)^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + 1)^{-1},$$

$$b_1 \quad = \quad -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0),$$

$$\begin{aligned} b_2 \quad = \quad & -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1)b_0 + \partial_2(b_0)\delta_2(a_1)b_0 + \\ & \partial_1(b_1)\delta_1(a_2)b_0 + \partial_2(b_1)\delta_2(a_2)b_0 + (1/2)\partial_{11}(b_0)\delta_1^2(a_2)b_0 + \\ & (1/2)\partial_{22}(b_0)\delta_2^2(a_2)b_0 + \partial_{12}(b_0)\delta_{12}(a_2)b_0) \end{aligned}$$

$$\begin{aligned} = \quad & 5\xi_1^2 b_0^2 k^3 \delta_1^2(k) b_0 + 2\xi_1^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k \\ & + \text{about 800 terms.} \end{aligned}$$

To integrate b_2 over the ξ-plane, pass to the coordinates

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta, \qquad \xi_2 = \frac{r}{\tau_2}\sin\theta.$$

After the integration with respect to θ, up to a factor of $\frac{r}{\tau_2}$ which is the Jacobian of the change of variables, one gets

$$4|\tau|^2\pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0 k + 4\tau_1 \pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0 k$$

$$+ \ldots \quad \text{(more than 80 terms)}$$

Terms with two b_0^i factors

These are the following terms

$$-4\pi r^4 b_0^3 k^4 \delta_1^2(k) b_0 k - 4|\tau|^2 \pi r^4 b_0^3 k^4 \delta_2^2(k) b_0 k + ...(23 \text{ terms})$$

where

$$b_0 = (r^2 k^2 + 1)^{-1}.$$

Rearrangement lemma (Connes)

The computation of $\int_0^\infty \bullet \, rdr$ of these terms is achieved by:

For all $\rho_j \in A_\theta^\infty$ and $m_j > 0$ one has

$$\int_0^\infty (k^2u+1)^{-m_0}\rho_1(k^2u+1)^{-m_1}\cdots\rho_\ell(k^2u+1)^{-m_\ell}u^{\sum m_j - 2}du$$

$$= k^{-2(\sum m_j - 1)}F_{m_0,m_1,\cdots,m_\ell}(\Delta_{(1)},\Delta_{(2)},\cdots,\Delta_{(\ell)})(\rho_1\rho_2\cdots\rho_\ell),$$

where

$$F_{m_0,m_1,\cdots,m_\ell}(u_1,u_2,\cdots,u_\ell) = \int_0^\infty (u+1)^{-m}\prod_1^\ell (u\prod_1^j u_h + 1)^{-m_j}u^{\sum m_j - 2}$$

and $\Delta_{(i)}$ signifies that Δ acts on the i-th factor.

Up to an overall factor of π, $\int_0^\infty \bullet \, r dr$ of the terms with two positive powers of b_0 is equal to

$$\begin{array}{rl} & f_1(\Delta)(k^{-1}\delta_1^2(k)) + f_2(\Delta)(k^{-2}\delta_1(k)^2) \\ + & |\tau|^2 f_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 f_2(\Delta)(k^{-2}\delta_2(k)^2) \\ + & \tau_1 f_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\ + & \tau_1 f_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)), \end{array}$$

where

$$\begin{array}{rcl} f_1(u) & := & -2\mathcal{L}_2(u)u^{1/2} - 2\mathcal{L}_2(u) + \mathcal{L}_1(u)u^{1/2} + 3\mathcal{L}_1(u) - \mathcal{L}_0(u) \\ & = & -\dfrac{u^{1/2}(2 - 2u + (1+u)\log u)}{(-1+u^{1/2})^3(1+u^{1/2})^2}, \end{array}$$

and

$$f_2(u) := -4\mathcal{L}_2(u) + 4\mathcal{L}_1(u) = 2\frac{-1 + u^2 - 2u \log u}{(-1+u)^3}.$$

Terms with three b_0^i

These terms are the following:

$$4|\tau|^2\pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0 k + 4\tau_1 \pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0 k$$

$$+ \cdots \quad \text{(62 terms)}$$

For computing $\int_0^\infty \bullet\, rdr$ of these terms we again use the *Rearrangement Lemma*.

After the integrations, up to an overall factor of π, we find the following expression

$$
\begin{aligned}
& f_1(\Delta)(k^{-1}\delta_1^2(k)) + f_2(\Delta)(k^{-2}\delta_1(k)^2) \\
+ \quad & F(\Delta_{(1)},\Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k))) \\
+ \quad & |\tau|^2 f_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 f_2(\Delta)(k^{-2}\delta_2(k)^2) \\
+ \quad & |\tau|^2 F(\Delta_{(1)},\Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \quad & \tau_1 f_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\
+ \quad & \tau_1 F(\Delta_{(1)},\Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \quad & \tau_1 f_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)) \\
+ \quad & \tau_1 F(\Delta_{(1)},\Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k))),
\end{aligned}
$$

where we have

$$f_1(u) = -\frac{u^{1/2}(2-2u+(1+u)\log u)}{(-1+u^{1/2})^3(1+u^{1/2})^2},$$

$$f_2(u) = 2\frac{-1+u^2-2u\log u}{(-1+u)^3},$$

$F(u,v) =$

$$\begin{aligned}
&2\mathcal{D}_{2,2}(u,v)u^{-1}v^{1/2} + 2\mathcal{D}_{2,2}(u,v)u^{-1} + 2\mathcal{D}_{2,2}(u,v)u^{-3/2}v^{1/2}\\
&+2\mathcal{D}_{2,2}(u,v)u^{-3/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2}\\
&+4\mathcal{D}_{3,1}(u,v)u^{-5/2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-5/2} - 2\mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2}\\
&-2\mathcal{D}_{1,2}(u,v)u^{-1/2} - 4\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} - 6\mathcal{D}_{2,1}(u,v)u^{-1}\\
&-6\mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - 8\mathcal{D}_{2,1}(u,v)u^{-3/2} + 2\mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2}\\
&+4\mathcal{D}_{1,1}(u,v)u^{-1/2}
\end{aligned}$$

$=$

$$(2u(-(((-1+uv)(1+\sqrt{u}(-1-\sqrt{v}-(-2+\sqrt{u}+u)v+uv^{3/2})))/$$
$$((-1+\sqrt{u})(-1+\sqrt{v}))+(\sqrt{u}\sqrt{v}(-1-\sqrt{u}+u+u(-2-\sqrt{u}+2u)$$
$$\sqrt{v}+u(-1+\sqrt{u}+u)v+u^{5/2}v^{3/2})\log u)/((-1+\sqrt{u})^2(1+\sqrt{u}))$$
$$+(\sqrt{v}(1-$$
$$\sqrt{u}\sqrt{v}(-1-\sqrt{v}+v+uv(-1+\sqrt{v}+v)+\sqrt{u}(-2+\sqrt{v}+2v)))\log v)/$$
$$((-1+\sqrt{v})^2(1+\sqrt{v}))))/(-1+uv)^3.$$

Computations for $\partial^* k^2 \partial$ on $(1,0)$-forms

The symbol of $\partial^* k^2 \partial$ is equal to $c_2(\xi) + c_1(\xi)$ where

$$c_2(\xi) = \xi_1^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + |\tau|^2 \xi_2^2 k^2,$$

$$c_1(\xi) = (\delta_1(k^2) + \overline{\tau}\delta_2(k^2))\xi_1 + (\tau\delta_1(k^2) + |\tau|^2 \delta_2(k^2))\xi_2.$$

After similar computations, the second component of the scalar curvature is:

$$
\begin{aligned}
& g_1(\Delta)(k^{-1}\delta_1^2(k)) + g_2(\Delta)(k^{-2}\delta_1(k)^2) \\
+ \quad & G(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k))) \\
+ \quad & |\tau|^2 g_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 g_2(\Delta)(k^{-2}\delta_2(k)^2) \\
+ \quad & |\tau|^2 G(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \quad & \tau_1 g_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\
+ \quad & \tau_1 G(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k))) \\
+ \quad & \tau_1 g_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)) \\
+ \quad & \tau_1 G(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k))) \\
+ \quad & i\tau_2 L(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k))) \\
- \quad & i\tau_2 L(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))
\end{aligned}
$$

$$g_1(u) = \frac{-1 + u^2 - 2u \log u}{(-1 + u^{1/2})^3 (1 + u^{1/2})^2},$$

$$g_2(u) = 2\frac{-1 + u^2 - 2u \log u}{(-1 + u)^3},$$

$G(u, v) =$

$$\begin{aligned}
&2\mathcal{D}_{2,2}(u,v)u^{-1}v^{1/2} + 2\mathcal{D}_{2,2}(u,v)u^{-1} + 2\mathcal{D}_{2,2}(u,v)u^{-3/2}v^{1/2} \\
&+2\mathcal{D}_{2,2}(u,v)u^{-3/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2} \\
&+4\mathcal{D}_{3,1}(u,v)u^{-5/2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-5/2} - 4\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
&-4\mathcal{D}_{2,1}(u,v)u^{-1} - 4\mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - 4\mathcal{D}_{2,1}(u,v)u^{-3/2} \\
&-\mathcal{D}_{1,2}(u,v)v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2} \\
&-\mathcal{D}_{1,2}(u,v) - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
&-\mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-1} - \mathcal{D}_{2,1}(u,v)u^{-1}
\end{aligned}$$

$$
\begin{aligned}
& -\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} \\
& +\mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2} + \mathcal{D}_{1,1}(u,v)v^{1/2} + \mathcal{D}_{1,1}(u,v)u^{-1/2} \\
& +\mathcal{D}_{1,1}(u,v)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -(\sqrt{u}(u(-1+v)^2(-1+uv(-4+u(4+v)))\log(1/u) + (-1+u) \\
& ((1+u(-2+v))(-1+v)(-1+uv)(1+uv) + (-1+u)v \\
& (-1+u(-4+v(4+uv)))\log v)))/((-1+\sqrt{u})^2(1+\sqrt{u})(-1+\sqrt{v})^2 \\
& (1+\sqrt{v})(-1+uv)^3),
\end{aligned}
$$

$$
\begin{aligned}
L(u,v) \quad := \quad & -\mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} - \mathcal{D}_{1,2}(u,v)v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2} \\
& -\mathcal{D}_{1,2}(u,v) - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
& -\mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-1} + \mathcal{D}_{2,1}(u,v)u^{-1} \\
& +\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} + \mathcal{D}_{2,1}(u,v)u^{-3/2} + \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} \\
& +\mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2} + \mathcal{D}_{1,1}(u,v)v^{1/2} + \mathcal{D}_{1,1}(u,v)u^{-1/2} \\
& +\mathcal{D}_{1,1}(u,v)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (\sqrt{u}(u(-1+v)^2\log(1/u) + (-1+u)((-1+v)(-1+uv) + (v-uv) \\
& \log v)))/((-1+\sqrt{u})^2(1+\sqrt{u})(-1+\sqrt{v})^2(1+\sqrt{v})(-1+uv)).
\end{aligned}
$$

Scalar curvature in terms of $\log(k)$

Lemma: For $i, j = 1, 2$, we have

$$k^{-2}\delta_i(k)\delta_j(k) = 4\frac{\Delta - \Delta^{1/2}}{\log \Delta}(\delta_i(\log k))\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_j(\log k));$$

$$k^{-1}\delta_i\delta_j(k) = 2\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_i\delta_j(\log k)) +$$

$$g(\Delta_{(1)}, \Delta_{(2)})(\delta_j(\log k)\delta_i(\log k)) + g(\Delta_{(1)}, \Delta_{(2)})(\delta_i(\log k)\delta_j(\log k)),$$

where

$$g(u, v) := 4\frac{(\sqrt{uv} - 1)\log u - (\sqrt{u} - 1)\log(uv)}{\log v \log u \log(uv)},$$

and $\Delta_{(i)}$ signifies the action of Δ on the i-th factor of the product.

Final formula for the scalar curvature (Connes-Moscovici, Fathizadeh-K.)

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\big(\triangle_0(\log k)\big) +$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2 + |\tau|^2 \delta_2(\log k)^2 + \tau_1 \big\{\delta_1(\log k), \delta_2(\log k)\big\}\Big) +$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2 \big[\delta_1(\log k), \delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1 + \cosh((s+t)/2))\times$$
$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

The limiting case

In the commutative case, the above modular curvature reduces to a constant multiple of the formula of Gauss:

$$\frac{1}{\tau_2}\delta_1^2(\log k) + \frac{|\tau|^2}{\tau_2}\delta_2^2(\log k) + 2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$

First application: the Gauss-Bonnet theorem for A_θ

Spectral formulation of the Gauss-Bonnet theorem:

$$\zeta(0) + 1 = \frac{1}{12\pi} \int_\Sigma R \,\mathrm{vol(g)} = \frac{1}{6}\chi(\Sigma)$$

Theorem (Connes-Tretkoff; Fathizadeh-K.): Let $\theta \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, $k \in A_\theta^\infty$ be a positive invertible element. Then

$$\mathrm{Trace}(\triangle^{-s})_{|_{s=0}} + 2 = \mathfrak{t}\,(R) = 0,$$

where $\triangle$ is the Laplacian and R is the scalar curvature of the spectral triple attached to (A_θ, τ, k).

Second application: conformal anomaly (Connes-Moscovici)

Polyakov's conformal anomaly formula:

$$\log\det(\triangle) = -\zeta'_{\triangle}(0)$$

$$\log\det(\triangle) - \log\det(\triangle_0) = \log\varphi_0(e^{-h}) + \varphi_0(\tilde{R}).$$

Scalar curvature for higher NC tori

- Let $\theta : V \otimes V \to \mathbb{R}$ alternating form, $\Lambda \subset V$ a cocompact lattice. $A_\theta = A_\theta(V, \Lambda)$ by generators and relations:

$$U_\alpha U_\beta = e^{\pi i \theta(\alpha,\beta)} U_{\alpha+\beta}, \quad \text{for all}\, \alpha, \beta \in \Lambda.$$

- The dual torus V/Λ^* acts on A_θ by

$$\lambda_s(U_\alpha) = e^{2\pi i \langle s,\alpha\rangle} U_\alpha.$$

- The differential of the action λ_s at identity defines a Lie algebra map

$$\delta : V \to \text{Der}\,(A_\theta^\infty, A_\theta^\infty)$$

where A_θ^∞ = space of smooth vectors of λ_s. It is a dense $*$-subalgebra of A_θ.

- Let $\mathfrak{t} : A_\theta \to \mathbb{C}$ denote the canonical tracial state. Its restriction on smooth elements is given by

$$\mathfrak{t}\,(\sum_{\alpha\in\mathbb{Z}^n} a_\alpha U_\alpha) = a_0.$$

Noncommutative complex tori

- Let $\Lambda \subset V$ be a lattice in a complex v.s.; $e_1, \cdots, e_n$ a basis of V, and $\lambda_1, \cdots, \lambda_{2n}$ a basis of Λ. Express $\lambda_1, \cdots, \lambda_{2n}$ in terms of $e_1, \cdots, e_n$ and obtain an n by $2n$ matrix $\mathcal{M} = (A, B)$ with $A, B \in M_n(\mathbb{C})$ with

$$\lambda_j = \sum_{i=1}^{n} \mathcal{M}_{ij} e_i, \quad j = 1, \cdots, 2n$$

- Let $dz_1, \cdots, dz_n$ denote the basis of $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, dual to $e_1, \cdots, e_n$, and let $dx_1, \cdots, dx_{2n}$ denote the basis of $V^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, dual to $\lambda_1, \cdots, \lambda_{2n}$.

So that

$$dz_i = \sum_{j=1}^{2n} \mathcal{M}_{ij} dx_j, \quad d\bar{z}_i = \sum_{j=1}^{2n} \bar{\mathcal{M}}_{ij} dx_j$$

We get the differential operators

$$\begin{aligned} \delta_i = \delta_{\lambda_i} \quad &: \quad A_\theta^\infty \to A_\theta^\infty, \quad i = 1, \cdots, 2n \\ \partial_i = \delta_{e_i} \quad &: \quad A_\theta^\infty \to A_\theta^\infty, \end{aligned}$$

These derivations satisfy the relations

$$\delta_j \quad = \quad \sum_{i=1}^{n} \mathcal{M}_{ij} \partial_i, \qquad 1 \leq j \leq n$$

Dolbeault complex

Let

$$\Omega^{p,q} := A_\theta^\infty \otimes \wedge^p V_{(1,0)}^* \otimes \wedge^q V_{(0,1)}^*$$

$$\partial_i : \Omega^{p,q} \to \Omega^{p+1,q}, \quad \bar{\partial}_i : \Omega^{p,q} \to \Omega^{p,q+1}.$$

$$\begin{aligned} \partial_i(a dz_I \wedge d\bar{z}_J) &= \sum_i \partial_i(a) dz_i \wedge dz_I \wedge d\bar{z}_J \\ \bar{\partial}_i(a dz_I \wedge d\bar{z}_J) &= \sum_i \bar{\partial}_i(a) d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J \end{aligned}$$

We have:

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Noncommutative abelian varieties

- Fix $\Omega \in \mathfrak{H}_n \subset M_n(\mathbb{C})$, *Siegel upper half space*:

$$\Omega^t = \Omega \quad \text{and} \quad \operatorname{Im} \Omega > 0$$

- A complex torus V/Λ is projective (hence Kaehler) iff there exists a basis $(e_1, \cdots, e_n)$ of V, and a basis $(\lambda_1, \cdots, \lambda_{2n})$ of Λ such that

$$(\lambda_1, \cdots, \lambda_{2n}) = (\Delta, \Omega)$$

where $\Delta = \operatorname{diag}(k_1, \cdots, k_n)$, $k_i \in \mathbb{Z}$, is a diagonal matrix, and $\Omega \in \mathfrak{H}_n$.

Dolbeault Laplacian v.s. de Rham Laplacian

If M is Kaehler then

$$\triangle_{de\ Rham} = 2\triangle_{Dolbeault}$$

So on Kaehler Manifolds Dolbeault Laplacian and de Rham Laplacian have same spectrum and same spectral zeta functions.

- Any Riemann surface is Kaehler.
- 4-tori with perturbed metric are not Kaehler.

- So we have to work directly with de Rham Laplacian.

Laplacian for $\Omega = \sqrt{-1}\, I_2$

In this case the de Rham differential can be written as

$$d = \partial + \bar{\partial} : A_\theta \to A_\theta^4,$$

where,

$$\partial_1 = \frac{1}{2}(\delta_1 - i\delta_3), \qquad \partial_2 = \frac{1}{2}(\delta_2 - i\delta_4),$$
$$\bar{\partial}_1 = \frac{1}{2}(\delta_1 + i\delta_3), \qquad \bar{\partial}_2 = \frac{1}{2}(\delta_2 + i\delta_4).$$

Flat Laplacian

$$d^* d = \partial_1^* \partial_1 + \partial_2^* \partial_2 + (\bar{\partial}_1)^* \bar{\partial}_1 + (\bar{\partial}_2)^* \bar{\partial}_2$$

Perturbing the metric

Perturb the volume form on A_θ by e^{-2h}, $h \in A_\theta$. We have to consider the factor e^{-h} for 1-forms.

$$(x,y)_0 = \mathfrak{t}(y^* x e^{-2h}), \quad H_0 = \overline{A_\theta}^{(\cdot,\cdot)_0}$$

$$(x,y)_1 = \mathfrak{t}(y^* x e^{-h}), \quad H_1 = \overline{A_\theta}^{(\cdot,\cdot)_1}$$

Lemma: The perturbed Laplacian $d^*d : H_0 \to H_0$ is anti-unitarily equivalent to $\triangle' =$

$$e^h \bar{\partial}_1 e^{-h} \partial_1 e^h + e^h \partial_1 e^{-h} \bar{\partial}_1 e^h + e^h \bar{\partial}_2 e^{-h} \partial_2 e^h + e^h \partial_2 e^{-h} \bar{\partial}_2 e^h.$$

Symbol of the perturbed Laplacian

With $k = e^{h/2}$, the symbol of $\triangle'$ is

$$a_2(\xi) + a_1(\xi) + a_0(\xi) : \mathbb{R}^4 \to A_\theta^4,$$

where

$$a_2(\xi) = \frac{1}{2} k^2 \sum_{i=1}^{4} \xi_i^2, \quad a_1(\xi) = \frac{1}{2} \sum_{i=1}^{4} \delta_i(k^2)\xi_i,$$

$$a_0(\xi) = \frac{1}{2} \sum_{i=1}^{4} \left(\delta_i^2(k^2) - \delta_i(k^2)k^{-2}\delta_i(k^2)\right).$$

Approximate the resolvent $(\triangle' - \lambda)^{-1}$ by a pseudodifferential operator B_λ whose symbol has an expansion of the form

$$b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots$$

where $b_j(\xi, \lambda)$ is a symbol of order $-2 - j$.

Computing $b_2(\xi, \lambda)$

$$b_2 = -\sum_{\substack{2+j+\ell_1+\ell_2+\ell_3+\ell_4-k=2,\\ 0\leq j<2,\, 0\leq k\leq 2}} \frac{1}{\ell!}\partial^{\ell}(b_j)\delta^{\ell}(a_k)b_0.$$

$$b_2(\xi, \lambda) =$$

$$-\frac{1}{2}b_0 k\delta_1\left[\delta_1[k]\right]b_0 - \frac{1}{2}b_0 k\delta_2\left[\delta_2[k]\right]b_0 - \frac{1}{2}b_0 k\delta_3\left[\delta_3[k]\right]b_0 - \frac{1}{2}b_0 k\delta_4\left[\delta_4[k]\right]b_0 - \frac{1}{2}b_0\delta_1[k]\delta_1[k]b_0 - \frac{1}{2}b_0\delta_1\left[\delta_1[k]\right]b_0 k - \frac{1}{2}b_0\delta_2[k]\delta_2[k]b_0 - \frac{1}{2}b_0\delta_2\left[\delta_2[k]\right]b_0 k - \frac{1}{2}b_0\delta_3[k]\delta_3[k]b_0 - \frac{1}{2}b_0 k\delta_1[k]\frac{1}{k}\delta_1[k]b_0 + (\text{more than 2200 similar terms})\cdots.$$

Homogeneity argument for contour integrals

$$R = \frac{1}{2\pi i} \int_{\mathbb{R}^4} \int_C e^{-\lambda} b_2(\xi, \lambda) d\lambda \, d\xi$$

Integration by parts for inner integral $\Longrightarrow$

$$\int_C b_2(\xi, \lambda) d(-e^{-\lambda}) = \int_C e^{-\lambda} D_\lambda(b_2(\xi, \lambda)) d\lambda$$

Define $\beta(\lambda) := \int_{\mathbb{R}^4} D_\lambda(b_2(\xi, \lambda)) d\lambda d\xi$. $\beta(\lambda)$ is homogenous of order -1 .

$$\beta(t\lambda) = t^{-1} \beta(\lambda).$$

So we have

$$R = \frac{1}{2\pi i} \int_C e^{-\lambda} \beta(\lambda) d\lambda = -\beta(-1) = \frac{-1}{2\pi i} \int_{\mathbb{R}^4} D_\lambda(b_2(\xi, -1)) d\xi.$$

Scalar curvature of NC 4-Tori (Fathizadeh-Ghorbanpour-k)

$$R = \frac{1}{2} e^{-h} K(\nabla)(\sum_{i=1}^{4} \delta_i^2(h)) + \frac{1}{4} e^{-h} H(\nabla_{(1)}, \nabla_{(2)})(\sum_{i=1}^{4} \delta_i(h)^2)$$

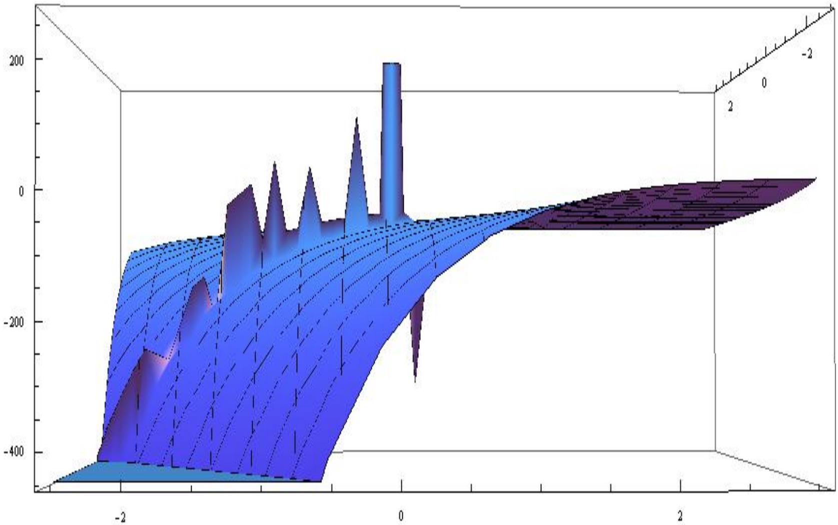
where $\Delta(x) = k^{-2} x k^2 = e^{-h} x e^h$, $\nabla(x) = \log \Delta(x) = [-h, x]$ for $x \in A_\theta$,

$$K(s) = \frac{2\pi^2 e^{-s} (e^s - 1)}{s},$$

and

$$H(s,t) = \frac{8\pi^2 e^{-3s/2-t}}{st}\Bigg(-\frac{e^{s/2}(e^{\frac{s}{2}+\frac{t}{2}}+1)(\frac{s}{2}-e^{t/2}(-\frac{1}{2}e^{s/2}t+\frac{s}{2}+\frac{t}{2}))}{\frac{s}{2}+\frac{t}{2}}$$

$$+e^{s+\frac{t}{2}}\sinh(\frac{s}{2})\sinh(\frac{t}{2})-2\sinh(\frac{s}{4})\sinh(\frac{t}{4})((\frac{s}{2}+\frac{t}{2})\coth(\frac{s}{2}+\frac{t}{2})+\frac{s}{2}+\frac{t}{2}-1)\Bigg)$$



Two variable function

A check: the classical limit

We have

$$\lim_{s \to 0} K(s) = 2\pi^2, \qquad \lim_{s,t \to 0} H(s,t) = -2\pi^2.$$

Therefore, in the commutative case, the above scalar curvature reduces to

$$\pi^2 e^{-h} \sum_{i=1}^{4} (\delta_i^2(h) - \frac{1}{2}\delta_i(h)^2),$$

which, up to a constant multiple, is precisely the scalar curvature of the ordinary 4-torus when the standard metric is perturbed by a factor of e^{-h}.