Gauss-Bonnet Theorem for the Noncommutative Torus

Noncommutative Torus

Fix $\theta \in \mathbb{R}$.

Let $A_\theta = C^*$-algebra generated by unitaries U and V satisfying

$$VU = e^{2\pi i\theta} UV$$

Dense subalgebra of *'smooth functions'*:

$$A_\theta^\infty \subset A_\theta$$

$a \in A_\theta^\infty$ iff

$$a = \sum a_{mn} U^m V^n$$

where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is rapidly decreasing:

$$\sup_{m,n} (1 + m^2 + n^2)^k \, |a_{mn}| < \infty$$

for all $k \in \mathbb{N}$.

If $\theta =$ *rational*, $A_\theta \sim C(T^2)$ (Morita equivalence).

For $\theta =$ *irrational*, A_θ is much more complicated; in particular it is a simple algebra; has a unique normalized trace (faithful and positive)

$$\tau_0 : A_\theta \to \mathbb{C}$$

$$\tau_0(\sum a_{mn} U^m V^n) = a_{00}$$

$$\tau_0(a^* a) \geq 0 \qquad \text{positivity}$$

$$\tau_0(a^* a) = 0 \quad \text{iff} \quad a = 0 \qquad \text{faithfulness}$$

Derivations

$$\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$$

uniquely defined by:

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V$$

We have

$$\delta_1 \delta_2 = \delta_2 \delta_1.$$

Invariance property:

$$\tau_0(\delta_i(a)) = 0, \qquad \delta_i(a^*) = -\delta_i(a)^*$$

GNS construction: the Hilbert space

$$\mathcal{H}_0 = L^2(A_\theta, \tau_0),$$

completion of A_θ w.r.t. inner product

$$\langle a\, ,\, b\rangle := \tau_0(b^* a).$$

Fact: δ_1, δ_2 have unique s.a. (unbounded) extensions

$$\delta_1, \delta_2 : \mathcal{H}_0 \to \mathcal{H}_0$$

Analogues of $\frac{1}{i}\frac{d}{dx}$, $\frac{1}{i}\frac{d}{dy}$.

The flat Laplacian

$$\Delta = \delta_1^2 + \delta_2^2 : A_\theta^\infty \to A_\theta^\infty$$

has a unique extension to a positive (unbounded) operator

$$\Delta : \mathcal{H}_0 \to \mathcal{H}_0$$

Complex structures

Fix

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0.$$

Connes-Tretkoff consider $\tau = i$, and define

$$\partial = \delta_1 + i\delta_2, \qquad \partial^* = \delta_1 - i\delta_2.$$

∂^* is the formal adjoint of ∂ w.r.t. $\langle\,,\,\rangle$ and

$$\Delta = \partial^*\partial = \delta_1^2 + \delta_2^2.$$

Define the Hilbert space

$$\mathcal{H}^{(1,0)} \subset \mathcal{H}_0$$

as the completion of the subspace spanned by $a\partial b$'s. Then

$$\partial : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$$

In general, let $\partial = \delta_1 + \tau\delta_2$. Considered as an unbounded operator,

$$\delta_1 + \tau\delta_2 : \mathcal{H}_0 \to \mathcal{H}^{(1,0)},$$

has an adjoint, given by

$$\partial^* = \delta_1 + \bar{\tau}\delta_2$$

Define

$$\Delta := \partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

Conformal perturbation of the metric

Fix $h = h^* \in A_\theta^\infty$. The new volume form:

$$\varphi(a) = \tau_0(ae^{-h}) \qquad a \in A_\theta.$$

φ is a positive linear functional on A_θ.

It is a twisted trace

$$\varphi(ba) = \varphi(a\sigma_i(b))$$

with the modular automorphism group

$$\sigma_t : A_\theta \to A_\theta,\ t \in \mathbb{R},$$

$$\sigma_t(x) = e^{ith} x e^{-ith}$$

and

$$\sigma_i(x) = e^{-h} x e^h$$

Let $\mathcal{H}_\varphi$ = completion of A_θ w.r.t. $\langle , \rangle_\varphi$, where

$$\langle a, b\rangle_\varphi = \varphi(b^* a), \qquad a, b \in A_\theta.$$

Let

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

It has a formal adjoint ∂_φ^*. Computation shows that

$$\partial_\varphi^* = R(e^h)\partial^*$$

where $R(e^h)$ is the right multiplication operator by e^h $(R(e^h)(x) = e^h x)$.

Perturbed Laplacian

$$\Delta' = \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi,$$

where $\partial = \delta_1 + i\delta_2$, or, in general, $\partial = \delta_1 + \tau\delta_2$.

Lemma (Connes-Tretkoff; continues to hold in the general case): The operator Δ' is anti-unitarily equivalent to the positive unbounded operator $k\Delta k$ acting on $\mathcal{H}_0$, where k is the operator of left multiplication by $e^{h/2}$.

Spectral zeta function

$$\zeta(s) = \sum \lambda_i^{-s} = \mathrm{Tr}\,(\Delta^{-s}), \qquad \mathrm{Re}(s) > 1.$$

Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt$$

gives us

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Trace}^+(e^{-t\Delta'}) t^{s-1} dt,$$

where

$$\mathrm{Trace}^+(e^{-t\Delta'}) = \mathrm{Trace}\,(e^{-t\Delta'}) - \mathrm{Dim\ Ker}(\Delta')$$

As we shall see soon, $\zeta(s)$ has a holomorphic extension to $\mathbb{C}\backslash\{1\}$ with a simple pole at $s = 1$.

Spectral form of the classical Gauss-Bonnet Theorem: Let Σ = compact connected oriented Riemannian surface. Then

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R = \frac{1}{6}\chi(\Sigma),$$

where R is the (scalar) curvature. In particular $\zeta(0)$ is a topological invariant; e.g. is invariant under conformal perturbations of the metric $g \mapsto e^f g$.

Gauss-Bonnet for NC Torus (Connes-Trekoff): For any positive invertible element $k \in A_\theta^\infty$, and $\Delta' \sim k\Delta k$, $\zeta_{\Delta'}(0)$ is independent of k. $(\Delta = \delta_1^2 + \delta_2^2.)$

Our goal: to extend this result to arbitrary complex structures on A_θ with

$$\Delta = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

Next: sketch Connes-Tretkoff's proof.

Pseudodifferential operators on A_θ^∞

Recall: Connes (1980).

Differential operators of order n:

$$P : A_\theta^\infty \to A_\theta^\infty$$

$$P = \sum_j a_j \delta_1^{j_1} \delta_2^{j_2}$$

with $a_j \in A_\theta^\infty,\ j = (j_1, j_2),\ |j| \leq n$.

Noncommutative symbols of order $n \in \mathbb{Z}$: smooth maps

$$\rho : \mathbb{R}^2 \to A_\theta^\infty$$

s.t.

$$||\delta_1^{i_1} \delta_2^{i_2} (\partial_1^{j_1} \partial_2^{j_2} \rho(\xi))|| \leq c(1 + |\xi|)^{n-|j|},$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$, and ρ is homogeneous of order n at infinity:

$$\lim \lambda^{-n} \rho(\lambda \xi_1, \lambda \xi_2) \qquad \lambda \to \infty$$

exists and is smooth for $\xi \neq 0$.

Algebra of symbols:

$$S = \cup_{n\in\mathbb{Z}} S_n$$

Given a symbol ρ, define a pseudodifferential operator

$$P_\rho : A_\theta^\infty \to A_\theta^\infty$$

by

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi,$$

where

$$\alpha_s(U^n V^m) = e^{is.(n,m)} U^n V^m$$

Def:

$$\rho \sim \rho' \quad \text{if} \quad \rho - \rho' \in \cap S_n$$

Smoothing symbols: $\rho \sim 0$.

Let Ψ= algebra of pseudodifferential operators P_ρ, $\rho \in S$. Symbol map

$$\sigma : \Psi \to S, \quad P \mapsto \sigma(P),$$

is well defined modulo smoothing operators and smoothing symbols. One has

$$\sigma(PQ) = \sum \frac{1}{\ell_1!\ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2}(\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2}(\rho'(\xi))$$

Elliptic Symbols: A symbol $\rho(\xi)$ of order n is called elliptic if $\rho(\xi)$ is invertible for $\xi \neq 0$, and, for $|\xi|$ large enough,

$$||\rho(\xi)^{-1}|| \le c(1 + |\xi|)^{-n}$$

Example:

$$\Delta = \partial^* \partial = \delta_1^2 + 2\tau_1 \delta_1 \delta_2 + |\tau|^2 \delta_2^2$$

is elliptic with an invertible symbol

$$\sigma(\Delta) = \xi_1^2 + 2\tau_1 \xi_1 \xi_2 + |\tau|^2 \xi_2^2.$$

Heat kernel expansion and zeta values

Spectral zeta function:

$$\zeta(s) = \sum \frac{1}{\lambda_i^s} = \text{Trace}(\Delta'^{-s}),$$

where $\Delta' = k\Delta k$.

Mellin transform:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt,$$

gives

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Trace}^+(e^{-t\Delta'}) t^{s-1} dt.$$

In the commutative case, $e^{-t\Delta'}$ is a smoothing pseudodifferential operator and so its trace can be computed from its kernel, or its symbol:

$$\mathrm{Tr}(e^{-t\Delta'}) = \int_{\mathbb{R}^n} k(x,x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x,\xi)dxd\xi$$

where $p = \sigma(e^{-t\Delta'})$.

In the NC torus case, the analogous formula is

$$\mathrm{Tr}(e^{-t\Delta'}) = \int_{\mathbb{R}^2} \tau_0(\sigma(e^{-t\Delta'})(\xi))d\xi.$$

Since $\Gamma(s)$ has a simple pole at $s = 0$ with Res $= 1$, we obtain

$$\zeta(0) =$$

$$\mathrm{Res}_{s=0} \int_0^\infty (\int_{\mathbb{R}^2} \tau_0(\sigma(e^{-t\Delta'}))d\xi - 1)t^{s-1}dt.$$

$(1 = \dim \mathrm{Ker}\Delta')$

Cauchy integral formula:

$$e^{-t\Delta'} = \frac{1}{2\pi i}\int_C e^{-t\lambda}(\Delta' - \lambda 1)^{-1} d\lambda$$

gives the asymptotic expansion: as $t \to 0^+$

$$\int_{\mathbb{R}^2} \tau_0(\sigma(e^{-t\Delta'}))d\xi \sim t^{-1}\sum_0^\infty B_{2n}(\Delta')t^n.$$

It follows that $\zeta(s)$ has analytic continuation to $\mathbb{C} \setminus \{1\}$.

For $Re(s) >> 0$:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 \sum_0^\infty B_{2n}(\Delta')t^{n-1+s-1}dt$$

+a holomorphic function.

It follows that:

$$\zeta(0) = B_2(\Delta').$$

Similar to the commutative case:

$$B_2(\Delta') = \frac{1}{2\pi i} \int \int_C e^{-\lambda} \tau_0(b_2(\xi, \lambda))d\lambda d\xi$$

where

$$(b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots)\sigma(\Delta' - \lambda) \sim 1,$$

$b_j(\xi, \lambda)$ is a symbol of order $-2-j$.

Can assume $\lambda = -1$, therefore

$$\zeta(0) = \int \tau_0(b_2(\xi))d\xi.$$

$$\sigma(\Delta' + 1) = \sigma(k\Delta k + 1) = (a_2 + 1) + a_1 + a_0$$

where

$$a_2 = k^2\xi_1^2 + 2\tau_1 k^2 \xi_1 \xi_2 + |\tau|^2 k^2 \xi_2^2$$

$$a_1 = (2k\delta_1(k) + 2\tau_1 k\delta_2(k))\xi_1 +$$

$$(2\tau_1 k\delta_1(k) + 2|\tau|^2 k\delta_2(k))\xi_2$$

$$a_0 = k\delta_1^2(k) + 2\tau_1 k\delta_1\delta_2(k) + |\tau|^2 k\delta_2^2(k).$$

Using the calculus for symbols:

$$b_0 = (a_2 + 1)^{-1}$$

$$b_1 = -(b_0 a_1 b_0 + \partial_i(b_0)\delta_i(a_2)b_0)$$

$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_i(b_0)\delta_i(a_1)b_0$$

$$+\partial_i(b_1)\delta_i(a_2)b_0 + (1/2)\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0).$$

$\tau_0(b_2(\xi))$ is equal to τ_0 of

$-b_0^2k\delta_1^2(k)+$
$(-2\tau_1)b_0^2k\delta_1\delta_2(k)+$
$((-1)|\tau|^2)b_0^2k\delta_2^2(k)+$
$((6)\xi_1^2+(12)\tau_1\xi_1\xi_2+(4)\tau_1^2\xi_2^2+(2)|\tau|^2\xi_2^2)b_0^3k^3\delta_1^2(k)+$
$((6)\xi_1^2+(12)\tau_1\xi_1\xi_2+(4)\tau_1^2\xi_2^2+(2)|\tau|^2\xi_2^2)b_0^3k^2\delta_1(k)^2+$
$((2)\xi_1^2+(4)\tau_1\xi_1\xi_2+(2)|\tau|^2\xi_2^2)b_0^2k^2\delta_1(k)b_0\delta_1(k)+$
$((6)\xi_1^2+(12)\tau_1\xi_1\xi_2+(4)\tau_1^2\xi_2^2+(2)|\tau|^2\xi_2^2)b_0^2k\delta_1(k)b_0k\delta_1(k)+$
$((12)\tau_1\xi_1^2+(16)\tau_1^2\xi_1\xi_2+(8)|\tau|^2\xi_1\xi_2+(12)\tau_1|\tau|^2\xi_2^2)b_0^3k^3\delta_1\delta_2(k)+$
$((6)\tau_1\xi_1^2+(8)\tau_1^2\xi_1\xi_2+(4)|\tau|^2\xi_1\xi_2+(6)\tau_1|\tau|^2\xi_2^2)b_0^3k^2\delta_1(k)\delta_2(k)+$
$((6)\tau_1\xi_1^2+(8)\tau_1^2\xi_1\xi_2+(4)|\tau|^2\xi_1\xi_2+(6)\tau_1|\tau|^2\xi_2^2)b_0^3k^2\delta_2(k)\delta_1(k)+$
$((2)\tau_1\xi_1^2+(4)\tau_1^2\xi_1\xi_2+(2)\tau_1|\tau|^2\xi_2^2)b_0^2k^2\delta_1(k)b_0\delta_2(k)+$
$((2)\tau_1\xi_1^2+(4)\tau_1^2\xi_1\xi_2+(2)\tau_1|\tau|^2\xi_2^2)b_0^2k^2\delta_2(k)b_0\delta_1(k)+$
$((6)\tau_1\xi_1^2+(8)\tau_1^2\xi_1\xi_2+(4)|\tau|^2\xi_1\xi_2+(6)\tau_1|\tau|^2\xi_2^2)b_0^2k\delta_1(k)b_0k\delta_2(k)+$
$((6)\tau_1\xi_1^2+(8)\tau_1^2\xi_1\xi_2+(4)|\tau|^2\xi_1\xi_2+(6)\tau_1|\tau|^2\xi_2^2)b_0^2k\delta_2(k)b_0k\delta_1(k)+$
$((4)\tau_1^2\xi_1^2+(2)|\tau|^2\xi_1^2+(12)\tau_1|\tau|^2\xi_1\xi_2+(6)|\tau|^4\xi_2^2)b_0^3k^3\delta_2^2(k)+$
$((4)\tau_1^2\xi_1^2+(2)|\tau|^2\xi_1^2+(12)\tau_1|\tau|^2\xi_1\xi_2+(6)|\tau|^4\xi_2^2)b_0^3k^2\delta_2(k)^2+$
$((4)\tau_1^2\xi_1^2+(2)|\tau|^2\xi_1^2+(12)\tau_1|\tau|^2\xi_1\xi_2+(6)|\tau|^4\xi_2^2)b_0^2k\delta_2(k)b_0k\delta_2(k)+$
$((2)|\tau|^2\xi_1^2+(4)\tau_1|\tau|^2\xi_1\xi_2+(2)|\tau|^4\xi_2^2)b_0^2k^2\delta_2(k)b_0\delta_2(k)+$
$((-8)\xi_1^4+(-32)\tau_1\xi_1^3\xi_2+(-40)\tau_1^2\xi_1^2\xi_2^2+(-8)|\tau|^2\xi_1^2\xi_2^2+(-16)\tau_1^3\xi_1\xi_2^3+(-16)\tau_1|\tau|^2\xi_1\xi_2^3+(-8)\tau_1^2|\tau|^2\xi_2^4)$
$b_0^4k^5\delta_1^2(k)+$
$((-8)\xi_1^4+(-32)\tau_1\xi_1^3\xi_2+(-40)\tau_1^2\xi_1^2\xi_2^2+(-8)|\tau|^2\xi_1^2\xi_2^2+(-16)\tau_1^3\xi_1\xi_2^3+(-16)\tau_1|\tau|^2\xi_1\xi_2^3+(-8)\tau_1^2|\tau|^2\xi_2^4)$
$b_0^4k^4\delta_1(k)^2+$
$((-10)\xi_1^4+(-40)\tau_1\xi_1^3\xi_2+(-48)\tau_1^2\xi_1^2\xi_2^2+(-12)|\tau|^2\xi_1^2\xi_2^2+(-16)\tau_1^3\xi_1\xi_2^3+(-24)\tau_1|\tau|^2\xi_1\xi_2^3+(-8)\tau_1^2|\tau|^2\xi_2^4$
$+(-2)|\tau|^4\xi_2^4)b_0^3k^4\delta_1(k)b_0\delta_1(k)+$
$((-20)\xi_1^4+(-80)\tau_1\xi_1^3\xi_2+(-96)\tau_1^2\xi_1^2\xi_2^2+(-24)|\tau|^2\xi_1^2\xi_2^2+(-32)\tau_1^3\xi_1\xi_2^3+(-48)\tau_1|\tau|^2\xi_1\xi_2^3+(-16)\tau_1^2|\tau|^2\xi_2^4$
$+(-4)|\tau|^4\xi_2^4)b_0^3k^3\delta_1(k)b_0k\delta_1(k)+$
$((-10)\xi_1^4+(-40)\tau_1\xi_1^3\xi_2+(-48)\tau_1^2\xi_1^2\xi_2^2+(-12)|\tau|^2\xi_1^2\xi_2^2+(-16)\tau_1^3\xi_1\xi_2^3+(-24)\tau_1|\tau|^2\xi_1\xi_2^3+(-8)\tau_1^2|\tau|^2\xi_2^4$
$+(-2)|\tau|^4\xi_2^4)b_0^3k^2\delta_1(k)b_0k^2\delta_1(k)+$
$((-4)\xi_1^4+(-16)\tau_1\xi_1^3\xi_2+(-20)\tau_1^2\xi_1^2\xi_2^2+(-4)|\tau|^2\xi_1^2\xi_2^2+(-8)\tau_1^3\xi_1\xi_2^3+(-8)\tau_1|\tau|^2\xi_1\xi_2^3+(-4)\tau_1^2|\tau|^2\xi_2^4)$
$b_0^2k^2\delta_1(k)b_0^2k^2\delta_1(k)+$
$((-4)\xi_1^4+(-16)\tau_1\xi_1^3\xi_2+(-20)\tau_1^2\xi_1^2\xi_2^2+(-4)|\tau|^2\xi_1^2\xi_2^2+(-8)\tau_1^3\xi_1\xi_2^3+(-8)\tau_1|\tau|^2\xi_1\xi_2^3+(-4)\tau_1^2|\tau|^2\xi_2^4)$
$b_0^2k\delta_1(k)b_0^2k^3\delta_1(k)+$
$((-16)\tau_1\xi_1^4+(-48)\tau_1^2\xi_1^3\xi_2+(-16)|\tau|^2\xi_1^3\xi_2+(-32)\tau_1^3\xi_1^2\xi_2^2+(-64)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-48)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-16)|\tau|^4\xi_1\xi_2^3+(-16)\tau_1|\tau|^4\xi_2^4)b_0^4k^5\delta_1\delta_2(k)+$
$((-8)\tau_1\xi_1^4+(-24)\tau_1^2\xi_1^3\xi_2+(-8)|\tau|^2\xi_1^3\xi_2+(-16)\tau_1^3\xi_1^2\xi_2^2+(-32)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-24)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-8)|\tau|^4\xi_1\xi_2^3+(-8)\tau_1|\tau|^4\xi_2^4)b_0^4k^4\delta_1(k)\delta_2(k)+$
$((-8)\tau_1\xi_1^4+(-24)\tau_1^2\xi_1^3\xi_2+(-8)|\tau|^2\xi_1^3\xi_2+(-16)\tau_1^3\xi_1^2\xi_2^2+(-32)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-24)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-8)|\tau|^4\xi_1\xi_2^3+(-8)\tau_1|\tau|^4\xi_2^4)b_0^4k^4\delta_2(k)\delta_1(k)+$
$((-10)\tau_1\xi_1^4+(-32)\tau_1^2\xi_1^3\xi_2+(-8)|\tau|^2\xi_1^3\xi_2+(-24)\tau_1^3\xi_1^2\xi_2^2+(-36)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-32)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-8)|\tau|^4\xi_1\xi_2^3+(-10)\tau_1|\tau|^4\xi_2^4)b_0^3k^4\delta_1(k)b_0\delta_2(k)+$
$((-10)\tau_1\xi_1^4+(-32)\tau_1^2\xi_1^3\xi_2+(-8)|\tau|^2\xi_1^3\xi_2+(-24)\tau_1^3\xi_1^2\xi_2^2+(-36)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-32)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-8)|\tau|^4\xi_1\xi_2^3+(-10)\tau_1|\tau|^4\xi_2^4)b_0^3k^4\delta_2(k)b_0\delta_1(k)+$
$((-20)\tau_1\xi_1^4+(-64)\tau_1^2\xi_1^3\xi_2+(-16)|\tau|^2\xi_1^3\xi_2+(-48)\tau_1^3\xi_1^2\xi_2^2+(-72)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-64)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-16)|\tau|^4\xi_1\xi_2^3+(-20)\tau_1|\tau|^4\xi_2^4)b_0^3k^3\delta_1(k)b_0k\delta_2(k)+$
$((-20)\tau_1\xi_1^4+(-64)\tau_1^2\xi_1^3\xi_2+(-16)|\tau|^2\xi_1^3\xi_2+(-48)\tau_1^3\xi_1^2\xi_2^2+(-72)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-64)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-16)|\tau|^4\xi_1\xi_2^3+(-20)\tau_1|\tau|^4\xi_2^4)b_0^3k^3\delta_2(k)b_0k\delta_1(k)+$
$((-10)\tau_1\xi_1^4+(-32)\tau_1^2\xi_1^3\xi_2+(-8)|\tau|^2\xi_1^3\xi_2+(-24)\tau_1^3\xi_1^2\xi_2^2+(-36)\tau_1|\tau|^2\xi_1^2\xi_2^2+(-32)\tau_1^2|\tau|^2\xi_1\xi_2^3+$
$(-8)|\tau|^4\xi_1\xi_2^3+(-10)\tau_1|\tau|^4\xi_2^4)b_0^3k^2\delta_1(k)b_0k^2\delta_2(k)+$

$((-10)\tau_1\xi_1^4 + (-32)\tau_1^2\xi_1^3\xi_2 + (-8)|\tau|^2\xi_1^3\xi_2 + (-24)\tau_1^3\xi_1^2\xi_2^2 + (-36)\tau_1|\tau|^2\xi_1^2\xi_2^2 + (-32)\tau_1^2|\tau|^2\xi_1\xi_2^3 +$
$(-8)|\tau|^4\xi_1\xi_2^3 + (-10)\tau_1|\tau|^4\xi_2^4)b_0^3k^2\delta_2(k)b_0k^2\delta_1(k) +$
$((-4)\tau_1\xi_1^4 + (-4)\tau_1\xi_1^4 + (-12)\tau_1^2\xi_1^3\xi_2 + (-12)\tau_1^2\xi_1^3\xi_2 + (-4)|\tau|^2\xi_1^3\xi_2 + (-4)|\tau|^2\xi_1^3\xi_2 + (-8)\tau_1^3\xi_1^2\xi_2^2 +$
$(-8)\tau_1^3\xi_1^2\xi_2^2 + (-16)\tau_1|\tau|^2\xi_1^2\xi_2^2 + (-16)\tau_1|\tau|^2\xi_1^2\xi_2^2 + (-12)\tau_1^2|\tau|^2\xi_1\xi_2^3 + (-12)\tau_1^2|\tau|^2\xi_1\xi_2^3 + (-4)|\tau|^4\xi_1\xi_2^3 +$
$(-4)|\tau|^4\xi_1\xi_2^3 + (-4)\tau_1|\tau|^4\xi_2^4 + (-4)\tau_1|\tau|^4\xi_2^4)b_0^2k^2\delta_1(k)b_0^2k^2\delta_2(k) +$
$((-4)\tau_1\xi_1^4 + (-12)\tau_1^2\xi_1^3\xi_2 + (-4)|\tau|^2\xi_1^3\xi_2 + (-8)\tau_1^3\xi_1^2\xi_2^2 + (-16)\tau_1|\tau|^2\xi_1^2\xi_2^2 + (-12)\tau_1^2|\tau|^2\xi_1\xi_2^3 +$
$(-4)|\tau|^4\xi_1\xi_2^3 + (-4)\tau_1|\tau|^4\xi_2^4)b_0^2k\delta_1(k)b_0^2k^3\delta_2(k) +$
$((-4)\tau_1\xi_1^4 + (-12)\tau_1^2\xi_1^3\xi_2 + (-4)|\tau|^2\xi_1^3\xi_2 + (-8)\tau_1^3\xi_1^2\xi_2^2 + (-16)\tau_1|\tau|^2\xi_1^2\xi_2^2 + (-12)\tau_1^2|\tau|^2\xi_1\xi_2^3 +$
$(-4)|\tau|^4\xi_1\xi_2^3 + (-4)\tau_1|\tau|^4\xi_2^4)b_0^2k\delta_2(k)b_0^2k^3\delta_1(k) +$
$((-8)\tau_1^2\xi_1^4 + (-16)\tau_1^3\xi_1^3\xi_2 + (-16)\tau_1|\tau|^2\xi_1^3\xi_2 + (-40)\tau_1^2|\tau|^2\xi_1^2\xi_2^2 + (-8)|\tau|^4\xi_1^2\xi_2^2 + (-32)\tau_1|\tau|^4\xi_1\xi_2^3 +$
$(-8)|\tau|^6\xi_2^4)b_0^4k^5\delta_2^2(k) +$
$((-8)\tau_1^2\xi_1^4 + (-16)\tau_1^3\xi_1^3\xi_2 + (-16)\tau_1|\tau|^2\xi_1^3\xi_2 + (-40)\tau_1^2|\tau|^2\xi_1^2\xi_2^2 + (-8)|\tau|^4\xi_1^2\xi_2^2 + (-32)\tau_1|\tau|^4\xi_1\xi_2^3 +$
$(-8)|\tau|^6\xi_2^4)b_0^4k^4\delta_2(k)^2 +$
$((-8)\tau_1^2\xi_1^4 + (-2)|\tau|^2\xi_1^4 + (-16)\tau_1^3\xi_1^3\xi_2 + (-24)\tau_1|\tau|^2\xi_1^3\xi_2 + (-48)\tau_1^2|\tau|^2\xi_1^2\xi_2^2 + (-12)|\tau|^4\xi_1^2\xi_2^2 +$
$(-40)\tau_1|\tau|^4\xi_1\xi_2^3 + (-10)|\tau|^6\xi_2^4)b_0^3k^4\delta_2(k)b_0\delta_2(k) +$
$((-16)\tau_1^2\xi_1^4 + (-4)|\tau|^2\xi_1^4 + (-32)\tau_1^3\xi_1^3\xi_2 + (-48)\tau_1|\tau|^2\xi_1^3\xi_2 + (-96)\tau_1^2|\tau|^2\xi_1^2\xi_2^2 + (-24)|\tau|^4\xi_1^2\xi_2^2 +$
$(-80)\tau_1|\tau|^4\xi_1\xi_2^3 + (-20)|\tau|^6\xi_2^4)b_0^3k^3\delta_2(k)b_0k\delta_2(k) +$
$((-8)\tau_1^2\xi_1^4 + (-2)|\tau|^2\xi_1^4 + (-16)\tau_1^3\xi_1^3\xi_2 + (-24)\tau_1|\tau|^2\xi_1^3\xi_2 + (-48)\tau_1^2|\tau|^2\xi_1^2\xi_2^2 + (-12)|\tau|^4\xi_1^2\xi_2^2 +$
$(-40)\tau_1|\tau|^4\xi_1\xi_2^3 + (-10)|\tau|^6\xi_2^4)b_0^3k^2\delta_2(k)b_0k^2\delta_2(k) +$
$((-4)\tau_1^2\xi_1^4 + (-8)\tau_1^3\xi_1^3\xi_2 + (-8)\tau_1|\tau|^2\xi_1^3\xi_2 + (-20)\tau_1^2|\tau|^2\xi_1^2\xi_2^2 + (-4)|\tau|^4\xi_1^2\xi_2^2 + (-16)\tau_1|\tau|^4\xi_1\xi_2^3 +$
$(-4)|\tau|^6\xi_2^4)b_0^2k^2\delta_2(k)b_0^2k^2\delta_2(k) +$
$((-4)\tau_1^2\xi_1^4 + (-8)\tau_1^3\xi_1^3\xi_2 + (-8)\tau_1|\tau|^2\xi_1^3\xi_2 + (-20)\tau_1^2|\tau|^2\xi_1^2\xi_2^2 + (-4)|\tau|^4\xi_1^2\xi_2^2 + (-16)\tau_1|\tau|^4\xi_1\xi_2^3 +$
$(-4)|\tau|^6\xi_2^4)b_0^2k\delta_2(k)b_0^2k^3\delta_2(k) +$
$((8)\xi_1^6 + (48)\tau_1\xi_1^5\xi_2 + (104)\tau_1^2\xi_1^4\xi_2^2 + (16)|\tau|^2\xi_1^4\xi_2^2 + (96)\tau_1^3\xi_1^3\xi_2^3 + (64)\tau_1|\tau|^2\xi_1^3\xi_2^3 + (32)\tau_1^4\xi_1^2\xi_2^4 +$
$(80)\tau_1^2|\tau|^2\xi_1^2\xi_2^4 + (8)|\tau|^4\xi_1^2\xi_2^4 + (32)\tau_1^3|\tau|^2\xi_1\xi_2^5 + (16)\tau_1|\tau|^4\xi_1\xi_2^5 + (8)\tau_1^2|\tau|^4\xi_2^6)b_0^4k^6\delta_1(k)b_0\delta_1(k) +$
$((16)\xi_1^6 + (96)\tau_1\xi_1^5\xi_2 + (208)\tau_1^2\xi_1^4\xi_2^2 + (32)|\tau|^2\xi_1^4\xi_2^2 + (192)\tau_1^3\xi_1^3\xi_2^3 + (128)\tau_1|\tau|^2\xi_1^3\xi_2^3 + (64)\tau_1^4\xi_1^2\xi_2^4 +$
$(160)\tau_1^2|\tau|^2\xi_1^2\xi_2^4 + (16)|\tau|^4\xi_1^2\xi_2^4 + (64)\tau_1^3|\tau|^2\xi_1\xi_2^5 + (32)\tau_1|\tau|^4\xi_1\xi_2^5 + (16)\tau_1^2|\tau|^4\xi_2^6)b_0^4k^5\delta_1(k)b_0k\delta_1(k) +$
$((8)\xi_1^6 + (48)\tau_1\xi_1^5\xi_2 + (104)\tau_1^2\xi_1^4\xi_2^2 + (16)|\tau|^2\xi_1^4\xi_2^2 + (96)\tau_1^3\xi_1^3\xi_2^3 + (64)\tau_1|\tau|^2\xi_1^3\xi_2^3 + (32)\tau_1^4\xi_1^2\xi_2^4 +$
$(80)\tau_1^2|\tau|^2\xi_1^2\xi_2^4 + (8)|\tau|^4\xi_1^2\xi_2^4 + (32)\tau_1^3|\tau|^2\xi_1\xi_2^5 + (16)\tau_1|\tau|^4\xi_1\xi_2^5 + (8)\tau_1^2|\tau|^4\xi_2^6)b_0^4k^4\delta_1(k)b_0k^2\delta_1(k) +$
$((4)\xi_1^6 + (24)\tau_1\xi_1^5\xi_2 + (52)\tau_1^2\xi_1^4\xi_2^2 + (8)|\tau|^2\xi_1^4\xi_2^2 + (48)\tau_1^3\xi_1^3\xi_2^3 + (32)\tau_1|\tau|^2\xi_1^3\xi_2^3 + (16)\tau_1^4\xi_1^2\xi_2^4 +$
$(40)\tau_1^2|\tau|^2\xi_1^2\xi_2^4 + (4)|\tau|^4\xi_1^2\xi_2^4 + (16)\tau_1^3|\tau|^2\xi_1\xi_2^5 + (8)\tau_1|\tau|^4\xi_1\xi_2^5 + (4)\tau_1^2|\tau|^4\xi_2^6)b_0^3k^4\delta_1(k)b_0^2k^2\delta_1(k) +$
$((8)\xi_1^6 + (48)\tau_1\xi_1^5\xi_2 + (104)\tau_1^2\xi_1^4\xi_2^2 + (16)|\tau|^2\xi_1^4\xi_2^2 + (96)\tau_1^3\xi_1^3\xi_2^3 + (64)\tau_1|\tau|^2\xi_1^3\xi_2^3 + (32)\tau_1^4\xi_1^2\xi_2^4 +$
$(80)\tau_1^2|\tau|^2\xi_1^2\xi_2^4 + (8)|\tau|^4\xi_1^2\xi_2^4 + (32)\tau_1^3|\tau|^2\xi_1\xi_2^5 + (16)\tau_1|\tau|^4\xi_1\xi_2^5 + (8)\tau_1^2|\tau|^4\xi_2^6)b_0^3k^3\delta_1(k)b_0^2k^3\delta_1(k) +$
$((4)\xi_1^6 + (24)\tau_1\xi_1^5\xi_2 + (52)\tau_1^2\xi_1^4\xi_2^2 + (8)|\tau|^2\xi_1^4\xi_2^2 + (48)\tau_1^3\xi_1^3\xi_2^3 + (32)\tau_1|\tau|^2\xi_1^3\xi_2^3 + (16)\tau_1^4\xi_1^2\xi_2^4 +$
$(40)\tau_1^2|\tau|^2\xi_1^2\xi_2^4 + (4)|\tau|^4\xi_1^2\xi_2^4 + (16)\tau_1^3|\tau|^2\xi_1\xi_2^5 + (8)\tau_1|\tau|^4\xi_1\xi_2^5 + (4)\tau_1^2|\tau|^4\xi_2^6)b_0^3k^2\delta_1(k)b_0^2k^4\delta_1(k) +$
$((8)\tau_1\xi_1^6 + (40)\tau_1^2\xi_1^5\xi_2 + (8)|\tau|^2\xi_1^5\xi_2 + (64)\tau_1^3\xi_1^4\xi_2^2 + (56)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (32)\tau_1^4\xi_1^3\xi_2^3 + (112)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(16)|\tau|^4\xi_1^3\xi_2^3 + (64)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (56)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (40)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (8)|\tau|^6\xi_1\xi_2^5 + (8)\tau_1|\tau|^6\xi_2^6)$
$b_0^4k^6\delta_1(k)b_0\delta_2(k) +$
$((8)\tau_1\xi_1^6 + (40)\tau_1^2\xi_1^5\xi_2 + (8)|\tau|^2\xi_1^5\xi_2 + (64)\tau_1^3\xi_1^4\xi_2^2 + (56)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (32)\tau_1^4\xi_1^3\xi_2^3 + (112)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(16)|\tau|^4\xi_1^3\xi_2^3 + (64)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (56)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (40)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (8)|\tau|^6\xi_1\xi_2^5 + (8)\tau_1|\tau|^6\xi_2^6)$
$b_0^4k^6\delta_2(k)b_0\delta_1(k) +$
$((16)\tau_1\xi_1^6 + (80)\tau_1^2\xi_1^5\xi_2 + (16)|\tau|^2\xi_1^5\xi_2 + (128)\tau_1^3\xi_1^4\xi_2^2 + (112)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (64)\tau_1^4\xi_1^3\xi_2^3 + (224)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(32)|\tau|^4\xi_1^3\xi_2^3 + (128)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (112)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (80)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (16)|\tau|^6\xi_1\xi_2^5 + (16)\tau_1|\tau|^6\xi_2^6)$
$b_0^4k^5\delta_1(k)b_0k\delta_2(k) +$
$((16)\tau_1\xi_1^6 + (80)\tau_1^2\xi_1^5\xi_2 + (16)|\tau|^2\xi_1^5\xi_2 + (128)\tau_1^3\xi_1^4\xi_2^2 + (112)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (64)\tau_1^4\xi_1^3\xi_2^3 + (224)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(32)|\tau|^4\xi_1^3\xi_2^3 + (128)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (112)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (80)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (16)|\tau|^6\xi_1\xi_2^5 + (16)\tau_1|\tau|^6\xi_2^6)$
$b_0^4k^5\delta_2(k)b_0k\delta_1(k) +$
$((8)\tau_1\xi_1^6 + (40)\tau_1^2\xi_1^5\xi_2 + (8)|\tau|^2\xi_1^5\xi_2 + (64)\tau_1^3\xi_1^4\xi_2^2 + (56)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (32)\tau_1^4\xi_1^3\xi_2^3 + (112)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$

$(16)|\tau|^4\xi_1^3\xi_2^3 + (64)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (56)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (40)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (8)|\tau|^6\xi_1\xi_2^5 + (8)\tau_1|\tau|^6\xi_2^6)$
$b_0^4k^4\delta_1(k)b_0k^2\delta_2(k) +$
$((8)\tau_1\xi_1^6 + (40)\tau_1^2\xi_1^5\xi_2 + (8)|\tau|^2\xi_1^5\xi_2 + (64)\tau_1^3\xi_1^4\xi_2^2 + (56)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (32)\tau_1^4\xi_1^3\xi_2^3 + (112)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(16)|\tau|^4\xi_1^3\xi_2^3 + (64)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (56)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (40)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (8)|\tau|^6\xi_1\xi_2^5 + (8)\tau_1|\tau|^6\xi_2^6)$
$b_0^4k^4\delta_2(k)b_0k^2\delta_1(k) +$
$((4)\tau_1\xi_1^6 + (20)\tau_1^2\xi_1^5\xi_2 + (4)|\tau|^2\xi_1^5\xi_2 + (32)\tau_1^3\xi_1^4\xi_2^2 + (28)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (16)\tau_1^4\xi_1^3\xi_2^3 + (56)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(8)|\tau|^4\xi_1^3\xi_2^3 + (32)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (28)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (20)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (4)|\tau|^6\xi_1\xi_2^5 + (4)\tau_1|\tau|^6\xi_2^6)$
$b_0^3k^4\delta_1(k)b_0^2k^2\delta_2(k) +$
$((4)\tau_1\xi_1^6 + (20)\tau_1^2\xi_1^5\xi_2 + (4)|\tau|^2\xi_1^5\xi_2 + (32)\tau_1^3\xi_1^4\xi_2^2 + (28)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (16)\tau_1^4\xi_1^3\xi_2^3 + (56)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(8)|\tau|^4\xi_1^3\xi_2^3 + (32)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (28)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (20)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (4)|\tau|^6\xi_1\xi_2^5 + (4)\tau_1|\tau|^6\xi_2^6)$
$b_0^3k^4\delta_2(k)b_0^2k^2\delta_1(k) +$
$((8)\tau_1\xi_1^6 + (40)\tau_1^2\xi_1^5\xi_2 + (8)|\tau|^2\xi_1^5\xi_2 + (64)\tau_1^3\xi_1^4\xi_2^2 + (56)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (32)\tau_1^4\xi_1^3\xi_2^3 + (112)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(16)|\tau|^4\xi_1^3\xi_2^3 + (64)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (56)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (40)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (8)|\tau|^6\xi_1\xi_2^5 + (8)\tau_1|\tau|^6\xi_2^6)$
$b_0^3k^3\delta_1(k)b_0^2k^3\delta_2(k) +$
$((8)\tau_1\xi_1^6 + (40)\tau_1^2\xi_1^5\xi_2 + (8)|\tau|^2\xi_1^5\xi_2 + (64)\tau_1^3\xi_1^4\xi_2^2 + (56)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (32)\tau_1^4\xi_1^3\xi_2^3 + (112)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(16)|\tau|^4\xi_1^3\xi_2^3 + (64)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (56)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (40)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (8)|\tau|^6\xi_1\xi_2^5 + (8)\tau_1|\tau|^6\xi_2^6)$
$b_0^3k^3\delta_2(k)b_0^2k^3\delta_1(k) +$
$((4)\tau_1\xi_1^6 + (20)\tau_1^2\xi_1^5\xi_2 + (4)|\tau|^2\xi_1^5\xi_2 + (32)\tau_1^3\xi_1^4\xi_2^2 + (28)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (16)\tau_1^4\xi_1^3\xi_2^3 + (56)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(8)|\tau|^4\xi_1^3\xi_2^3 + (32)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (28)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (20)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (4)|\tau|^6\xi_1\xi_2^5 + (4)\tau_1|\tau|^6\xi_2^6)$
$b_0^3k^2\delta_1(k)b_0^2k^4\delta_2(k) +$
$((4)\tau_1\xi_1^6 + (20)\tau_1^2\xi_1^5\xi_2 + (4)|\tau|^2\xi_1^5\xi_2 + (32)\tau_1^3\xi_1^4\xi_2^2 + (28)\tau_1|\tau|^2\xi_1^4\xi_2^2 + (16)\tau_1^4\xi_1^3\xi_2^3 + (56)\tau_1^2|\tau|^2\xi_1^3\xi_2^3 +$
$(8)|\tau|^4\xi_1^3\xi_2^3 + (32)\tau_1^3|\tau|^2\xi_1^2\xi_2^4 + (28)\tau_1|\tau|^4\xi_1^2\xi_2^4 + (20)\tau_1^2|\tau|^4\xi_1\xi_2^5 + (4)|\tau|^6\xi_1\xi_2^5 + (4)\tau_1|\tau|^6\xi_2^6)$
$b_0^3k^2\delta_2(k)b_0^2k^4\delta_1(k) +$
$((8)\tau_1^2\xi_1^6 + (32)\tau_1^3\xi_1^5\xi_2 + (16)\tau_1|\tau|^2\xi_1^5\xi_2 + (32)\tau_1^4\xi_1^4\xi_2^2 + (80)\tau_1^2|\tau|^2\xi_1^4\xi_2^2 + (8)|\tau|^4\xi_1^4\xi_2^2 + (96)\tau_1^3|\tau|^2\xi_1^3\xi_2^3 +$
$(64)\tau_1|\tau|^4\xi_1^3\xi_2^3 + (104)\tau_1^2|\tau|^4\xi_1^2\xi_2^4 + (16)|\tau|^6\xi_1^2\xi_2^4 + (48)\tau_1|\tau|^6\xi_1\xi_2^5 + (8)|\tau|^8\xi_2^6)b_0^4k^6\delta_2(k)b_0\delta_2(k) +$
$((16)\tau_1^2\xi_1^6 + (64)\tau_1^3\xi_1^5\xi_2 + (32)\tau_1|\tau|^2\xi_1^5\xi_2 + (64)\tau_1^4\xi_1^4\xi_2^2 + (160)\tau_1^2|\tau|^2\xi_1^4\xi_2^2 + (16)|\tau|^4\xi_1^4\xi_2^2 + (192)\tau_1^3|\tau|^2\xi_1^3\xi_2^3$
$+ (128)\tau_1|\tau|^4\xi_1^3\xi_2^3 + (208)\tau_1^2|\tau|^4\xi_1^2\xi_2^4 + (32)|\tau|^6\xi_1^2\xi_2^4 + (96)\tau_1|\tau|^6\xi_1\xi_2^5 + (16)|\tau|^8\xi_2^6)b_0^4k^5\delta_2(k)b_0k\delta_2(k) +$
$((8)\tau_1^2\xi_1^6 + (32)\tau_1^3\xi_1^5\xi_2 + (16)\tau_1|\tau|^2\xi_1^5\xi_2 + (32)\tau_1^4\xi_1^4\xi_2^2 + (80)\tau_1^2|\tau|^2\xi_1^4\xi_2^2 + (8)|\tau|^4\xi_1^4\xi_2^2 + (96)\tau_1^3|\tau|^2\xi_1^3\xi_2^3 +$
$(64)\tau_1|\tau|^4\xi_1^3\xi_2^3 + (104)\tau_1^2|\tau|^4\xi_1^2\xi_2^4 + (16)|\tau|^6\xi_1^2\xi_2^4 + (48)\tau_1|\tau|^6\xi_1\xi_2^5 + (8)|\tau|^8\xi_2^6)b_0^4k^4\delta_2(k)b_0k^2\delta_2(k) +$
$((4)\tau_1^2\xi_1^6 + (16)\tau_1^3\xi_1^5\xi_2 + (8)\tau_1|\tau|^2\xi_1^5\xi_2 + (16)\tau_1^4\xi_1^4\xi_2^2 + (40)\tau_1^2|\tau|^2\xi_1^4\xi_2^2 + (4)|\tau|^4\xi_1^4\xi_2^2 + (48)\tau_1^3|\tau|^2\xi_1^3\xi_2^3 +$
$(32)\tau_1|\tau|^4\xi_1^3\xi_2^3 + (52)\tau_1^2|\tau|^4\xi_1^2\xi_2^4 + (8)|\tau|^6\xi_1^2\xi_2^4 + (24)\tau_1|\tau|^6\xi_1\xi_2^5 + (4)|\tau|^8\xi_2^6)b_0^3k^4\delta_2(k)b_0^2k^2\delta_2(k) +$
$((8)\tau_1^2\xi_1^6 + (32)\tau_1^3\xi_1^5\xi_2 + (16)\tau_1|\tau|^2\xi_1^5\xi_2 + (32)\tau_1^4\xi_1^4\xi_2^2 + (80)\tau_1^2|\tau|^2\xi_1^4\xi_2^2 + (8)|\tau|^4\xi_1^4\xi_2^2 + (96)\tau_1^3|\tau|^2\xi_1^3\xi_2^3 +$
$(64)\tau_1|\tau|^4\xi_1^3\xi_2^3 + (104)\tau_1^2|\tau|^4\xi_1^2\xi_2^4 + (16)|\tau|^6\xi_1^2\xi_2^4 + (48)\tau_1|\tau|^6\xi_1\xi_2^5 + (8)|\tau|^8\xi_2^6)b_0^3k^3\delta_2(k)b_0^2k^3\delta_2(k) +$
$((4)\tau_1^2\xi_1^6 + (16)\tau_1^3\xi_1^5\xi_2 + (8)\tau_1|\tau|^2\xi_1^5\xi_2 + (16)\tau_1^4\xi_1^4\xi_2^2 + (40)\tau_1^2|\tau|^2\xi_1^4\xi_2^2 + (4)|\tau|^4\xi_1^4\xi_2^2 + (48)\tau_1^3|\tau|^2\xi_1^3\xi_2^3 +$
$(32)\tau_1|\tau|^4\xi_1^3\xi_2^3 + (52)\tau_1^2|\tau|^4\xi_1^2\xi_2^4 + (8)|\tau|^6\xi_1^2\xi_2^4 + (24)\tau_1|\tau|^6\xi_1\xi_2^5 + (4)|\tau|^8\xi_2^6)b_0^3k^2\delta_2(k)b_0^2k^4\delta_2(k)$

In the case of $\tau = i$, Connes and Tretkoff by passing to polar coordinates and integrating the angular variable, obtain term such as

$$8\pi r^2 b_0^3 k^3 \delta_1^2(k)$$

all b_0 on the left,

terms such as

$$-4\pi r^4 b_0^2 k \delta_2(k) b_0^2 k^3 \delta_2(k)$$

with b_0^2 in the middle,

and terms such as

$$16\pi r^6 b_0^4 k^5 \delta_1(k) b_0 k \delta_1(k)$$

with b_0 in the middle.

Using

$$\partial_r(b_0) = -2rk^2 b_0^2$$

and integration by parts terms with b_0^2 in the middle can be converted to terms with b_0 in the middle and for integrating the terms of the latter type the following lemma is used.

Lemma (Connes-Tretkoff). For $\rho \in A_\theta^\infty$ and every non-negative integer m:

$$\int_0^\infty \frac{k^{2m+2}u^m}{(k^2u+1)^{m+1}}\rho\frac{1}{(k^2u+1)}du = \mathcal{D}_m(\rho)$$

where

$$\mathcal{D}_m = \mathcal{L}_m(\Delta),$$

$$\Delta = \text{the modular automorphism},$$

$$\mathcal{L}_m(u) = \int_0^\infty \frac{x^m}{(x+1)^{m+1}}\frac{1}{(xu+1)}dx =$$

$$(-1)^m(u-1)^{-(m+1)}\Big(\log u - \sum_{j=1}^{m}(-1)^{j+1}\frac{(u-1)^j}{j}\Big)$$

(modified logarithm).

Theorem (Connes-Tretkoff). The value $\zeta(0)$ of the zeta function of the operator $\Delta' \sim k\Delta k$ is independent of k.

Lemma

$$\zeta(0) + 1 = 2\pi\tau_0(f(\Delta)(\delta_j(k))\delta_j(k)k^{-2})$$

where

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1 + u^{1/2})\mathcal{L}_2(u)$$
$$+(1 + u^{1/2})^2\mathcal{L}_3(u).$$

If we write $f(u) = h(\log(u))$, it follows that:

$$\zeta(0) + 1 = 2\pi\tau_0(K(\log\Delta)(\delta_j(\log k))\delta_j(\log k))$$

where

$$K(x) = 4(-1 + e^{x/2})^2 x^{-2} h(x).$$

and $\log\Delta(x) = -[h, x]$.

Proof of the theorem: From the fact that K is an odd function it follows that

$$\tau_0(K(\log\Delta)(\delta_j(\log k))\delta_j(\log k)) = 0.$$

What we have done so far:
Case $\tau = \tau_2 i$ is done now,
General case is in progress!

For the case of complex parameter $\tau = \tau_2 i$ with $\tau_2 > 0$, after computing the integral of $b_2(\xi)$ over the plane, we find the following lemma which is the analogue of Lemma 3.2 in the paper by Connes and Tretkoff.

Lemma 1. *$\zeta(0)$ of the zeta function of the operator $\Delta' \sim k\Delta k$ is given by*

$$\zeta(0) + 1 = \frac{2\pi}{\tau_2}\varphi(f(\Delta)(\delta_1(k))\delta_1(k)) + 2\pi\tau_2\varphi(f(\Delta)(\delta_2(k))\delta_2(k))$$

where $\varphi(x) = \tau_0(xk^{-2})$, Δ is the modular operator,

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1+u^{1/2})\mathcal{L}_2(u) + (1+u^{1/2})^2\mathcal{L}_3(u),$$

and

$$\mathcal{L}_m(u) = (-1)^m(u-1)^{-(m+1)}\big(log\, u - \sum_{j=1}^{m}(-1)^{j+1}\frac{(u-1)^j}{j}\big).$$

Then in their paper, Connes and Tretkoff have proved that both terms $\varphi(f(\Delta)(\delta_j(k))\delta_j(k))$ vanish for $j = 1, 2$ (Lemma 3.3 and proof of Theorem 3.1 in page 8). Therefore from the above lemma it follows that $\zeta(0) + 1 = 0$, in particular it is independent of k.