Weyl Law at 101

Masoud Khalkhali
Colloquium, UWO, 2012

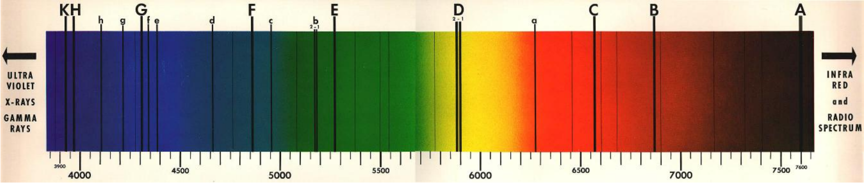

Figure: Sun's Spectrum; notice the black lines

$$\boxed{AX = \lambda X}$$

Dramatis Personae:

- Physics: Planck, Einstein, Lorentz, Sommerfeld, among others;

quantum mechanics $\leftrightarrow$ classical mechanics

- Mathematics: Hilbert, Weyl.

spectrum $\leftrightarrow$ geometry

- In 1910 H. A. Lorentz gave a series of lectures in Göttingen under the title "old and new problems of physics". Weyl and Hilbert were in attendance. In particular he mentioned attempts to drive Planck's radiation formula in a mathematically satisfactory way and remarked:

- *‘ It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between* ν *and* $\nu + d\nu$ *is independent of the shape of the enclosure and is simply proportional to its volume. ....There is no doubt that it holds in general even for multiply connected spaces’.*

- Hilbert was not very optimistic to see a solution in his lifetime. His bright student Hermann Weyl solved this conjecture of Lorentz and Sommerfeld within a year and announced a proof in 1911! All he needed was Hilbert’s theory of integral equations and compact operators developed by Hilbert and his students in 1900-1910.

Figure: Hermann Weyl in Göttingen

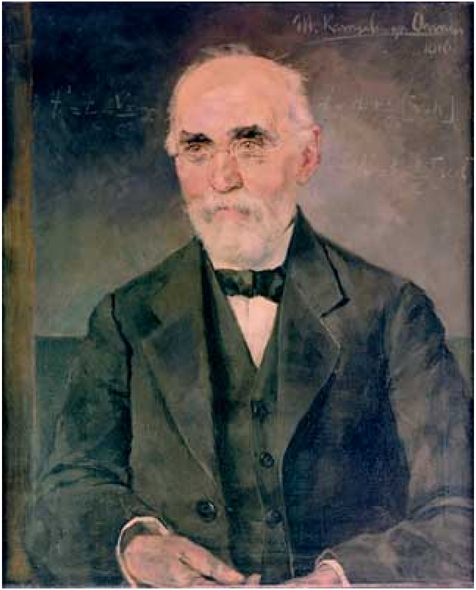

Figure: H. A. Lorentz

Black body radiation

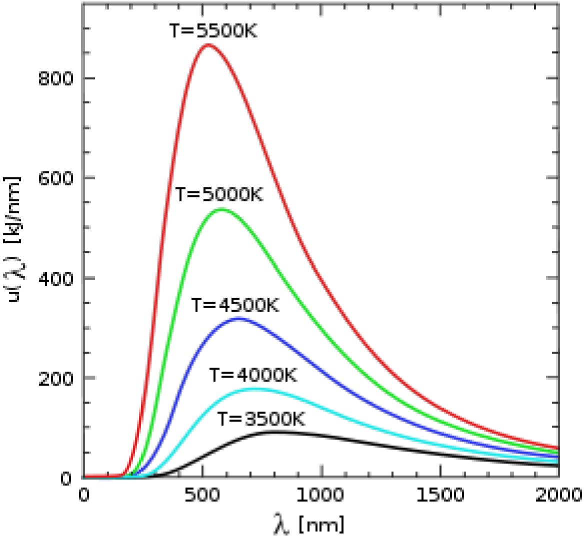

Figure: Black body spectrum

Planck's Radiation Law

- From 1859 (Kirchhoff) till 1900 (Planck) a great effort went into finding the right formula for spectral energy density function of a radiating black body

$$\rho(\nu, T)$$

- Kirchhoff predicted: ρ will be independent of the shape of the cavity and should only depend on its volume.

- Planck's formula:

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

Limits of Planck's Law

- Quantum Limit (high-frequency or low temperature regime; $h\nu/kT \gg 1$)

$$\rho(\nu, T) \sim A\nu^3 e^{-B\nu/T} \quad (T \to 0)$$

- Semiclassical Limit (low frequency or high temperature; $h\nu/kT \ll 1$)

$$\rho(\nu, T) = \frac{8\pi\nu^2}{c^3}(kT)(1 + O(h)) \quad (T \to \infty)$$

- RHS is the Rayleigh-Jeans-Einstein radiation formula. It can be established, assuming Weyl's Law: *"For high frequencies there are approximately $V(8\pi\nu^3 d\nu/c^3)$ modes of oscillations in the frequency interval $v, \nu + d\nu$."*

- Moral: To relate classical and quantum worlds, Weyl's law is needed:

  One can hear the volume of a cavity.

- But the ultimate question is

  What else can one hear about the shape of a cavity?

Dirichlet eigenvalues and Weyl Law

- Let $\Omega \subset \mathbb{R}^2$ be a compact connected domain with a piecewise smooth boundary.

$$\begin{cases} \Delta u = \lambda u \\ u|\partial\Omega = 0 \end{cases}$$

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

$$\langle u_i, u_j \rangle = \delta_{ij} \quad \text{o.n. basis for } L^2(\Omega)$$

- Weyl Law for planar domains $\Omega \subset \mathbb{R}^2$

$$N(\lambda) \sim \frac{\text{Area}(\Omega)}{4\pi}\lambda \qquad \lambda \to \infty$$

where $N(\lambda)$ is the eigenvalue counting function.

- In general, for $\Omega \subset \mathbb{R}^n$

$$N(\lambda) \sim \frac{\omega_n \text{Vol}(\Omega)}{(2\pi)^n}\lambda^{\frac{n}{2}} \qquad \lambda \to \infty$$

Weyl Law and acoustics

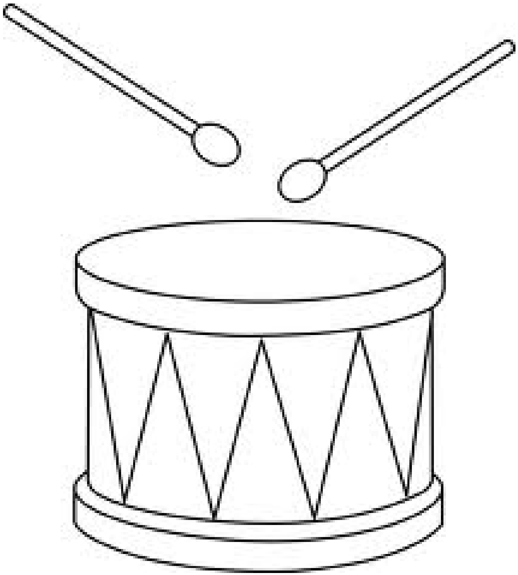

Figure: One can hear the area of a drum

• One can hear the Volume and dimension of a compact Riemannian manifold.

• In his book, *The Theory of Sound* (1877), Lord Rayleigh writes: *Some of the natural notes of the air contained within a room may generally be detected on singing the scale. Probably it is somewhat in this way that blind people are able to estimate the size of rooms....... A remarkable instance is in Darwin's Zoonomia: "The late blind Justice Fielding walked for the first time into my room, when he once visited me, and after speaking a few words said, This room is about 22 feet long, 18 wide, and 12 high; all which he guessed by the ear with great accuracy"*

Weyl Law with remainder; oscillations

- Weyl's conjecture for planar domains $\Omega \subset \mathbb{R}^2$

$$N(\lambda) \sim \frac{\mathrm{Area}(\Omega)}{4\pi}\lambda - \frac{\mathrm{L}(\partial\Omega)}{4\pi}\sqrt{\lambda} + \mathrm{o}(\sqrt{\lambda}) \qquad \lambda \to \infty$$

- Weyl's conjecture for spatial domains $\Omega \subset \mathbb{R}^3$

$$N(\lambda) \sim \frac{\mathrm{Vol}(\Omega)}{6\pi^2}\lambda^{\frac{3}{2}} - \frac{\mathrm{A}(\partial\Omega)}{16\pi}\lambda + \mathrm{o}(\lambda) \qquad \lambda \to \infty$$

- As innocent as they look, they were only proved in early 1980's after a long time effort by many mathematicians.

Wild oscillations!

- To understand the nature of the spectrum, beyond Weyl's term, one needs to understand the highly oscillatory remainder term

$$R(\lambda) = N(\lambda) - \frac{\omega_n \mathrm{Vol}(\Omega)}{(2\pi)^n} \lambda^{\frac{n}{2}}$$

- Hormander-Avakumovic

$$R(\lambda) = \mathrm{O}(\lambda^{\frac{n-1}{2}})$$

(An improvement over Weyl's result $\mathrm{o}(\lambda^{\frac{n}{2}})$

- Compare with Prime number theorem $\pi(x) \sim \frac{x}{\log x}$ and the Riemann hypothesis which gives the best estimate for $\pi(x) - \mathrm{Li}(x)$. In fact the two problems are quite related, thanks to *Connes' trace formula*.

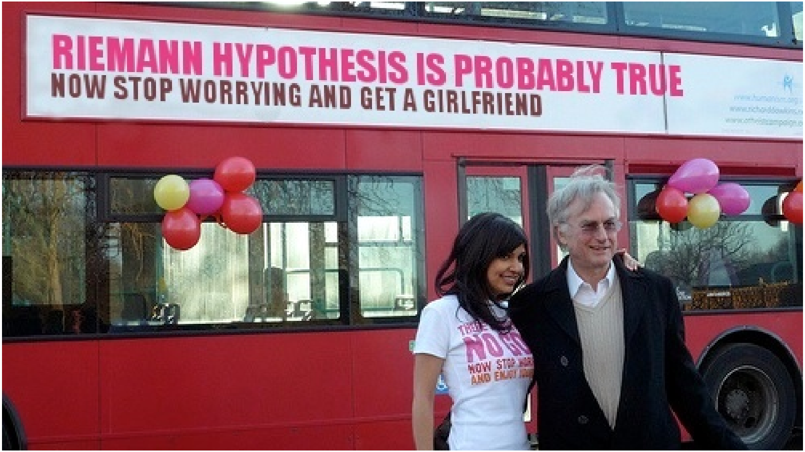

Figure: RH is probably true!

Gauss circle problem

- For $M = \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with flat metric, the spectrum is

$$\lambda_{m,n} = m^2 + n^2, \qquad m, n \in \mathbb{Z}$$

$$N(\lambda) = \#\{(m,n) : m^2 + n^2 \leq \lambda\}$$

is the number of *integral lattice points* in a circle of radius $\sqrt{\lambda}$.

- Gauss:

$$N(\lambda) = \pi\lambda + O\sqrt{\lambda}$$

- Circle problem: let

$$R(\lambda) = N(\lambda) - \pi\lambda$$

Find $\alpha_0 = \inf(\alpha)$ such that

$$R(\lambda) = O(\lambda^\alpha)$$

- Hardy's conjecture: $\alpha_0 = \frac{1}{4}$. He showed that $\alpha_0 \geq \frac{1}{4}$.
- van der Corput (1923): $\alpha_0 \leq \frac{37}{112} = 0.330\ldots$
- Iwaniec and Mozzochi (1988): $\alpha_0 \leq \frac{7}{22} = 0.318\ldots$
- Huxley (1992):

$$R(x) = \mathrm{O}(x^{\frac{23}{73}}(\log x)^{\frac{315}{146}})$$

- We are still far off from Hardy's conjectured value of $\frac{1}{4}$ for α_0!

- F. Fathizadeh and M. Khalkhali, *The Gauss-Bonnet Theorem for noncommutative two tori with a general conformal structure*, May 2010.
- F. Fathizadeh and M. Khalkhali, *Scalar Curvature for the Noncommutative Two Torus*, Oct. 2011.
- F. Fathizadeh and M. Khalkhali, *Weyl's law and Connes' trace theorem for the Noncommutative Two Torus*, Nov. 2011.
- A. Connes and P. Tretkoff, *The Gauss-Bonnet Theorem for the noncommutative two torus* , 1992, and Sept. 2009.
- A. Connes and H. Moscovici, *Modular curvature for noncommutative two-tori,* Oct. 2011.

A simple example in spectral geometry: flat tori

- Let $\Gamma \subset \mathbb{R}^m$ be a cocompact lattice.

$$M = \mathbb{R}^m/\Gamma \qquad \text{flat torus}$$

$$\triangle = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} : \mathrm{Dom}(\Delta) \subset L^2(M) \to L^2(M) \qquad \text{flat Laplacian}$$

is an unbounded, s.a., positive operator with pure point spectrum:

$$\mathrm{spec}(\triangle) = \{4\pi^2 ||\gamma^*||^2; \, \gamma^* \in \Gamma^*\},$$

- Heat equation: $(\partial_t + \triangle)\varphi = 0$

- Let $k(t,x,y)$ = fundamental solution of the heat equation= kernel of $e^{-t\triangle}$. Then:

$$K(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \sum_{\gamma\in\Gamma} e^{-4\pi^2||x-y+\gamma||^2/4t}$$

$$\begin{aligned} \mathrm{Tr} e^{-t\Delta} = \int_M k(t,x,x) &= \sum_{\gamma^*\in\Gamma^*} e^{-4\pi^2||\gamma^*||^2 t} \\ &= \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \sum_{\gamma\in\Gamma} e^{-4\pi^2||\gamma||^2/4t} \end{aligned}$$

And from this we obtain the asymptotic expansion of the heat trace near $t = 0$

$$\mathrm{Tr} e^{-t\Delta} \sim \frac{\mathrm{Vol}(M)}{(4\pi t)^{m/2}} \qquad (t \to 0)$$

Can one hear the shape of a drum ?

- Cor: One can hear the dimension and volume of M. In fact more is true: a Tauberian theorem + asymptotic expansion $\Rightarrow$ Weyl's law:

$$N(\lambda) \sim \frac{\text{Vol}\,(M)}{(4\pi)^{m/2}\Gamma(1+m/2)}\lambda^{m/2} \qquad \lambda \to \infty$$

- Cor: One can hear the total scalar curvature of M $(=0)$.

From heat trace and zeta functions to spectral invariants

- (M, g) = closed Riemannian manifold. Laplacian on functions

$$\triangle = d^* d : C^\infty(M) \to C^\infty(M)$$

is an unbounded positive operator with pure point spectrum

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- The spectrum contains a wealth of geometric and topological informations about M. In particular the dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of M are fully determined by the spectrum of Δ (on all p-forms).

The heat engine

- Let $k(t, x, y) =$ kernel of $e^{-t\triangle}$. Restrict to the diagonal: as $t \to 0$, we have (Minakshisundaram-Plejel; MacKean-Singer)

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, \triangle) + a_1(x, \triangle)t + a_2(x, \Delta)t^2 + \cdots)$$

- $a_i(x, \triangle)$: the Seeley-De Witt-Schwinger coefficients.

- Functions $a_i(x, \triangle)$: expressed by universal polynomials in curvature tensor R and its covariant derivatives:

$$
\begin{aligned}
a_0(x, \triangle) &= 1 \\
a_1(x, \triangle) &= \frac{1}{6} S(x) \qquad \text{scalar curvature} \\
a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\
a_3(x, \triangle) &= \cdots\cdots
\end{aligned}
$$

- For flat torus $a_i(x, \triangle) = 0$ for all $i \geq 1$.

Short time asymptotics of the heat trace

$$\begin{aligned}\mathrm{Trace}(e^{-t\triangle}) &= \sum e^{-t\lambda_i} = \int_M k(t,x,x)\,d\mathrm{vol}_x \\ &\sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)\end{aligned}$$

So

$$a_j = \int_M a_j(x, \triangle)\,d\mathrm{vol}_x,$$

are manifestly spectral invariants.

$$\begin{aligned} a_0 &= \int_M d\mathrm{vol}_x = \mathrm{Vol}(M), && \Longrightarrow \text{Weyl's law} \\ a_1 &= \frac{1}{6}\int_M S(x)\,d\mathrm{vol}_x, && \text{total scalar curvature}\end{aligned}$$

Abelian-Tauberian Theorem

Assume $\sum_1^\infty e^{-\lambda_n t}$ is convergent for all $t > 0$. TFAE:

$$\lim_{t\to 0^+} t^r \sum_1^\infty e^{-\lambda_n t} = a,$$

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^r} = \frac{a}{\Gamma(r+1)}$$

Spectral zeta functions

$$\zeta_{\triangle}(s) := \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > \frac{m}{2}$$

Mellin transform + asymptotic expansion:

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

$$\begin{aligned} \zeta_{\triangle}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(e^{-t\triangle}) - \mathrm{Dim\,Ker}\,\triangle) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \{ \int_0^c \cdots + \int_c^\infty \cdots \} \end{aligned}$$

The second term defines an entire function, while the first term has a meromorphic extension to $\mathbb{C}$ with simple poles within the set

$$\frac{m}{2} - j, \qquad j = 0, 1, \cdots$$

Also: 0 is always a regular point.

Simplest example: For $M = S^1$ with round metric, we have

$$\zeta_\triangle(s) = 2\zeta(2s) \qquad \text{Riemann zeta function}$$

The spectral invariants a_i in the heat asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle}) \sim (4\pi t)^{\frac{-m}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad (t \to 0)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_{\triangle}(s) = (4\pi)^{-\frac{m}{2}} \frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)}, \qquad \alpha = \frac{m}{2} - j > 0$$

Focusing on subleading pole $s = \frac{m}{2} - 1$ and using $a_1 = \frac{1}{6}\int_M S(x) dvol_x$, we obtain a formula for scalar curvature density.

Let $\zeta_f(s) := \mathrm{Tr}\,(f\triangle^{-s}),\ \ f \in C^\infty(M)$.

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1} = \frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)} \int_M fS(x)dvol_x, \quad m \geq 3$$

$$\zeta_f(s)|_{s=0} = \frac{1}{4\pi} \int_M fS(x)dvol_x - \mathrm{Tr}(fP) \qquad m = 2$$

$\log\det(\triangle) = -\zeta'(0)$, Ray-Singer regularized determinant

Spectral Triples: $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A}$= involutive unital algebra, $\mathcal{H}$ = Hilbert space,

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \qquad D : \mathcal{H} \to \mathcal{H}$$

  D has compact resolvent and all commutators $[D, \pi(a)]$ are bounded.

- An asymptotic expansion holds

$$\mathrm{Trace}\,(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad (t \to 0)$$

- Let $\triangle = D^2$. Spectral zeta function

$$\zeta_D(s) = \mathrm{Tr}\,(|D|^{-s}) = \mathrm{Tr}\,(\Delta^{-s/2}), \quad \mathrm{Re}(s) \gg 0.$$

- The metric dimension and dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$.

Noncommutative Torus

- Fix $\theta \in \mathbb{R}$. $A_\theta = C^*$-algebra generated by unitaries U and V satisfying
$$VU = e^{2\pi i\theta} UV.$$
- Dense subalgebra of *'smooth functions'*:
$$A_\theta^\infty \subset A_\theta,$$
$a \in A_\theta^\infty$ iff
$$a = \sum a_{mn} U^m V^n$$
where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is rapidly decreasing:
$$\sup_{m,n} (1 + m^2 + n^2)^k \, |a_{mn}| < \infty$$
for all $k \in \mathbb{N}$.

- Differential operators on A_θ

$$\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty,$$

Infinitesimal generators of the action

$$\alpha_s(U^m V^n) = e^{is.(m,n)} U^m V^n \quad s \in \mathbb{R}^2.$$

Analogues of $\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}$ on 2-torus.

- Canonical trace $\mathfrak{t} : A_\theta \to \mathbb{C}$ on smooth elements:

$$\mathfrak{t}(\sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n) = a_{0,0}.$$

Complex structures on A_θ

- Let $\mathcal{H}_0 = L^2(A_\theta)$= GNS completion of A_θ w.r.t. $\mathfrak{t}$.
- Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 = \Im(\tau) > 0$, and define

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

- Hilbert space of $(1,0)$-forms:

  $\mathcal{H}^{(1,0)} :=$ completion of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, w.r.t.

$$\langle a\partial b, a'\partial b'\rangle := \mathfrak{t}((a'\partial b')^* a\partial b).$$

- Flat Dolbeault Laplacian: $\partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$.

Conformal perturbation of metric

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form $\mathfrak{t}$ by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \mathfrak{t}(ae^{-h}), \quad a \in A_\theta.$$

- It is a KMS state with modular group

$$\sigma_t(x) = e^{ith} x e^{-ith},$$

and modular automorphism (Tomita-Takesaki theory)

$$\sigma_i(x) = \Delta(x) = e^{-h} x e^h.$$

$$\varphi(ab) = \varphi(b\Delta(a)), \quad \forall a, b \in A_\theta.$$

- Warning: $\triangle$ and Δ are very different operators!

Connes-Tretkoff spectral triple

- Hilbert space $\mathcal{H}_\varphi := GNS$ completion of A_θ w.r.t. $\langle , \rangle_\varphi$,

$$\langle a, b \rangle_\varphi := \varphi(b^* a), \quad a, b \in A_\theta$$

- View $\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$. and let

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi$$

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Full perturbed Laplacian:

$$\triangle := D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Lemma: $\partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$, and $\partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}$ are anti-unitarily equivalent to

$$k\partial^* \partial k : \mathcal{H}_0 \to \mathcal{H}_0,$$

$$\partial^* k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)},$$

where $k = e^{h/2}$.

The Tomita anti-unitary map J is used.

Scalar curvature for A_θ

- The scalar curvature of the curved nc torus $(\mathbb{T}^2_\theta, \tau, k)$ is the unique element $R \in A^\infty_\theta$ satisfying

$$\text{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \text{Trace}\,(aP) = \mathfrak{t}\,(aR), \qquad \forall a \in A^\infty_\theta,$$

  where P is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansin for the kernel of the operator $e^{-t\triangle}$, using Connes' pseudodifferential calculus.

Connes' pseudodifferential calculus

- Symbols: $\rho : \mathbb{R}^2 \to A_\theta^\infty$.
- ΨDO's: $P_\rho : A_\theta^\infty \to A_\theta^\infty$,

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is.\xi} \rho(\xi) \alpha_s(a) ds d\xi.$$

- For example:

$$\rho(\xi_1, \xi_2) = \sum a_{ij} \xi_1^i \xi_2^j, \quad a_{ij} \in A_\theta^\infty \quad \Rightarrow \quad P_\rho = \sum a_{ij} \delta_1^i \delta_2^j.$$

- Multiplication of symbol.

$$\sigma(PQ) \sim \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1! \ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2}(\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2}(\rho'(\xi)).$$

Local expression for the scalar curvature

- Cauchy integral formula:

$$e^{-t\triangle} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\triangle - \lambda)^{-1} \, d\lambda.$$

- $B_\lambda \approx (\triangle - \lambda)^{-1}$:

$$\sigma(B_\lambda) \sim b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots ,$$

  each $b_j(\xi, \lambda)$ is a symbol of order $-2 - j$, and

$$\sigma(B_\lambda(\triangle - \lambda)) \sim 1.$$

  (Note: λ is considered of order 2.)

Proposition: The scalar curvature of the spectral triple attached to (A_θ, τ, k) is equal to

$$\frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_C e^{-\lambda} b_2(\xi, \lambda)\, d\lambda\, d\xi,$$

where b_2 is defined as above.

Final formula for the scalar curvature (Connes-Moscovici, Fathizadeh-K, Oct. 2011)

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\left(\triangle_0(\log k)\right)+$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2+|\tau|^2\delta_2(\log k)^2+\tau_1\left\{\delta_1(\log k), \delta_2(\log k)\right\}\Big)+$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2\left[\delta_1(\log k), \delta_2(\log k)\right]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1 + \cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s-t+t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

The limiting case

In the commutative case, the above modular curvature reduces to a constant multiple of the formula of Gauss:

$$\frac{1}{\tau_2}\delta_1^2(\log k) + \frac{|\tau|^2}{\tau_2}\delta_2^2(\log k) + 2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$

First application: conformal anomaly (Connes-Moscovici)

Polyakov's conformal anomaly formula:

$$\log\det(\triangle) = -\zeta'_{\triangle}(0)$$

$$\log\det(\triangle) - \log\det(\triangle_0) = \log\varphi_0(e^{-h}) + \varphi_0(\tilde{R}).$$

Second application: the Gauss-Bonnet theorem for A_θ

- How to relate geometry (short term asymptotics) to topology (long term asymptotics)? MacKean-Singer formula:

$$\sum_{p=0}^{m}(-1)^p \mathrm{Tr}\left(e^{-t\Delta_p}\right) = \sum_{p=0}^{m}(-1)^p \beta_p = \chi(M) \qquad \forall t > 0$$

- Spectral formulation of the Gauss-Bonnet theorem:

$$\zeta(0) + 1 = \frac{1}{12\pi}\int_\Sigma R \, \mathrm{vol}(\mathrm{g}) = \frac{1}{6}\chi(\Sigma)$$

Theorem (Connes-Tretkoff; Fathizadeh-K.): Let $\theta \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, $k \in A_\theta^\infty$ be a positive invertible element. Then

$$\mathrm{Trace}(\triangle^{-s})_{|_{s=0}} + 2 = \mathfrak{t}(R) = 0,$$

where $\triangle$ is the Laplacian and R is the scalar curvature of the spectral triple attached to (A_θ, τ, k).

Third application: Weyl Law for A_θ

$$N(\lambda) \sim \frac{\pi}{\Im(\tau)} \varphi(1) \lambda \qquad \text{as} \qquad \lambda \to \infty.$$

Equivalently:

$$\lambda_j \sim \frac{\Im(\tau)}{\pi \varphi(1)} j \qquad \text{as} \qquad j \to \infty.$$

- This suggests:

$$\mathrm{Vol}(\mathbb{T}_\theta^2) := \frac{4\pi^2}{\Im(\tau)} \varphi(1) = \frac{4\pi^2}{\Im(\tau)} \varphi_0(k^{-2}).$$

The geometry in noncommutative geometry

- Geometry starts with metric and curvature. While there are a good number of 'soft' topological tools in NCG, like cyclic cohomology, K and KK-theory, and index theory, a truly noncommutative theory of curvature is still illusive. The situation is better with scalar curvature, but computations are quite tough at the moment.

- Metric aspects of NCG are informed by Spectral Geometry. Spectral invariants are the only means by which we can formulate metric ideas of NCG.