

# The Gauss-Bonnet Theorem for the Noncommutative Torus

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# Noncommutative Torus

Fix  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ .  $A_\theta = C^*$ -algebra generated by unitaries  $U$  and  $V$  satisfying

$$VU = e^{2\pi i\theta} UV.$$

Dense subalgebra of 'smooth functions':

$$A_\theta^\infty \subset A_\theta,$$

$a \in A_\theta^\infty$  iff

$$a = \sum a_{mn} U^m V^n$$

where  $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$  is rapidly decreasing:

$$\sup_{m,n} (1 + m^2 + n^2)^k |a_{mn}| < \infty$$

for all  $k \in \mathbb{N}$ .

$A_\theta$  has a unique normalized trace (faithful and positive):

$$\tau_0 : A_\theta \rightarrow \mathbb{C}$$

$$\tau_0\left(\sum a_{mn} U^m V^n\right) = a_{00}.$$

Derivations  $\delta_1, \delta_2 : A_\theta^\infty \rightarrow A_\theta^\infty$ ; uniquely defined by:

$$\delta_1(U) = U, \quad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \quad \delta_2(V) = V.$$

We have

$$\delta_1\delta_2 = \delta_2\delta_1, \quad \delta_i(a^*) = -\delta_i(a)^*,$$

Invariance property:

$$\tau_0(\delta_i(a)) = 0.$$

The Hilbert space

$$\mathcal{H}_0 = L^2(A_\theta, \tau_0),$$

completion of  $A_\theta$  w.r.t. inner product

$$\langle a, b \rangle = \tau_0(b^* a).$$

The derivations

$$\delta_1, \delta_2 : \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

are formally selfadjoint unbounded operators (analogues of  $\frac{1}{i} \frac{d}{dx}$ ,  $\frac{1}{i} \frac{d}{dy}$ ).

# Complex structure

Fix

$$\tau = \tau_1 + i\tau_2, \quad \tau_2 > 0,$$

and define

$$\partial = \delta_1 + \tau\delta_2, \quad \partial^* = \delta_1 + \bar{\tau}\delta_2.$$

Define the Hilbert space (analogue of  $(1, 0)$ -forms)

$$\mathcal{H}^{(1,0)} \subset \mathcal{H}_0$$

as the completion of the subspace spanned by finite sums  $\sum a\partial b$ ,  
 $a, b \in A_\theta^\infty$ .

Connes and Tretkoff consider  $\tau = i$ .

View

$$\partial = \delta_1 + \tau\delta_2 : \mathcal{H}_0 \rightarrow \mathcal{H}^{(1,0)}$$

as an unbounded operator with the adjoint given by

$$\partial^* = \delta_1 + \bar{\tau}\delta_2.$$

Define the **Laplacian**

$$\Delta := \partial^*\partial = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2.$$

# Conformal perturbation of the metric

To investigate the analogue of the Gauss-Bonnet theorem, vary the conformal class of the metric by  $h = h^* \in A_\theta^\infty$ : Define a positive linear functional  $\varphi : A_\theta \rightarrow \mathbb{C}$  by

$$\varphi(a) = \tau_0(ae^{-h}), \quad a \in A_\theta.$$

It is a twisted trace

$$\varphi(ba) = \varphi(a\sigma_i(b))$$

which is the KMS condition at  $\beta = 1$  for 1PG of automorphisms

$\sigma_t : A_\theta \rightarrow A_\theta$ ,  $t \in \mathbb{R}$ ,

$$\sigma_t(x) = e^{ith}xe^{-ith}.$$

In fact

$$\sigma_t = \Delta^{-it}$$

with the **modular operator**

$$\Delta(x) = e^{-h}xe^h.$$

# The perturbed Laplacian

Let  $\mathcal{H}_\varphi =$  completion of  $A_\theta$  w.r.t.  $\langle \cdot, \cdot \rangle_\varphi$ , where

$$\langle a, b \rangle_\varphi = \varphi(b^* a), \quad a, b \in A_\theta.$$

Let

$$\partial_\varphi = \partial = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \rightarrow \mathcal{H}^{(1,0)}.$$

It has a formal adjoint  $\partial_\varphi^*$  given by

$$\partial_\varphi^* = R(e^h)\partial^*$$

where  $R(e^h)$  is the right multiplication operator by  $e^h$  ( $R(e^h)(x) = e^h x$ ).

Define the new Laplacian:

$$\Delta' = \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \rightarrow \mathcal{H}_\varphi.$$

**Lemma (Connes-Tretkoff; continues to hold)**

$\Delta'$  is anti-unitarily equivalent to the positive unbounded operator  $k\Delta k$  acting on  $\mathcal{H}_0$ , where  $k = e^{h/2}$ .



# Spectral zeta function

$$\zeta(s) = \sum \lambda_i^{-s} = \text{Trace}(\Delta'^{-s}), \quad \text{Re}(s) > 1.$$

Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt$$

gives us

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Trace}^+(e^{-t\Delta'}) t^{s-1} dt,$$

where

$$\text{Trace}^+(e^{-t\Delta'}) = \text{Trace}(e^{-t\Delta'}) - \text{Dim Ker}(\Delta').$$

$\zeta$  has a meromorphic extension to  $\mathbb{C} \setminus 1$  with a simple pole at  $s = 1$ .

# The Gauss-Bonnet theorem

## Theorem (Gauss-Bonnet for classical Riemann surface)

Let  $\Sigma =$  compact connected oriented Riemann surface with metric  $g$ .  
Then

$$\zeta(0) + 1 = \frac{1}{12\pi} \int_{\Sigma} R = \frac{1}{6} \chi(\Sigma),$$

where  $\zeta$  is the zeta function associated to the Laplacian  $\Delta_g = d^*d$ , and  $R$  is the (scalar) curvature. In particular  $\zeta(0)$  is a topological invariant; e.g. is invariant under conformal perturbations of the metric  $g \mapsto e^f g$ .

## Theorem (Gauss-Bonnet for NC torus)

Let  $k \in A_{\theta}^{\infty}$  be an invertible positive element. Then the value  $\zeta(0)$  of the zeta function  $\zeta$  of the operator  $\Delta' \sim k\Delta k$  is independent of  $k$ .

# Pseudodifferential calculus

Recall: Connes (1980), Baaj (1988).

**Differential operators** of order  $n$ :

$$P : A_\theta^\infty \rightarrow A_\theta^\infty, \quad P = \sum_j a_j \delta_1^{j_1} \delta_2^{j_2}$$

with  $a_j \in A_\theta^\infty$ ,  $j = (j_1, j_2) \mid |j| \leq n$ .

**Operator valued symbols** of order  $n \in \mathbb{Z}$ : smooth maps

$$\rho : \mathbb{R}^2 \rightarrow A_\theta^\infty$$

s.t.

$$\|\delta_1^{i_1} \delta_2^{i_2} (\partial_1^{j_1} \partial_2^{j_2} \rho(\xi))\| \leq c(1 + |\xi|)^{n-|j|},$$

where  $\partial_i = \frac{\partial}{\partial \xi_i}$ , and  $\rho$  is homogeneous of order  $n$  at infinity:

$$\lim \lambda^{-n} \rho(\lambda \xi_1, \lambda \xi_2), \quad \lambda \rightarrow \infty$$

exists and is smooth.

Given a symbol  $\rho$ , define a **pseudodifferential operator**

$$P_\rho : A_\theta^\infty \rightarrow A_\theta^\infty$$

by

$$P_\rho(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is \cdot \xi} \rho(\xi) \alpha_s(a) ds d\xi,$$

where

$$\alpha_s(U^n V^m) = e^{is \cdot (n,m)} U^n V^m.$$

For pseudodifferential operators  $P, Q$ , with symbols  $\sigma(P) = \rho, \sigma(Q) = \rho'$ :

$$\sigma(PQ) \sim \sum \frac{1}{l_1! l_2!} \partial_1^{l_1} \partial_2^{l_2} (\rho(\xi)) \delta_1^{l_1} \delta_2^{l_2} (\rho'(\xi)).$$

**Elliptic Symbols:** A symbol  $\rho(\xi)$  of order  $n$  is called elliptic if  $\rho(\xi)$  is invertible for  $\xi \neq 0$ , and, for  $|\xi|$  large enough,

$$\|\rho(\xi)^{-1}\| \leq c(1 + |\xi|)^{-n}.$$

Example:

$$\Delta = \delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$$

is an **elliptic operator** with an elliptic symbol

$$\sigma(\Delta) = \xi_1^2 + 2\tau_1\xi_1\xi_2 + |\tau|^2\xi_2^2.$$

# Computing $\zeta(0)$

Recall:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\text{Trace}(e^{-t\Delta'}) t^{s-1} - 1) dt,$$

$1 = \text{Dim Ker}(\Delta')$ .

Cauchy integral formula:

$$e^{-t\Delta'} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\Delta' - \lambda 1)^{-1} d\lambda$$

gives the asymptotic expansion as  $t \rightarrow 0^+$ :

$$\text{Trace}(e^{-t\Delta'}) \sim t^{-1} \sum_0^\infty B_{2n}(\Delta') t^n.$$

It follows that:

$$\zeta(0) = B_2(\Delta'),$$

$$B_2(\Delta') = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_C e^{-\lambda} \tau_0(b_2(\xi, \lambda)) d\lambda d\xi$$

where

$$(b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \cdots) \sigma(\Delta' - \lambda) \sim 1,$$

$b_j(\xi, \lambda)$  is a symbol of order  $-2 - j$ .

Can assume  $\lambda = -1$ :

$$\zeta(0) = - \int \tau_0(b_2(\xi, -1)) d\xi.$$

$$\sigma(\Delta' + 1) = \sigma(k\Delta k + 1) = (a_2 + 1) + a_1 + a_0$$

where

$$a_2 = k^2\xi_1^2 + 2\tau_1 k^2\xi_1\xi_2 + |\tau|^2 k^2\xi_2^2$$

$$a_1 = (2k\delta_1(k) + 2\tau_1 k\delta_2(k))\xi_1 +$$

$$(2\tau_1 k\delta_1(k) + 2|\tau|^2 k\delta_2(k))\xi_2$$

$$a_0 = k\delta_1^2(k) + 2\tau_1 k\delta_1\delta_2(k) + |\tau|^2 k\delta_2^2(k).$$

Using the calculus for symbols:

$$b_0 = (a_2 + 1)^{-1}$$

$$b_1 = -(b_0 a_1 b_0 + \partial_i(b_0)\delta_i(a_2)b_0)$$

$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_i(b_0)\delta_i(a_1)b_0 \\ + \partial_i(b_1)\delta_i(a_2)b_0 + (1/2)\partial_i\partial_j(b_0)\delta_i\delta_j(a_2)b_0).$$



# Integrating $b_2(\xi, -1)$ over the plane

Pass to these coordinates:

$$\xi_1 = r \cos \theta - r \frac{\tau_1}{\tau_2} \sin \theta$$

$$\xi_2 = \frac{r}{\tau_2} \sin \theta$$

where  $\theta$  ranges from 0 to  $2\pi$  and  $r$  ranges from 0 to  $\infty$ .

After integrating  $\int_0^{2\pi} \bullet d\theta$  we have terms such as

$$4\tau_1 r^3 b_0^3 k^2 \delta_2(k) \delta_1(k),$$

$$2r^3 b_0^2 k^2 \delta_1(k) b_0 \delta_1(k),$$

$$-2r^5 b_0^2 k^2 \delta_1(k) b_0^2 k^2 \delta_1(k),$$

where

$$b_0 = (1 + r^2 k^2)^{-1}.$$

## Lemma (Connes-Tretkoff)

For  $\rho \in A_\theta^\infty$  and every non-negative integer  $m$ :

$$\int_0^\infty \frac{k^{2m+2} u^m}{(k^2 u + 1)^{m+1}} \rho \frac{1}{(k^2 u + 1)} du = \mathcal{D}_m(\rho)$$

where

$$\mathcal{D}_m = \mathcal{L}_m(\Delta),$$

$\Delta =$  the modular automorphism,

$$\begin{aligned} \mathcal{L}_m(u) &= \int_0^\infty \frac{x^m}{(x+1)^{m+1}} \frac{1}{(xu+1)} dx = \\ &(-1)^m (u-1)^{-(m+1)} \left( \log u - \sum_{j=1}^m (-1)^{j+1} \frac{(u-1)^j}{j} \right) \\ &\text{(modified logarithm).} \end{aligned}$$

## Lemma

Let  $k$  be an invertible positive element of  $A_\theta^\infty$ . Then the value  $\zeta(0)$  of the zeta function  $\zeta$  of the operator  $\Delta' \sim k\Delta k$  is given by

$$\zeta(0) + 1 = \frac{2\pi}{\tau_2} \varphi(f(\Delta)(\delta_1(k))\delta_1(k)) + \frac{2\pi|\tau|^2}{\tau_2} \varphi(f(\Delta)(\delta_2(k))\delta_2(k)) + \frac{2\pi\tau_1}{\tau_2} \varphi(f(\Delta)(\delta_1(k))\delta_2(k)) + \frac{2\pi\tau_1}{\tau_2} \varphi(f(\Delta)(\delta_2(k))\delta_1(k)),$$

where  $\varphi(x) = \tau_0(xk^{-2})$ ,  $\tau_0$  is the unique trace on  $A_\theta$ ,  $\Delta$  is the modular automorphism, and

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1 + u^{1/2})\mathcal{L}_2(u) + (1 + u^{1/2})^2\mathcal{L}_3(u).$$

( $\mathcal{L}_m$  is the modified logarithm.)

## Theorem (Gauss-Bonnet for NC torus)

Let  $k \in A_\theta^\infty$  be an invertible positive element. Then the value  $\zeta(0)$  of the zeta function  $\zeta$  of the operator  $\Delta' \sim k\Delta k$  is independent of  $k$ .

Proof.

$$\begin{aligned}\varphi(f(\Delta)(\delta_j(k))\delta_j(k)) &= 0 \text{ for } j = 1, 2, \\ \varphi(f(\Delta)(\delta_1(k))\delta_2(k)) &= -\varphi(f(\Delta)(\delta_2(k))\delta_1(k)).\end{aligned}$$

Therefore

$$\begin{aligned}\zeta(0) + 1 &= \frac{2\pi}{\tau_2} \varphi(f(\Delta)(\delta_1(k))\delta_1(k)) + \frac{2\pi|\tau|^2}{\tau_2} \varphi(f(\Delta)(\delta_2(k))\delta_2(k)) + \\ &\quad \frac{2\pi\tau_1}{\tau_2} \varphi(f(\Delta)(\delta_1(k))\delta_2(k)) + \frac{2\pi\tau_1}{\tau_2} \varphi(f(\Delta)(\delta_2(k))\delta_1(k)) \\ &= 0\end{aligned}$$



$$\begin{aligned}
& -b_0k\delta_1^2(k)b_0 - 2\tau_1b_0k\delta_1\delta_2(k)b_0 - |\tau|^2b_0k\delta_2^2(k)b_0 + \\
& 6\xi_1^2b_0^2k^2\delta_1(k)^2b_0 + \xi_1^2b_0^2k^2\delta_1^2(k)b_0k + 5\xi_1^2b_0^2k^3\delta_1^2(k)b_0 + \\
& 2\xi_1^2b_0k\delta_1(k)b_0\delta_1(k)b_0k + 6\xi_1^2b_0k\delta_1(k)b_0k\delta_1(k)b_0 + \\
& 6\tau_1\xi_1^2b_0^2k^2\delta_1(k)\delta_2(k)b_0 + 6\tau_1\xi_1^2b_0^2k^2\delta_2(k)\delta_1(k)b_0 + \\
& 2\tau_1\xi_1^2b_0^2k^2\delta_1\delta_2(k)b_0k + 10\tau_1\xi_1^2b_0^2k^3\delta_1\delta_2(k)b_0 + \\
& 2\tau_1\xi_1^2b_0k\delta_1(k)b_0\delta_2(k)b_0k + 6\tau_1\xi_1^2b_0k\delta_1(k)b_0k\delta_2(k)b_0 + \\
& 2\tau_1\xi_1^2b_0k\delta_2(k)b_0\delta_1(k)b_0k + 6\tau_1\xi_1^2b_0k\delta_2(k)b_0k\delta_1(k)b_0 + \\
& 12\tau_1\xi_1\xi_2b_0^2k^2\delta_1(k)^2b_0 + 2\tau_1\xi_1\xi_2b_0^2k^2\delta_1^2(k)b_0k + \\
& 10\tau_1\xi_1\xi_2b_0^2k^3\delta_1^2(k)b_0 + 4\tau_1\xi_1\xi_2b_0k\delta_1(k)b_0\delta_1(k)b_0k + \\
& 12\tau_1\xi_1\xi_2b_0k\delta_1(k)b_0k\delta_1(k)b_0 + 4\tau_1^2\xi_1^2b_0^2k^2\delta_2(k)^2b_0 + \\
& 4\tau_1^2\xi_1^2b_0^2k^3\delta_2^2(k)b_0 + 4\tau_1^2\xi_1^2b_0k\delta_2(k)b_0k\delta_2(k)b_0 + \\
& 8\tau_1^2\xi_1\xi_2b_0^2k^2\delta_1(k)\delta_2(k)b_0 + 8\tau_1^2\xi_1\xi_2b_0^2k^2\delta_2(k)\delta_1(k)b_0 + \\
& 4\tau_1^2\xi_1\xi_2b_0^2k^2\delta_1\delta_2(k)b_0k + 12\tau_1^2\xi_1\xi_2b_0^2k^3\delta_1\delta_2(k)b_0 + \\
& 4\tau_1^2\xi_1\xi_2b_0k\delta_1(k)b_0\delta_2(k)b_0k + 8\tau_1^2\xi_1\xi_2b_0k\delta_1(k)b_0k\delta_2(k)b_0 + \\
& 4\tau_1^2\xi_1\xi_2b_0k\delta_2(k)b_0\delta_1(k)b_0k + 8\tau_1^2\xi_1\xi_2b_0k\delta_2(k)b_0k\delta_1(k)b_0 + \\
& 4\tau_1^2\xi_2^2b_0^2k^2\delta_1(k)^2b_0 + 4\tau_1^2\xi_2^2b_0^2k^3\delta_1^2(k)b_0 + \\
& 4\tau_1^2\xi_2^2b_0k\delta_1(k)b_0k\delta_1(k)b_0 + 2|\tau|^2\xi_1^2b_0^2k^2\delta_2(k)^2b_0 + \\
& |\tau|^2\xi_2^2b_0^2k^2\delta_2^2(k)b_0k + |\tau|^2\xi_1^2b_0^2k^3\delta_2^2(k)b_0 + \\
& 2|\tau|^2\xi_1^2b_0k\delta_2(k)b_0\delta_2(k)b_0k + 2|\tau|^2\xi_1^2b_0k\delta_2(k)b_0k\delta_2(k)b_0 + \\
& 4|\tau|^2\xi_1\xi_2b_0^2k^2\delta_1(k)\delta_2(k)b_0 + 4|\tau|^2\xi_1\xi_2b_0^2k^2\delta_2(k)\delta_1(k)b_0 + \\
& 8|\tau|^2\xi_1\xi_2b_0^2k^3\delta_1\delta_2(k)b_0 + 4|\tau|^2\xi_1\xi_2b_0k\delta_1(k)b_0k\delta_2(k)b_0 + \\
& 4|\tau|^2\xi_1\xi_2b_0k\delta_2(k)b_0k\delta_1(k)b_0 + 2|\tau|^2\xi_2^2b_0^2k^2\delta_1(k)^2b_0 + \\
& |\tau|^2\xi_2^2b_0^2k^2\delta_1^2(k)b_0k + |\tau|^2\xi_2^2b_0^2k^3\delta_1^2(k)b_0 + \\
& 2|\tau|^2\xi_2^2b_0k\delta_1(k)b_0\delta_1(k)b_0k + 2|\tau|^2\xi_2^2b_0k\delta_1(k)b_0k\delta_1(k)b_0 + \\
& 12\tau_1|\tau|^2\xi_1\xi_2b_0^2k^2\delta_2(k)^2b_0 + 2\tau_1|\tau|^2\xi_1\xi_2b_0^2k^2\delta_2^2(k)b_0k + \\
& 10\tau_1|\tau|^2\xi_1\xi_2b_0^2k^3\delta_2^2(k)b_0 + 4\tau_1|\tau|^2\xi_1\xi_2b_0k\delta_2(k)b_0\delta_2(k)b_0k + \\
& 12\tau_1|\tau|^2\xi_1\xi_2b_0k\delta_2(k)b_0k\delta_2(k)b_0 + 6\tau_1|\tau|^2\xi_2^2b_0^2k^2\delta_1(k)\delta_2(k)b_0 + \\
& 6\tau_1|\tau|^2\xi_2^2b_0^2k^2\delta_2(k)\delta_1(k)b_0 + 2\tau_1|\tau|^2\xi_2^2b_0^2k^2\delta_1\delta_2(k)b_0k + \\
& 10\tau_1|\tau|^2\xi_2^2b_0^2k^3\delta_1\delta_2(k)b_0 + 2\tau_1|\tau|^2\xi_2^2b_0k\delta_1(k)b_0\delta_2(k)b_0k + \\
& 6\tau_1|\tau|^2\xi_2^2b_0k\delta_1(k)b_0k\delta_2(k)b_0 + 2\tau_1|\tau|^2\xi_2^2b_0k\delta_2(k)b_0\delta_1(k)b_0k + \\
& 6\tau_1|\tau|^2\xi_2^2b_0k\delta_2(k)b_0k\delta_1(k)b_0 + 6|\tau|^4\xi_2^2b_0^2k^2\delta_2(k)^2b_0 + \\
& |\tau|^4\xi_2^2b_0^2k^2\delta_2^2(k)b_0k + 5|\tau|^4\xi_2^2b_0^2k^3\delta_2^2(k)b_0 + \\
& 2|\tau|^4\xi_2^2b_0k\delta_2(k)b_0\delta_2(k)b_0k + 6|\tau|^4\xi_2^2b_0k\delta_2(k)b_0k\delta_2(k)b_0 - \\
& 8\xi_1^4b_0^3k^4\delta_1(k)^2b_0 - 4\xi_1^4b_0^3k^4\delta_1^2(k)b_0k - \\
& 4\xi_1^4b_0^3k^5\delta_1^2(k)b_0 - 6\xi_1^4b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k - \\
& 10\xi_1^4b_0^2k^2\delta_1(k)b_0k^2\delta_1(k)b_0 - 10\xi_1^4b_0^2k^3\delta_1(k)b_0\delta_1(k)b_0k - \\
& 14\xi_1^4b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0 - 4\xi_1^4b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k - \\
& 4\xi_1^4b_0k\delta_1(k)b_0^2k^3\delta_1(k)b_0 - 8\tau_1\xi_1^4b_0^3k^4\delta_1(k)\delta_2(k)b_0 - \\
& 8\tau_1\xi_1^4b_0^3k^4\delta_2(k)\delta_1(k)b_0 - 8\tau_1\xi_1^4b_0^3k^4\delta_1\delta_2(k)b_0k - \\
& 8\tau_1\xi_1^4b_0^3k^5\delta_1\delta_2(k)b_0 - 6\tau_1\xi_1^4b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k - \\
& 10\tau_1\xi_1^4b_0^2k^2\delta_1(k)b_0k^2\delta_2(k)b_0 - 6\tau_1\xi_1^4b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k - \\
& 10\tau_1\xi_1^4b_0^2k^2\delta_2(k)b_0k^2\delta_1(k)b_0 - 10\tau_1\xi_1^4b_0^2k^3\delta_1(k)b_0\delta_2(k)b_0k - \\
& 14\tau_1\xi_1^4b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0 - 10\tau_1\xi_1^4b_0^2k^3\delta_2(k)b_0\delta_1(k)b_0k - \\
& 14\tau_1\xi_1^4b_0^2k^3\delta_2(k)b_0k\delta_1(k)b_0 - 4\tau_1\xi_1^4b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k - \\
& 4\tau_1\xi_1^4b_0k\delta_1(k)b_0^2k^3\delta_2(k)b_0 - 4\tau_1\xi_1^4b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k - \\
& 4\tau_1\xi_1^4b_0k\delta_2(k)b_0^2k^3\delta_1(k)b_0 - 32\tau_1\xi_1^3\xi_2b_0^3k^4\delta_1(k)^2b_0 - \\
& 16\tau_1\xi_1^3\xi_2b_0^3k^4\delta_1^2(k)b_0k - 16\tau_1\xi_1^3\xi_2b_0^3k^5\delta_1^2(k)b_0 - \\
& 24\tau_1\xi_1^3\xi_2b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k - 40\tau_1\xi_1^3\xi_2b_0^2k^2\delta_1(k)b_0k^2\delta_1(k)b_0 -
\end{aligned}$$

$$\begin{aligned}
& 40\tau_1\xi_1^3\xi_2b_0^2k^3\delta_1(k)b_0\delta_1(k)b_0k - 56\tau_1\xi_1^3\xi_2b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0 - \\
& 16\tau_1\xi_1^3\xi_2b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k - 16\tau_1\xi_1^3\xi_2b_0k\delta_1(k)b_0^2k^3\delta_1(k)b_0 - \\
& 8\tau_1^2\xi_1^4b_0^3k^4\delta_2(k)^2b_0 - 4\tau_1^2\xi_1^4b_0^3k^4\delta_2^2(k)b_0k - \\
& 4\tau_1^2\xi_1^4b_0^3k^5\delta_2^2(k)b_0 - 4\tau_1^2\xi_1^4b_0^3k^2\delta_2(k)b_0k\delta_2(k)b_0k - \\
& 8\tau_1^2\xi_1^4b_0^2k^2\delta_2(k)b_0k^2\delta_2(k)b_0 - 8\tau_1^2\xi_1^4b_0^2k^3\delta_2(k)b_0\delta_2(k)b_0k - \\
& 12\tau_1^2\xi_1^4b_0^2k^3\delta_2(k)b_0k\delta_2(k)b_0 - 4\tau_1^2\xi_1^4b_0k\delta_2(k)b_0^2k^2\delta_2(k)b_0k - \\
& 4\tau_1^2\xi_1^4b_0k\delta_2(k)b_0^2k^3\delta_2(k)b_0 - 24\tau_1^2\xi_1^3\xi_2b_0^3k^4\delta_1(k)\delta_2(k)b_0 - \\
& 24\tau_1^2\xi_1^3\xi_2b_0^3k^4\delta_2(k)\delta_1(k)b_0 - 24\tau_1^2\xi_1^3\xi_2b_0^3k^4\delta_1\delta_2(k)b_0k - \\
& 24\tau_1^2\xi_1^3\xi_2b_0^3k^5\delta_1\delta_2(k)b_0 - 20\tau_1^2\xi_1^3\xi_2b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k - \\
& 32\tau_1^2\xi_1^3\xi_2b_0^2k^2\delta_1(k)b_0k^2\delta_2(k)b_0 - 20\tau_1^2\xi_1^3\xi_2b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k - \\
& 32\tau_1^2\xi_1^3\xi_2b_0^2k^2\delta_2(k)b_0k^2\delta_1(k)b_0 - 32\tau_1^2\xi_1^3\xi_2b_0^2k^3\delta_1(k)b_0\delta_2(k)b_0k - \\
& 44\tau_1^2\xi_1^3\xi_2b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0 - 32\tau_1^2\xi_1^3\xi_2b_0^2k^3\delta_2(k)b_0\delta_1(k)b_0k - \\
& 44\tau_1^2\xi_1^3\xi_2b_0^2k^3\delta_2(k)b_0k\delta_1(k)b_0 - 12\tau_1^2\xi_1^3\xi_2b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k - \\
& 12\tau_1^2\xi_1^3\xi_2b_0k\delta_1(k)b_0^2k^3\delta_2(k)b_0 - 12\tau_1^2\xi_1^3\xi_2b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k - \\
& 12\tau_1^2\xi_1^3\xi_2b_0k\delta_2(k)b_0^2k^3\delta_1(k)b_0 - 40\tau_1^2\xi_1^2\xi_2^2b_0^3k^4\delta_1(k)^2b_0 - \\
& 20\tau_1^2\xi_1^2\xi_2^2b_0^3k^4\delta_1^2(k)b_0k - 20\tau_1^2\xi_1^2\xi_2^2b_0^3k^3\delta_1^2(k)b_0 - \\
& 28\tau_1^2\xi_1^2\xi_2^2b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k - 48\tau_1^2\xi_1^2\xi_2^2b_0^2k^2\delta_1(k)b_0k^2\delta_1(k)b_0 - \\
& 48\tau_1^2\xi_1^2\xi_2^2b_0^2k^3\delta_1(k)b_0\delta_1(k)b_0k - 68\tau_1^2\xi_1^2\xi_2^2b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0 - \\
& 20\tau_1^2\xi_1^2\xi_2^2b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k - 20\tau_1^2\xi_1^2\xi_2^2b_0k\delta_1(k)b_0^2k^3\delta_1(k)b_0 - \\
& 2|\tau|^2\xi_1^4b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k - 2|\tau|^2\xi_1^4b_0^2k^2\delta_2(k)b_0k^2\delta_2(k)b_0 - \\
& 2|\tau|^2\xi_1^4b_0^2k^3\delta_2(k)b_0\delta_2(k)b_0k - 2|\tau|^2\xi_1^4b_0^2k^3\delta_2(k)b_0k\delta_2(k)b_0 - \\
& 8|\tau|^2\xi_1^3\xi_2b_0^3k^4\delta_1(k)\delta_2(k)b_0 - 8|\tau|^2\xi_1^3\xi_2b_0^3k^4\delta_2(k)\delta_1(k)b_0 - \\
& 8|\tau|^2\xi_1^3\xi_2b_0^3k^4\delta_1\delta_2(k)b_0k - 8|\tau|^2\xi_1^3\xi_2b_0^3k^5\delta_1\delta_2(k)b_0 - \\
& 4|\tau|^2\xi_1^3\xi_2b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k - 8|\tau|^2\xi_1^3\xi_2b_0^2k^2\delta_1(k)b_0k^2\delta_2(k)b_0 - \\
& 4|\tau|^2\xi_1^3\xi_2b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k - 8|\tau|^2\xi_1^3\xi_2b_0^2k^2\delta_2(k)b_0k^2\delta_1(k)b_0 - \\
& 8|\tau|^2\xi_1^3\xi_2b_0^2k^3\delta_1(k)b_0\delta_2(k)b_0k - 12|\tau|^2\xi_1^3\xi_2b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0 - \\
& 8|\tau|^2\xi_1^3\xi_2b_0^2k^3\delta_2(k)b_0\delta_1(k)b_0k - 12|\tau|^2\xi_1^3\xi_2b_0^2k^3\delta_2(k)b_0k\delta_1(k)b_0 - \\
& 4|\tau|^2\xi_1^3\xi_2b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k - 4|\tau|^2\xi_1^3\xi_2b_0k\delta_1(k)b_0^2k^3\delta_2(k)b_0 - \\
& 4|\tau|^2\xi_1^3\xi_2b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k - 4|\tau|^2\xi_1^3\xi_2b_0k\delta_2(k)b_0^2k^3\delta_1(k)b_0 - \\
& 8|\tau|^2\xi_1^2\xi_2^2b_0^3k^4\delta_1(k)^2b_0 - 4|\tau|^2\xi_1^2\xi_2^2b_0^3k^4\delta_1^2(k)b_0k - \\
& 4|\tau|^2\xi_1^2\xi_2^2b_0^3k^5\delta_1^2(k)b_0 - 8|\tau|^2\xi_1^2\xi_2^2b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k - \\
& 12|\tau|^2\xi_1^2\xi_2^2b_0^2k^2\delta_1(k)b_0k^2\delta_1(k)b_0 - 12|\tau|^2\xi_1^2\xi_2^2b_0^2k^3\delta_1(k)b_0\delta_1(k)b_0k - \\
& 16|\tau|^2\xi_1^2\xi_2^2b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0 - 4|\tau|^2\xi_1^2\xi_2^2b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k - \\
& 4|\tau|^2\xi_1^2\xi_2^2b_0k\delta_1(k)b_0^2k^3\delta_1(k)b_0 - 16\tau_1^3\xi_1^3\xi_2b_0^3k^4\delta_2(k)^2b_0 - \\
& 8\tau_1^3\xi_1^3\xi_2b_0^3k^4\delta_2^2(k)b_0k - 8\tau_1^3\xi_1^3\xi_2b_0^3k^5\delta_2^2(k)b_0 - \\
& 8\tau_1^3\xi_1^3\xi_2b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k - 16\tau_1^3\xi_1^3\xi_2b_0^2k^2\delta_2(k)b_0k^2\delta_2(k)b_0 - \\
& 16\tau_1^3\xi_1^3\xi_2b_0^2k^3\delta_2(k)b_0\delta_2(k)b_0k - 24\tau_1^3\xi_1^3\xi_2b_0^2k^3\delta_2(k)b_0k\delta_2(k)b_0 - \\
& 8\tau_1^3\xi_1^3\xi_2b_0k\delta_2(k)b_0^2k^2\delta_2(k)b_0k - 8\tau_1^3\xi_1^3\xi_2b_0k\delta_2(k)b_0^2k^3\delta_2(k)b_0 - \\
& 16\tau_1^3\xi_1^2\xi_2^2b_0^3k^4\delta_1(k)\delta_2(k)b_0 - 16\tau_1^3\xi_1^2\xi_2^2b_0^3k^4\delta_2(k)\delta_1(k)b_0 - \\
& 16\tau_1^3\xi_1^2\xi_2^2b_0^3k^4\delta_1\delta_2(k)b_0k - 16\tau_1^3\xi_1^2\xi_2^2b_0^3k^5\delta_1\delta_2(k)b_0 - \\
& 16\tau_1^3\xi_1^2\xi_2^2b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k - 24\tau_1^3\xi_1^2\xi_2^2b_0^2k^2\delta_1(k)b_0k^2\delta_2(k)b_0 - \\
& 16\tau_1^3\xi_1^2\xi_2^2b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k - 24\tau_1^3\xi_1^2\xi_2^2b_0^2k^2\delta_2(k)b_0k^2\delta_1(k)b_0 - \\
& 24\tau_1^3\xi_1^2\xi_2^2b_0^2k^3\delta_1(k)b_0\delta_2(k)b_0k - 32\tau_1^3\xi_1^2\xi_2^2b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0 - \\
& 24\tau_1^3\xi_1^2\xi_2^2b_0^2k^3\delta_2(k)b_0\delta_1(k)b_0k - 32\tau_1^3\xi_1^2\xi_2^2b_0^2k^3\delta_2(k)b_0k\delta_1(k)b_0 - \\
& 8\tau_1^3\xi_1^2\xi_2^2b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k - 8\tau_1^3\xi_1^2\xi_2^2b_0k\delta_1(k)b_0^2k^3\delta_2(k)b_0 - \\
& 8\tau_1^3\xi_1^2\xi_2^2b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k - 8\tau_1^3\xi_1^2\xi_2^2b_0k\delta_2(k)b_0^2k^3\delta_1(k)b_0 - \\
& 16\tau_1^3\xi_1\xi_2^3b_0^3k^4\delta_1(k)^2b_0 - 8\tau_1^3\xi_1\xi_2^3b_0^3k^4\delta_1^2(k)b_0k - \\
& 8\tau_1^3\xi_1\xi_2^3b_0^3k^5\delta_1^2(k)b_0 - 8\tau_1^3\xi_1\xi_2^3b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k -
\end{aligned}$$













