Curvature of the determinant line bundle for noncommutative tori

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Future directions for noncommutative geometry
Warm up: zeta regularized determinants

Given a sequence

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \]

spec(\Delta)

How one defines

\[ \prod_{\lambda_i} = \det \Delta? \]

Define the spectral zeta function:

\[ \zeta_{\Delta}(s) = \sum_{1}^{\lambda_i s^i}, \quad \text{Re}(s) \gg 0 \]

Assume:

\[ \zeta_{\Delta}(s) \text{ has meromorphic extension to } \mathbb{C} \]

and is regular at 0.

Zeta regularized determinant:

\[ \prod_{\lambda_i} = e^{-\zeta'_{\Delta}(0)} = \det \Delta \]
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Usually $\Delta = D^* D$. The determinant map $D \mapsto \sqrt{\det D^* D}$ is not holomorphic. How to define a holomorphic regularized determinant? This is hard.
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Quillen’s approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.
Curved noncommutative tori $A_{\theta}$

$$A_{\theta} = C(\mathbb{T}^2_{\theta}) = \text{universal } C^*\text{-algebra generated by unitaries } U \text{ and } V$$

$$VU = e^{2\pi i \theta} UV.$$
Differential operators $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$

\[
\delta_1(U) = U, \quad \delta_1(V) = 0
\]
\[
\delta_2(U) = 0, \quad \delta_2(V) = V
\]

Integration $\varphi_0 : A_\theta \to \mathbb{C}$ on smooth elements:

\[
\varphi_0\left(\sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n\right) = a_{0,0}.
\]

Complex structures: Fix $\tau = \tau_1 + i\tau_2$, $\tau_2 > 0$. Dolbeault operators

\[
\partial := \delta_1 + \tau \delta_2, \quad \partial^* := \delta_1 + \bar{\tau} \delta_2.
\]
Conformal perturbation of the metric (Connes-Tretkoff)

- Fix \( h = h^* \in A^\infty_\theta \). Replace the volume form \( \varphi_0 \) by \( \varphi : A_\theta \to \mathbb{C} \),
  \[
  \varphi(a) := \varphi_0(ae^{-h}).
  \]

- It is a twisted trace (KMS state):
  \[
  \varphi(ab) = \varphi(b\Delta(a)),
  \]
  where
  \[
  \Delta(x) = e^{-h}xe^h.
  \]
Perturbed Dolbeault operator

- Hilbert space $\mathcal{H}_\varphi = L^2(A_\theta, \varphi)$, GNS construction.

- Let $\partial \varphi = \delta_1 + \tau \delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$,

$$\partial^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi.$$  

and $\Delta = \partial^* \partial \varphi$, perturbed non-flat Laplacian.
Scalar curvature for $A_\theta$

- Gilkey-De Witt-Seeley formulae in spectral geometry motivates the following definition:

The scalar curvature of the curved nc torus $(A_\theta, \tau, h)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\text{Trace} \left( a \triangle^{-s} \right)_{s=0} + \text{Trace} \left( aP \right) = \varphi_0 \left( aR \right), \quad \forall a \in A_\theta^\infty,$$

where $P$ is the projection onto the kernel of $\triangle$. 

In practice this is done by finding an asymptotic expansion for the kernel of the operator $e^{-t \triangle}$, using Connes' pseudodifferential calculus for nc tori.
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Final formula for the scalar curvature (Connes-Moscovici; Fathizadeh-K)

**Theorem:** The scalar curvature of \((A_\theta, \tau, k)\), up to an overall factor of \(-\frac{\pi}{\tau_2}\), is equal to

\[
R_1(\log \Delta)(\triangle_0(\log k)) + \\
R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\left(\delta_1(\log k)^2 + |\tau|^2 \delta_2(\log k)^2 + \tau_1 \{\delta_1(\log k), \delta_2(\log k)\}\right) + \\
iW(\log \Delta_{(1)}, \log \Delta_{(2)})\left(\tau_2 [\delta_1(\log k), \delta_2(\log k)]\right)
\]
where

\[ R_1(x) = -\frac{1}{2} \frac{x \sinh(x/2)}{\sinh^2(x/4)}, \]

\[ R_2(s, t) = (1 + \cosh((s + t)/2)) \times \]

\[ -t(s + t) \cosh s + s(s + t) \cosh t - (s - t)(s + t + \sinh s + \sinh t - \sinh(s + t)) \]

\[ \frac{st(s + t) \sinh(s/2) \sinh(t/2) \sinh^2((s + t)/2)}{st(s + t) \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)} , \]

\[ W(s, t) = -\frac{(-s - t + t \cosh s + s \cosh t + \sinh s + \sinh t - \sinh(s + t))}{st \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)} . \]
What remains to be done

- Define new curved NC spaces and extend these spectral computations to them.

- Other curvature related work: Marcolli-Buhyan, Dabrowski-Sitarz, Lesch, Rosenberg, and Arnlind. Recently Fathizadeh has simplified the four dimensional calculations and its Einstein-Hilbert action.
Holomorphic determinants

- **Logdet is not a holomorphic function.** How to define a holomorphic determinant $\det : \mathcal{A} \to \mathbb{C}$.
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- Recall: Space of Fredholm operators:

  $$ F = \text{Fred}(H_0, H_1) = \{ T : H_0 \to H_1; \ T \text{ is Fredholm} \} $$

  $$ K_0(X) = [X, F], \quad \text{classifying space for K-theory} $$
The determinant line bundle

- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.
The determinant line bundle

- Let $\lambda = \wedge^{\text{max}}$ denote the top exterior power functor.

- **Theorem (Quillen)** 1) There is a holomorphic line bundle $\text{DET} \to F$ s.t.

$$ (\text{DET})_T = \lambda(\text{Ker} T)^* \otimes \lambda(\text{Ker} T^*) $$
The determinant line bundle

- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.

Theorem (Quillen) 1) There is a holomorphic line bundle $DET \to F$ s.t.

$$(DET)_T = \lambda(KerT)^* \otimes \lambda(KerT^*)$$

2) There map $\sigma : F_0 \to DET$

$$\sigma(T) = \begin{cases} 
1 & T \text{ invertible} \\
0 & \text{otherwise}
\end{cases}$$

is a holomorphic section of $DET$ over $F_0$. 
Families of spectral triples

\[ \mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, \quad \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix}, \]

with \( \alpha \in \mathcal{A}_\theta \), \( \bar{\partial} = \delta_1 + \tau \delta_2 \).

Let \( \mathcal{A} = \) space of elliptic operators \( D = \bar{\partial} + \alpha \).
Cauchy-Riemann operators on $\mathcal{A}_\theta$

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- Let $\mathcal{A}$ = space of elliptic operators $D = \bar{\partial} + \alpha$.

- Pull back DET to a holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{A}$ with

$$\mathcal{L}_D = \lambda(\text{Ker}D)^* \otimes \lambda(\text{Ker}D^*).$$
From det section to det function

- If $\mathcal{L}$ admits a canonical global holomorphic frame $s$, then

\[ \sigma(D) = \det(D)s \]

defines a holomorphic determinant function $\det(D)$. A canonical frame is defined once we have a canonical flat holomorphic connection.
Quillen’s metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $\text{Index}(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta, \quad \Delta = D^* D.$$ 

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$. 

A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from $\bar{\partial}\partial \log ||s||^2$, where $s$ is any local holomorphic frame.
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Connes’ pseudodifferential calculus

- To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.

- Symbols of order $m$: smooth maps $\sigma : \mathbb{R}^2 \to A^{\infty}_{\theta}$ with

$$\|\delta^{(i_1,i_2)} \partial^{(j_1,j_2)} \sigma(\xi)\| \leq c(1 + |\xi|)^{m-j_1-j_2}.$$ 

The space of symbols of order $m$ is denoted by $S^m(A_{\theta})$. 
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To a symbol \( \sigma \) of order \( m \), one associates an operator

\[
P_{\sigma}(a) = \int \int e^{-is\cdot\xi} \sigma(\xi) \alpha_s(a) \, ds \, d\xi.
\]

The operator \( P_{\sigma} : A_{\theta} \to A_{\theta} \) is said to be a pseudodifferential operator of order \( m \).
Classical symbols

- Classical symbol of order $\alpha \in \mathbb{C}$:

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{\alpha-j} \quad \text{ord } \sigma_{\alpha-j} = \alpha - j.$$ 

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.$$ 

- We denote the set of classical symbols of order $\alpha$ by $\mathcal{S}_{\text{cl}}^{\alpha}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi_{\text{cl}}^{\alpha}(\mathcal{A}_\theta)$. 
A cutoff integral

- Any pseudo $P_\sigma$ of order $<-2$ is trace-class with

$$\text{Tr}(P_\sigma) = \varphi_0 \left( \int_{\mathbb{R}^2} \sigma(\xi) d\xi \right).$$
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- For $\text{ord}(P) \geq -2$ the integral is divergent, but, assuming $P$ is classical, and of non-integral order, one has an asymptotic expansion as $R \to \infty$

\[
\int_{B(R)} \sigma(\xi) d\xi \sim \sum_{j=0, \alpha-j+2 \neq 0}^\infty \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma) \log R + c(\sigma),
\]

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi = \text{Wodzicki residue of } P$ (Fathizadeh).
The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}_c^\alpha(A_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $\int \sigma(\xi) d\xi$. 

NC residue in terms of $\text{TR}$: 

$$\text{Res}_{z=0} \text{TR}(A_\theta - z) = 1.$$
The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in S^\alpha_{cl}(A_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $\int \sigma(\xi) d\xi$.

- The canonical trace of a classical pseudo $P \in \Psi^\alpha_{cl}(A_\theta)$ of non-integral order $\alpha$ is defined as

$$\text{TR}(P) := \varphi_0 \left( \int \sigma_P(\xi) d\xi \right).$$
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- NC residue in terms of TR:
  \[ \text{Res}_{z=0} TR(AQ^{-z}) = \frac{1}{q} \text{Res}(A). \]
Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in $\xi$ of degree $\alpha - j$. 

Example: $\log Q$ where $Q \in \Psi^q_{\text{cl}}(A\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$. 

Wodzicki residue: $\text{Res}(A) = \phi_0(\text{res}(A))$, $\text{res}(A) = \int_{|\xi|=1} \sigma^{-2}_{0,0}(\xi) d\xi$. 
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Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section $\sigma$,

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta,0}}$$
Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section $\sigma$,

$$\|\sigma\|^2 = e^{-\zeta_{\Delta}(0)}$$

- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$. Want to compute

$$\bar{\partial}\partial \log \|\sigma\|^2 = \delta_w \delta_w \zeta'_{\Delta}(0) = \delta_w \delta_w \frac{d}{dz} \text{TR}(\Delta^{-z})|_{z=0}. $$
The second variation of logDet

- **Prop 1**: For a holomorphic family of Cauchy-Riemann operators $D_w$, the second variation of $\zeta'(0)$ is given by:

$$
\delta \bar{w} \delta w \zeta'(0) = \frac{1}{2} \varphi_0 \left( \delta w D \delta \bar{w} \text{res}(\log \Delta D^{-1}) \right).
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▶ Prop 2: The residue density of $\log \Delta D^{-1}$:

$$\sigma_{-2,0}(\log \Delta D^{-1}) = \frac{(\alpha + \alpha^*)\xi_1 + (\bar{\tau} \alpha + \tau \alpha^*)\xi_2}{(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)(\xi_1 + \tau \xi_2)} \cdot$$

$$- \log \left( \frac{\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2}{|\xi|^2} \right) \frac{\alpha}{\xi_1 + \tau \xi_2},$$

and

$$\delta \bar{w} \text{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi \Im(\tau)} (\delta w D)^*.$$
Curvature of the determinant line bundle

- **Theorem** (A. Fathi, A. Ghorbanpour, MK.): The curvature of the determinant line bundle for the noncommutative two torus is given by

\[ \delta \bar{w} \delta w \zeta'(0) = \frac{1}{4\pi \Im(\tau)} \varphi_0 (\delta w D (\delta w D)^*) \cdot \]

- **Remark**: For \( \theta = 0 \) this reduces to Quillen’s theorem (for elliptic curves).
A holomorphic determinant a la Quillen

Modify the metric to get a flat connection:

\[ ||s||^2_f = e^{||D-D_0||^2} ||s||^2 \]
A holomorphic determinant a la Quillen

- Modify the metric to get a flat connection:

  \[ \|s\|_f^2 = e^{\|D - D_0\|^2} \|s\|^2 \]

- Get a flat holomorphic global section. This gives a holomorphic determinant function

  \[ \det(D, D_0) : \mathcal{A} \to \mathbb{C} \]

  It satisfies

  \[ |\det(D, D_0)|^2 = e^{\|D - D_0\|^2} \det_\zeta(D^*D) \]