Curvature of the determinant line bundle for noncommutative tori

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Future directions for noncommutative geometry
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- Zeta regularized determinant:

$$\prod \lambda_i := e^{-\zeta'_\Delta(0)} = \det \Delta$$

Holomorphic determinants

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- Quillen's approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.

Curved noncommutative tori A_θ

$A_\theta = C(\mathbb{T}^2_\theta) =$ universal C^*-algebra generated by unitaries U and V

$$VU = e^{2\pi i\theta} UV.$$

$$A_\theta^\infty = C^\infty(\mathbb{T}^2_\theta) = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n \,:\, (a_{m,n}) \,\text{Schwartz class}\Big\}.$$

- Differential operators $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$

$$\delta_1(U) = U, \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0, \qquad \delta_2(V) = V$$

- Integration $\varphi_0 : A_\theta \to \mathbb{C}$ on smooth elements:

$$\varphi_0\Big(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\Big) = a_{0,0}.$$

- Complex structures: Fix $\tau = \tau_1 + i\tau_2, \qquad \tau_2 > 0$. Dolbeault operators

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

Conformal perturbation of the metric (Connes-Tretkoff)

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form φ_0 by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \varphi_0(ae^{-h}).$$

- It is a twisted trace (KMS state):

$$\varphi(ab) = \varphi(b\Delta(a)),$$

where

$$\Delta(x) = e^{-h}xe^{h}.$$

Perturbed Dolbeault operator

- Hilbert space $\mathcal{H}_\varphi = L^2(A_\theta, \varphi)$, *GNS* construction.

- Let $\partial_\varphi = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$,

$$\partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi.$$

  and $\triangle = \partial_\varphi^* \partial_\varphi$, perturbed non-flat Laplacian.

Scalar curvature for A_θ

- Gilkey-De Witt-Seeley formulae in spectral geometry motivates the following definition:

  The scalar curvature of the curved nc torus (A_θ, τ, h) is the unique element $R \in A_\theta^\infty$ satisfying

  $$\mathrm{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \mathrm{Trace}\,(aP) = \varphi_0\,(aR), \qquad \forall a \in A_\theta^\infty,$$

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- In practice this is done by finding an asymptotic expansion for the kernel of the operator $e^{-t\triangle}$, using Connes' pseudodifferential calculus for nc tori.

Final formula for the scalar curvature (Connes-Moscovici; Fathizadeh-K

Theorem: The scalar curvature of (A_θ, τ, k), up to an overall factor of $\frac{-\pi}{\tau_2}$, is equal to

$$R_1(\log \Delta)\big(\triangle_0(\log k)\big)+$$

$$R_2(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\delta_1(\log k)^2+|\tau|^2\delta_2(\log k)^2+\tau_1\big\{\delta_1(\log k), \delta_2(\log k)\big\}\Big)+$$

$$iW(\log \Delta_{(1)}, \log \Delta_{(2)})\Big(\tau_2\big[\delta_1(\log k), \delta_2(\log k)\big]\Big)$$

where

$$R_1(x) = -\frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) = (1+\cosh((s+t)/2))\times$$

$$\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

$$W(s,t) = -\frac{(-s-t+t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

What remains to be done

- Define new curved NC spaces and extend these spectral computations to them.

- Other curvature related work: Marcolli-Buhyan, Dabrowski-Sitarz, Lesch, Rosenberg, and Arnlind. Recently Fathizadeh has simplified the four dimensional calculations and its Einstein-Hilbert action.

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- Recall: Space of Fredholm operators:

$$F = \mathrm{Fred}(H_0, H_1) = \{T : H_0 \to H_1; \quad T \text{ is Fredholm}\}$$

$$K_0(X) = [X, F], \qquad \text{classifying space for K-theory}$$

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- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.
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- Let $\lambda = \wedge^{max}$ denote the top exterior power functor.
- Theorem (Quillen) 1) There is a holomorphic line bundle DET $\to F$ s.t.

$$(DET)_T = \lambda(KerT)^* \otimes \lambda(KerT^*)$$

2) There map $\sigma : F_0 \to DET$

$$\sigma(T) = \begin{cases} 1 & T \quad invertible \\ 0 & otherwise \end{cases}$$

is a holomorphic section of DET over F_0.

Cauchy-Riemann operators on $\mathcal{A}_\theta$

- Families of spectral triples

$$\mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, \quad \begin{pmatrix} 0 & \bar\partial^* + \alpha^* \\ \bar\partial + \alpha & 0 \end{pmatrix},$$

with $\alpha \in \mathcal{A}_\theta, \ \bar\partial = \delta_1 + \tau\delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar\partial + \alpha$.

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- Let $\mathcal{A} =$ space of elliptic operators $D = \bar{\partial} + \alpha$.

- Pull back DET to a holomorphic line bundle $\mathcal{L} \to \mathcal{A}$ with

$$\mathcal{L}_D = \lambda(KerD)^* \otimes \lambda(KerD^*).$$

From det section to det function

- If $\mathcal{L}$ admits a canonical global holomorphic frame s, then

$$\sigma(D) = det(D)s$$

defines a holomorphic determinant function $\det(D)$. A canonical frame is defined once we have a canonical flat holomorphic connection.

Quillen's metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $Index(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta'_\Delta(0)) = \det\Delta, \qquad \Delta = D^* D.$$

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

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- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

- A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from

$$\bar{\partial}\partial \log ||s||^2,$$

where s is any local holomorphic frame.

Connes' pseudodifferential calculus

- To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.
- Symbols of order m: smooth maps $\sigma : \mathbb{R}^2 \to A_\theta^\infty$ with

$$||\delta^{(i_1,i_2)}\partial^{(j_1,j_2)}\sigma(\xi)|| \leq c(1+|\xi|)^{m-j_1-j_2}.$$

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The space of symbols of order m is denoted by $\mathcal{S}^m(\mathcal{A}_\theta)$.
- To a symbol σ of order m, one associates an operator
$$P_\sigma(a) = \int\int e^{-is\cdot\xi}\sigma(\xi)\alpha_s(a)\, ds\, d\xi.$$
The operator $P_\sigma : \mathcal{A}_\theta \to \mathcal{A}_\theta$ is said to be a pseudodifferential operator of order m.

Classical symbols

- Classical symbol of order $\alpha \in \mathbb{C}$:

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{\alpha-j} \qquad \operatorname{ord} \sigma_{\alpha-j} = \alpha - j.$$

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.$$

- We denote the set of classical symbols of order α by $\mathcal{S}_{cl}^{\alpha}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$.

A cutoff integral

- Any pseudo P_σ of order < -2 is trace-class with

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- For $\mathrm{ord}(P) \geq -2$ the integral is divergent, but, assuming P is classical, and of non-integral order, one has an asymptotic expansion as $R \to \infty$

$$\int_{B(R)} \sigma(\xi) d\xi \sim \sum_{j=0, \alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma) \log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi =$ Wodzicki residue of P (Fathizadeh).

The Kontsevich-Vishik trace

- The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi) d\xi$.

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- The canonical trace of a classical pseudo $P \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ of non-integral order α is defined as

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- NC residue in terms of TR:

$$\mathrm{Res}_{z=0} \mathrm{TR}(AQ^{-z}) = \frac{1}{q} \mathrm{Res}(A).$$

Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$\sigma(\xi) \sim \sum_{j\geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

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- Wodzicki residue: $\mathrm{Res}(A) = \varphi_0(\mathrm{res}(A))$,

$$\mathrm{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.$$

Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section σ,

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- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$. Want to compute

$$\bar{\partial}\partial \log \|\sigma\|^2 = \delta_{\bar{w}}\delta_w \zeta'_\Delta(0) = \delta_{\bar{w}}\delta_w \frac{d}{dz}\mathrm{TR}(\Delta^{-z})|_{z=0}.$$

The second variation of logDet

- Prop 1: For a holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ is given by :

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{2}\varphi_0\left(\delta_w D\delta_{\bar{w}}\mathrm{res}(\log\Delta\, D^{-1})\right).$$

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- Prop 2: The residue density of $\log\Delta\, D^{-1}$:

$$\sigma_{-2,0}(\log\Delta\, D^{-1}) = \frac{(\alpha+\alpha^*)\xi_1 + (\bar{\tau}\alpha+\tau\alpha^*)\xi_2}{(\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2)(\xi_1+\tau\xi_2)}$$

$$- \log\left(\frac{\xi_1^2+2\Re(\tau)\xi_1\xi_2+|\tau|^2\xi_2^2}{|\xi|^2}\right)\frac{\alpha}{\xi_1+\tau\xi_2},$$

and

$$\delta_{\bar{w}}\mathrm{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Curvature of the determinant line bundle

- Theorem (A. Fathi, A. Ghorbanpour, MK.): The curvature of the determinant line bundle for the noncommutative two torus is given by
$$\delta_{\bar{w}}\delta_w \zeta'(0) = \frac{1}{4\pi \Im(\tau)} \varphi_0 \left(\delta_w D (\delta_w D)^*\right).$$

- Remark: For $\theta = 0$ this reduces to Quillen's theorem (for elliptic curves).

A holomorphic determinant a la Quillen

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- Get a flat holomorphic global section. This gives a holomorphic determinant function

$$det(D, D_0) : \mathcal{A} \to \mathbb{C}$$

It satisfies

$$|det(D, D_0)|^2 = e^{||D-D_0||^2} det_\zeta(D^* D)$$