

# **Rationality of Spectral Action for Robertson-Walker Metrics**

Masoud Khalkhali

(Joint work with F. Fathizadeh and A. Ghorbanpour)

## Hard calculations made easy

- ▶ Compute

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}, \quad t = 0.01.$$

You have to add 21 terms to get it to one decimal digit.

## Hard calculations made easy

- ▶ Compute

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}, \quad t = 0.01.$$

You have to add 21 terms to get it to one decimal digit.

- ▶ But: you can compute it to 130 decimal digits without any calculation!

## Modular equation is the key

- ▶ Modular equation (Jacobi, 1828)

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right).$$

## Modular equation is the key

- ▶ Modular equation (Jacobi, 1828)

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right).$$

- ▶ In particular

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} + O\left(e^{-\frac{1}{t}}\right) \quad (t \rightarrow 0)$$

$$\theta(0.01) = 10.000000000000000000 \dots$$

## Modular equation is the key

- ▶ Modular equation (Jacobi, 1828)

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right).$$

- ▶ In particular

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} + O\left(e^{-\frac{1}{t}}\right) \quad (t \rightarrow 0)$$

$$\theta(0.01) = 10.000000000000000000 \dots$$

- ▶ So: Jacobi computed the first heat trace asymptotic expansion and in fact the first trace formula.

## Poisson summation formula

- ▶ Poisson summation formula:

$$\sum_{n \in \mathbb{Z}} \delta_n(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x}$$

## Poisson summation formula

- ▶ Poisson summation formula:

$$\sum_{n \in \mathbb{Z}} \delta_n(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x}$$

- ▶ Application: for any lattice  $\Gamma \subset \mathbb{R}^n$ :

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \|\gamma^*\|^2 t} = \frac{\text{Vol}(M)}{(4\pi t)^{d/2}} \sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2 / 4t}$$



## Zeta Values

- ▶ To evaluate

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

to 6 decimal places you need to add 1000,000 terms!

## Zeta Values

- ▶ To evaluate

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

to 6 decimal places you need to add 1000,000 terms!

- ▶ Euler:  $\zeta(2) = 1.644944$

## Zeta Values

- ▶ To evaluate

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

to 6 decimal places you need to add 1000,000 terms!

- ▶ Euler:  $\zeta(2) = 1.644944$

$$\zeta(2) = (\log 2)^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

## Zeta Values

- ▶ To evaluate

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

to 6 decimal places you need to add 1000,000 terms!

- ▶ Euler:  $\zeta(2) = 1.644944$

$$\zeta(2) = (\log 2)^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

- ▶ Dilogarithm function  $Li_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}$

$$Li_2(x) + Li_2(1-x) + \log x \log(1-x) = Li_2(1)$$

## Zeta values

- ▶ Euler computed zeta values  $\zeta(2), \zeta(3), \dots, \zeta(23)$  with at least 15 decimal digits! How? Dilogarithm identities are not useful for finding

$$\zeta(3) = 1,2020569031595942853997 \dots$$

## Zeta values

- ▶ Euler computed zeta values  $\zeta(2), \zeta(3), \dots, \zeta(23)$  with at least 15 decimal digits! How? Dilogarithm identities are not useful for finding

$$\zeta(3) = 1,2020569031595942853997 \dots$$

- ▶ Euler-Maclaurin Summation Formula:

$$\begin{aligned} \sum_{k=a}^b g(k) &= \int_a^b g(x) dx + \frac{g(a) + g(b)}{2} \\ &+ \sum_{j=2}^m \frac{B_j}{j!} (g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m \end{aligned}$$

## Friedmann-Lemaître-Robertson-Walker metric

- ▶ (Euclidean) FLRW metric with the scale factor  $a(t)$ :

$$ds^2 = dt^2 + a^2(t) d\sigma^2.$$

Where  $d\sigma^2$  is the round metric on 3-sphere. It describes a homogeneous, isotropic (expanding or contracting) universe with spatially closed universe.

## Friedmann-Lemaître-Robertson-Walker metric

- ▶ (Euclidean) FLRW metric with the scale factor  $a(t)$ :

$$ds^2 = dt^2 + a^2(t) d\sigma^2.$$

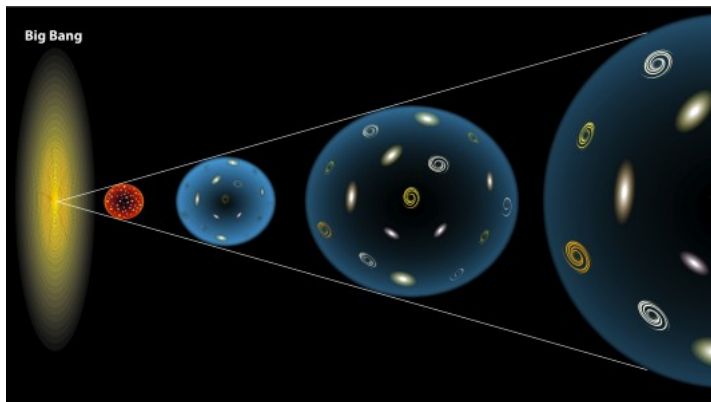
Where  $d\sigma^2$  is the round metric on 3-sphere. It describes a homogeneous, isotropic (expanding or contracting) universe with spatially closed universe.

- ▶ For  $a(t) = \sin(t)$  one obtains the round metric on  $S^4$ .

$$ds^2 = dt^2 + a^2(t) \left( d\chi^2 + \sin^2(\chi) (d\theta^2 + \sin^2(\theta) d\varphi^2) \right)$$



# FLRW Metric



## References

1. Chamseddine and Connes: Spectral Action for Robertson-Walker metrics (2012)
2. Fathizadeh, Ghorbanpour, and Khalkhali: Rationality of Spectral Action for Robertson-Walker Metrics (2014)

# Euler Maclaurin formula and Heat kernel for $S^4$

Euler Maclaurin formula

$$\sum_{k=a}^b g(k) = \int_a^b g(x) dx + \frac{g(a) + g(b)}{2} + \sum_{j=2}^m \frac{B_j}{j!} (g^{(j-1)}(b) - g^{(j-1)}(a)) - R_m$$

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

## Dirac spectrum

- ▶ Spectrum of Dirac for round  $S^4$  :

	eigenvalues	multiplicity
$D$	$\pm k$	$\frac{2}{3}(k^3 - k)$
$D^2$	$k^2$	$\frac{4}{3}(k^3 - k)$

- ▶ To find heat kernel coefficients of  $D^2$  we apply the Euler Maclaurin formula for  $a = 0$ ,  $b = \infty$  and

$$g(x) = \frac{4}{3}(x^3 - x)f(x) = \frac{4}{3}(x^3 - x)e^{-tx^2}$$

The integral term gives

$$\int_a^b g(x) dx = \frac{4}{3} \int_0^\infty (x^3 - x) e^{-tx^2} dx = \frac{2}{3} (t^{-2} - t^{-1})$$

The term  $\frac{g(a)+g(b)}{2}$  is zero since  $g(0) = g(\infty) = 0$ .

And

$$g^{(2m-1)}(0)/(2m-1)! = (-1)^m \frac{4}{3} \left( \frac{t^{m-2}}{(m-2)!} + \frac{t^{m-1}}{(m-1)!} \right)$$

Putting all these together we get

$$\frac{3}{4} \text{Tr}(e^{-tD^2}) = \frac{1}{2t^2} - \frac{1}{2t} + \frac{11}{120} + \sum_{k=1}^m (-1)^k \left( \frac{B_{2k+2}}{2k+2} + \frac{B_{2k+4}}{2k+4} \right) \frac{t^k}{k!} + o(t^m)$$

## Euler Maclaurin formula and spectral action for $S^4$

For general  $f$  the Euler Maclaurin formula gives

$$\begin{aligned} \frac{3}{4} \text{Tr}(f(tD^2)) &= \int_0^\infty f(tx^2)(x^3 - x)dx + \frac{11f(0)}{120} - \frac{31f'(0)}{2520}t \\ &+ \frac{41f''(0)}{10080}t^2 - \frac{31f^{(3)}(0)}{15840}t^3 + \frac{10331f^{(4)}(0)}{8648640}t^4 + \dots + R_m \end{aligned}$$

## Levi-Civita Connection and the Spin Connection

Fix a frame  $\{\theta_\alpha\}$  and coframe  $\{\theta^\alpha\}$ . Connection 1-forms

$$\nabla\theta^\alpha = \omega_\beta^\alpha\theta^\beta.$$

Metric connection:

$$\omega_\beta^\alpha = -\omega_\alpha^\beta.$$

Cartan equations: torsion and curvature 2-forms

$$T^\alpha = d\theta^\alpha - \omega_\beta^\alpha \wedge \theta^\beta$$

For torsion free connections:

$$d\theta^\beta = \omega_\alpha^\beta \wedge \theta^\alpha.$$

## Connection one-form for Levi-civita connection

Orthonormal basis for the cotangent space

$$\theta^1 = dt,$$

$$\theta^2 = a(t) d\chi,$$

$$\theta^3 = a(t) \sin \chi d\theta,$$

$$\theta^4 = a(t) \sin \chi \sin \theta d\varphi.$$

The computation by Chamseddin-Connes shows that the connection one-form is given by

$$\omega = \begin{bmatrix} 0 & -\frac{a'(t)}{a(t)}\theta^2 & -\frac{a'(t)}{a(t)}\theta^3 & -\frac{a'(t)}{a(t)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^2 & 0 & -\frac{\cot(\chi)}{a(t)}\theta^3 & -\frac{\cot(\chi)}{a(t)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^3 & \frac{\cot(\chi)}{a(t)}\theta^3 & 0 & -\frac{\cot(\theta)}{a(t)\sin(\chi)}\theta^4 \\ \frac{a'(t)}{a(t)}\theta^4 & \frac{\cot(\chi)}{a(t)}\theta^4 & \frac{\cot(\theta)}{a\sin(\chi)}\theta^4 & 0 \end{bmatrix}$$



## The Spin Connection

The spin connection is the lift of the Levi-Civita connection defined on  $T^*M$ . Now we have the connection one-forms  $\omega$ , which is a skew symmetric matrix, i.e.  $\omega \in \mathfrak{so}(4)$ . Using the Lie algebra isomorphism  $\mu : \mathfrak{so}(4) \rightarrow \mathfrak{spin}(4)$  given by

$$A \mapsto \frac{1}{4} \sum_{\alpha, \beta} \langle A\theta^\alpha, \theta^\beta \rangle c(\theta^\alpha) c(\theta^\beta)$$

Since  $\omega$  is written in the orthonormal basis  $\theta^\alpha$  so  $\langle \omega\theta^\alpha, \theta^\beta \rangle = \omega_\beta^\alpha$ . So the connection one forms for the spinor connection is given by

$$\tilde{\omega} = \frac{1}{2}\omega_2^1\gamma^{12} + \frac{1}{2}\omega_3^1\gamma^{13} + \frac{1}{2}\omega_4^1\gamma^{14} + \frac{1}{2}\omega_3^2\gamma^{23} + \frac{1}{2}\omega_4^2\gamma^{24} + \frac{1}{2}\omega_4^3\gamma^{34}$$

## Gilkey's local formulae

For an operator of Laplace type  $P = \nabla^* \nabla - E$ ,

$$a_0 = (4\pi)^{-m/2} \operatorname{Tr}(1).$$

$$a_2 = (4\pi)^{-m/2} \operatorname{Tr}\left(E - \frac{1}{6}R\right).$$

$$a_4 = \frac{(4\pi)^{-m/2}}{360} \operatorname{Tr}\left(-12R_{ijij;kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\right).$$

$$\begin{aligned}
a_6 = & (4\pi)^{-m/2} \text{Tr} \left( \frac{1}{7!} \left( -18R_{ijj;kkll} + 17R_{ijj;kR_{ulul;k}} - 2R_{ijik;l}R_{ujuk;l} \right. \right. \\
& -4R_{ijik;l}R_{ujul;k} + 9R_{ijk;l}R_{ijk;l} + 28R_{ijj}R_{kuku;ll} \\
& \left. \left. -8R_{ijjk}R_{ujuk;ll} + 24R_{ijik}R_{ujul;kl} + 12R_{ijkl}R_{ijkl;uu} \right) \right. \\
& + \frac{1}{9 \cdot 7!} \left( -35R_{ijj}R_{klkl}R_{\rho q\rho q} + 42R_{ijj}R_{klkp}R_{qlqp} \right. \\
& -42R_{ijj}R_{kl\rho q}R_{kl\rho q} + 208R_{ijik}R_{julu}R_{kplp} - 192R_{ijik}R_{uplp}R_{jukl} \\
& \left. + 48R_{ijik}R_{julp}R_{kulp} - 44R_{ijku}R_{ijlp}R_{kulp} - 80R_{ijku}R_{ilkp}R_{jilup} \right) \\
& + \frac{1}{360} \left( 8\Omega_{ij;k}\Omega_{ij;k} + 2\Omega_{ij;j}\Omega_{ik;k} + 12\Omega_{ij}\Omega_{ij;kk} - 12\Omega_{ij}\Omega_{jk}\Omega_{ki} \right. \\
& \left. - 6R_{ijkl}\Omega_{ij}\Omega_{kl} + 4R_{ijik}\Omega_{jl}\Omega_{kl} - 5R_{ijj}\Omega_{kl}\Omega_{kl} \right) \\
& + \frac{1}{360} \left( 6E_{;ijj} + 60EE_{;ii} + 30E_{;i}E_{;i} + 60E^3 + 30E\Omega_{ij}\Omega_{ij} \right. \\
& - 10R_{ijj}E_{;kk} - 4R_{ijik}E_{;jk} - 12R_{ijj;k}E_{;k} - 30R_{ijj}E^2 \\
& \left. \left. - 12R_{ijj;kk}E + 5R_{ijj}R_{klkl}E - 2R_{ijik}R_{ijkl}E + 2R_{ijkl}R_{ijkl}E \right) \right).
\end{aligned}$$

For the Dirac operator  $D^2 = \nabla^* \nabla - \frac{1}{4} R$ , so

$$E = \frac{1}{4} R.$$

## Chamseddine-Connes Computations

They used Gilkey's local formulae to obtain

$$a_0 = \frac{a(t)^3}{2}$$

$$a_2 = \frac{1}{4} a(t) (a(t)a''(t) + a'(t)^2 - 1)$$

$$a_4 = \frac{1}{120} (3a^{(4)}(t)a(t)^2 + 3a(t)a''(t)^2 - 5a''(t) + 9a^{(3)}(t)a(t)a'(t) - 4a'(t)^2a''(t))$$

$$a_6 = \frac{1}{5040a(t)^2} (9a^{(6)}(t)a(t)^4 - 21a^{(4)}(t)a(t)^2 - 3a^{(3)}(t)^2a(t)^3 - 56a(t)^2a''(t)^3 + 42a(t)a''(t)^2 + 36a^{(5)}(t)a(t)^3a'(t) + 6a^{(4)}(t)a(t)^3a''(t) - 42a^{(4)}(t)a(t)^2a'(t)^2 + 60a^{(3)}(t)a(t)a'(t)^3 + 21a^{(3)}(t)a(t)a'(t) + 240a(t)a'(t)^2a''(t)^2 - 60a'(t)^4a''(t) - 21a'(t)^2a''(t) - 252a^{(3)}(t)a(t)^2a'(t)a''(t))$$

# Chamseddine-Connes Computations

Using Euler-Maclaurin summation and Feynman-Kac formula they computed up to  $a_{10}$ :

$$\begin{aligned} a_8 = & -\frac{1}{10080a(t)^4} (-a^{(8)}(t)a(t)^6 + 3a^{(6)}(t)a(t)^4 + 13a^{(4)}(t)^2a(t)^5 - 24a^{(3)}(t)^2a(t)^3 - 114a(t)^3a''(t)^4 + \\ & 43a(t)^2a''(t)^3 - 5a^{(7)}(t)a(t)^5a'(t) + 2a^{(6)}(t)a(t)^5a''(t) + 9a^{(6)}(t)a(t)^4a'(t)^2 + 16a^{(3)}(t)a^{(5)}(t)a(t)^5 - \\ & 24a^{(5)}(t)a(t)^3a'(t)^3 - 6a^{(5)}(t)a(t)^3a'(t) + 69a^{(4)}(t)a(t)^4a''(t)^2 - 36a^{(4)}(t)a(t)^3a''(t) + 60a^{(4)}(t)a(t)^2a'(t)^4 + \\ & 15a^{(4)}(t)a(t)^2a'(t)^2 + 90a^{(3)}(t)^2a(t)^4a''(t) - 216a^{(3)}(t)^2a(t)^3a'(t)^2 - 108a^{(3)}(t)a(t)a'(t)^5 - 27a^{(3)}(t)a(t)a'(t)^3 + \\ & 801a(t)^2a'(t)^2a''(t)^3 - 588a(t)a'(t)^4a''(t)^2 - 87a(t)a'(t)^2a''(t)^2 + 108a'(t)^6a''(t) + 27a'(t)^4a''(t) + \\ & 78a^{(5)}(t)a(t)^4a'(t)a''(t) + 132a^{(3)}(t)a^{(4)}(t)a(t)^4a'(t) - 312a^{(4)}(t)a(t)^3a'(t)^2a''(t) - 819a^{(3)}(t)a(t)^3a'(t)a''(t)^2 + \\ & 768a^{(3)}(t)a(t)^2a'(t)^3a''(t) + 102a^{(3)}(t)a(t)^2a'(t)a''(t) \end{aligned}$$

$$\begin{aligned}
a_{10} = & \frac{1}{665280a(t)^6} (3a^{(10)}(t)a(t)^8 - 222a^{(5)}(t)^2a(t)^7 - 348a^{(4)}(t)a^{(6)}(t)a(t)^7 - 147a^{(3)}(t)a^{(7)}(t)a(t)^7 - \\
& 18a''(t)a^{(8)}(t)a(t)^7 + 18a'(t)a^{(9)}(t)a(t)^7 - 482a''(t)a^{(4)}(t)^2a(t)^6 - 331a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - \\
& 1110a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 1556a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 448a''(t)^2a^{(6)}(t)a(t)^6 - \\
& 1074a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 476a'(t)a''(t)a^{(7)}(t)a(t)^6 - 43a'(t)^2a^{(8)}(t)a(t)^6 - 11a^{(8)}(t)a(t)^6 + \\
& 8943a'(t)a^{(3)}(t)^3a(t)^5 + 21846a''(t)^2a^{(3)}(t)^2a(t)^5 + 4092a'(t)^2a^{(4)}(t)^2a(t)^5 + 396a^{(4)}(t)^2a(t)^5 + \\
& 10560a''(t)^3a^{(4)}(t)a(t)^5 + 39402a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 11352a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + \\
& 6336a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 594a^{(3)}(t)a^{(5)}(t)a(t)^5 + 2904a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 264a''(t)a^{(6)}(t)a(t)^5 + \\
& 165a'(t)^3a^{(7)}(t)a(t)^5 + 33a'(t)a^{(7)}(t)a(t)^5 - 10338a''(t)^5a(t)^4 - 95919a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - \\
& 3729a''(t)a^{(3)}(t)^2a(t)^4 - 117600a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - 68664a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - \\
& 2772a''(t)^2a^{(4)}(t)a(t)^4 - 23976a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - 2640a'(t)a^{(3)}(t)a^{(4)}(t)a(t)^4 - \\
& 12762a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 1386a'(t)a''(t)a^{(5)}(t)a(t)^4 - 651a'(t)^4a^{(6)}(t)a(t)^4 - 132a'(t)^2a^{(6)}(t)a(t)^4 + \\
& 111378a''(t)^2a''(t)^4a(t)^3 + 2354a''(t)^4a(t)^3 + 31344a'(t)^4a^{(3)}(t)^2a(t)^3 + 3729a'(t)^2a^{(3)}(t)^2a(t)^3 + \\
& 236706a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 13926a'(t)a''(t)^2a^{(3)}(t)a(t)^3 + 43320a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + \\
& 5214a'(t)^2a''(t)a^{(4)}(t)a(t)^3 + 2238a'(t)^5a^{(5)}(t)a(t)^3 + 462a'(t)^3a^{(5)}(t)a(t)^3 - 162162a'(t)^4a''(t)^3a(t)^2 - \\
& 11880a'(t)^2a''(t)^3a(t)^2 - 103884a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 13332a'(t)^3a''(t)a^{(3)}(t)a(t)^2 - \\
& 6138a'(t)^6a^{(4)}(t)a(t)^2 - 1287a'(t)^4a^{(4)}(t)a(t)^2 + 76440a'(t)^6a''(t)^2a(t) + 10428a'(t)^4a''(t)^2a(t) + \\
& 11700a'(t)^7a^{(3)}(t)a(t) + 2475a'(t)^5a^{(3)}(t)a(t) - 11700a'(t)^8a''(t) - 2475a'(t)^6a''(t))
\end{aligned}$$

## Conjectures and question about coefficients (CC):

- ▶ Check the agreement between the above formulas for  $a_8$  and  $a_{10}$  and the universal formulas.
- ▶ Show that the term  $a_{2n}$  of the asymptotic expansion of the spectral action for Robertson-Walker metric is of the form  $P_n(a, \dots, a^{(2n)})/a^{2n-4}$  where  $P_n$  is a polynomial with rational coefficients and compute  $P_n$ .

## Our approach: spectral analysis via pseudodifferential calculus

$$\begin{aligned} D &= \gamma^\alpha \nabla_{\theta_\alpha} = \gamma^\alpha (\theta_\alpha + \omega(\theta_\alpha)) \\ &= \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{1}{a} \frac{\partial}{\partial \chi} + \gamma^2 \frac{1}{a \sin \chi} \frac{\partial}{\partial \theta} + \gamma^3 \frac{1}{a \sin \chi \sin \theta} \frac{\partial}{\partial \varphi} \\ &\quad + \frac{3a'}{2a} \gamma^0 + \frac{\cot(\chi)}{a} \gamma^1 + \frac{\cot(\theta)}{2a \sin(\chi)} \gamma^2 \end{aligned}$$

So the symbol of the Dirac operator would be

$$\begin{aligned} \sigma_D(\mathbf{x}, \xi) &= i\gamma^0 \xi_1 + \frac{i}{a} \gamma^1 \xi_2 + \frac{i}{a \sin(\chi)} \gamma^2 \xi_3 + \frac{i}{a \sin(\chi) \sin(\theta)} \gamma^3 \xi_4 \\ &\quad + \frac{3a'}{2a} \gamma^0 + \frac{\cot(\chi)}{a} \gamma^1 + \frac{\cot(\theta)}{2a \sin(\chi)} \gamma^2 \end{aligned}$$



## Symbol of $D^2$

Using the symbol multiplication rule one can compute the symbol of the square of the Dirac operator. The symbol of  $D^2$  has following homogeneous parts.

$$p_2 = \xi_1^2 + \frac{1}{a(t)^2} \xi_2^2 + \frac{1}{a(t)^2 \sin^2(\chi)} \xi_3^2 + \frac{1}{a(t)^2 \sin^2(\theta) \sin^2(\chi)} \xi_4^2,$$

$$\begin{aligned} p_1 = & -\frac{3ia'(t)}{a(t)} \xi_1 - \frac{i}{a(t)^2} \left( \gamma^{12} a'(t) + 2 \cot(\chi) \right) \xi_2 \\ & - \frac{i}{a(t)^2} \left( \gamma^{13} \csc(\chi) a'(t) + \cot(\theta) \csc^2(\chi) + \gamma^{23} \cot(\chi) \csc(\chi) \right) \xi_3 \\ & - \frac{i}{a(t)^2} \left( \csc(\theta) \csc(\chi) a'(t) \gamma^{14} + \cot(\theta) \csc(\theta) \csc^2(\chi) \gamma^{34} + \csc(\theta) \cot(\chi) \csc(\chi) \gamma^{24} \right) \xi_4, \end{aligned}$$

$$\begin{aligned} p_0 = & +\frac{1}{8a(t)^2} \left( -12a(t)a''(t) - 6a'(t)^2 + 3 \csc^2(\theta) \csc^2(\chi) - \cot^2(\theta) \csc^2(\chi) \right. \\ & \left. + 4i \cot(\theta) \cot(\chi) \csc(\chi) - 4i \cot(\theta) \cot(\chi) \csc(\chi) - 4 \cot^2(\chi) + 5 \csc^2(\chi) + 4 \right) \\ & - \frac{(\cot(\theta) \csc(\chi) a'(t))}{2a(t)^2} \gamma^{13} - \frac{(\cot(\chi) a'(t))}{a(t)^2} \gamma^{12} - \frac{(\cot(\theta) \cot(\chi) \csc(\chi))}{2a(t)^2} \gamma^{23} \end{aligned}$$

## Symbol of the parametrix

Parametrix:  $(P - \lambda)\tilde{R}(\lambda) = I.$

$$\sigma(\tilde{R}(\lambda)) = r_0 + r_1 + r_2 + \dots$$

Recursive formulas:

$$r_n = -r_0 \sum_{|\alpha| + j + 2 - k = n} (-i)^{|\alpha|} d_\xi^\alpha p_k \cdot d_x^\alpha r_j / \alpha!,$$

where  $r_0 = (p_2 - \lambda)^{-1} = (\|\xi\|^2 - \lambda)^{-1}$ . So the summation, for  $n > 1$ , will only have the following possible summands.

$$k = 0, |\alpha| = 0, j = n - 2 \quad - r_0 p_0 r_{n-2}$$

$$k = 1, |\alpha| = 0, j = n - 1 \quad - r_0 p_1 r_{n-1}$$

$$k = 1, |\alpha| = 0, j = n - 2 \quad ir_0 \frac{\partial}{\partial \xi_0} p_1 \cdot \frac{\partial}{\partial t} r_{n-2} + ir_0 \frac{\partial}{\partial \xi_1} p_1 \cdot \frac{\partial}{\partial x} r_{n-2} + ir_0 \frac{\partial}{\partial \xi_2} p_1 \cdot \frac{\partial}{\partial \theta} r_{n-2}$$

$$k = 2, |\alpha| = 1, j = n - 1 \quad ir_0 \frac{\partial}{\partial \xi_0} p_2 \cdot \frac{\partial}{\partial t} r_{n-1} + ir_0 \frac{\partial}{\partial \xi_1} p_2 \cdot \frac{\partial}{\partial x} r_{n-1} + ir_0 \frac{\partial}{\partial \xi_2} p_2 \cdot \frac{\partial}{\partial \theta} r_{n-1}$$

$$k = 2, |\alpha| = 2, j = n - 2 \quad \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_0^2} p_2 \cdot \frac{\partial^2}{\partial t^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_1^2} p_2 \cdot \frac{\partial^2}{\partial x^2} r_{n-2} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_2^2} p_2 \cdot \frac{\partial^2}{\partial \theta^2} r_{n-2}$$

## Heat Kernel of $D^2$ in terms of symbols of the parametrix.

Let

$$\begin{aligned} e_n &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} r_n(x, \xi, \lambda) d\lambda d\xi \\ &= \frac{1}{2\pi i (2\pi)^4} \sum r_{n,j,\alpha}(x) \int_{\mathbb{R}^4} \xi^\alpha \int_{\gamma} e^{-t\lambda} r_0^j d\lambda d\xi \\ &= \sum c_\alpha \frac{1}{(j-1)!} r_{n,j,\alpha} a(t)^{\alpha_2+\alpha_3+\alpha_4+3} \sin(\chi)^{\alpha_3+\alpha_4+2} \sin(\theta)^{\alpha_4+1} \end{aligned}$$

Where  $c_\alpha = \frac{1}{(2\pi)^4} \prod_k \Gamma\left(\frac{\alpha_k+1}{2}\right) \frac{(-1)^{\alpha_k+1}}{2}$ .

where

$$a_n = \int_0^{2\pi} \int_0^\pi \int_0^\pi e_n d\chi d\theta d\phi$$

## new term $a_{12}$

$$\begin{aligned}
 a_{12} = & \frac{1}{17297280a(t)^8} \left( 3a^{(12)}(t)a(t)^{10} - 1057a^{(6)}(t)^2a(t)^9 - 1747a^{(5)}(t)a^{(7)}(t)a(t)^9 - 970a^{(4)}(t)a^{(8)}(t)a(t)^9 - \right. \\
 & 317a^{(3)}(t)a^{(9)}(t)a(t)^9 - 34a''(t)a^{(10)}(t)a(t)^9 + 21a'(t)a^{(11)}(t)a(t)^9 + 5001a^{(4)}(t)^3a(t)^8 + 2419a''(t)a^{(5)}(t)^2a(t)^8 + \\
 & 19174a^{(3)}(t)a^{(4)}(t)a^{(5)}(t)a(t)^8 + 4086a^{(3)}(t)^2a^{(6)}(t)a(t)^8 + 2970a''(t)a^{(4)}(t)a^{(6)}(t)a(t)^8 - \\
 & 5520a'(t)a^{(5)}(t)a^{(6)}(t)a(t)^8 - 511a''(t)a^{(3)}(t)a^{(7)}(t)a(t)^8 - 4175a'(t)a^{(4)}(t)a^{(7)}(t)a(t)^8 - \\
 & 745a''(t)^2a^{(8)}(t)a(t)^8 - 2289a'(t)a^{(3)}(t)a^{(8)}(t)a(t)^8 - 828a'(t)a''(t)a^{(9)}(t)a(t)^8 - 62a'(t)^2a^{(10)}(t)a(t)^8 - \\
 & 13a^{(10)}(t)a(t)^8 + 45480a^{(3)}(t)^4a(t)^7 + 152962a''(t)^2a^{(4)}(t)^2a(t)^7 + 203971a'(t)a^{(3)}(t)a^{(4)}(t)^2a(t)^7 + \\
 & 21369a'(t)^2a^{(5)}(t)^2a(t)^7 + 1885a^{(5)}(t)^2a(t)^7 + 410230a''(t)a^{(3)}(t)^2a^{(4)}(t)a(t)^7 + 163832a'(t)a^{(3)}(t)^2a^{(5)}(t)a(t)^7 + \\
 & 250584a''(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^7 + 244006a'(t)a''(t)a^{(4)}(t)a^{(5)}(t)a(t)^7 + 42440a''(t)^3a^{(6)}(t)a(t)^7 + \\
 & 163390a'(t)a''(t)a^{(3)}(t)a^{(6)}(t)a(t)^7 + 35550a'(t)^2a^{(4)}(t)a^{(6)}(t)a(t)^7 + 3094a^{(4)}(t)a^{(6)}(t)a(t)^7 + \\
 & 34351a'(t)a''(t)^2a^{(7)}(t)a(t)^7 + 19733a'(t)^2a^{(3)}(t)a^{(7)}(t)a(t)^7 + 1625a^{(3)}(t)a^{(7)}(t)a(t)^7 + \\
 & 6784a'(t)^2a''(t)a^{(8)}(t)a(t)^7 + 520a''(t)a^{(8)}(t)a(t)^7 + 308a'(t)^3a^{(9)}(t)a(t)^7 + 52a'(t)a^{(9)}(t)a(t)^7 - \\
 & 2056720a'(t)a''(t)a^{(3)}(t)^3a(t)^6 - 1790580a''(t)^3a^{(3)}(t)^2a(t)^6 - 900272a'(t)^2a''(t)a^{(4)}(t)^2a(t)^6 - \\
 & 31889a''(t)a^{(4)}(t)^2a(t)^6 - 643407a''(t)^4a^{(4)}(t)a(t)^6 - 1251548a'(t)^2a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - \\
 & 43758a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 4452042a'(t)a''(t)^2a^{(3)}(t)a^{(4)}(t)a(t)^6 - 836214a'(t)a''(t)^3a^{(5)}(t)a(t)^6 - \\
 & 1400104a'(t)^2a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 48620a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 181966a'(t)^3a^{(4)}(t)a^{(5)}(t)a(t)^6 - \\
 & 18018a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 319996a'(t)^2a''(t)^2a^{(6)}(t)a(t)^6 - 11011a''(t)^2a^{(6)}(t)a(t)^6 - \\
 & 115062a'(t)^3a^{(3)}(t)a^{(6)}(t)a(t)^6 - 11154a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 42764a'(t)^3a''(t)a^{(7)}(t)a(t)^6 - \\
 & 4004a'(t)a''(t)a^{(7)}(t)a(t)^6 - 1649a'(t)^4a^{(8)}(t)a(t)^6 - 286a'(t)^2a^{(8)}(t)a(t)^6 + 460769a''(t)^6a(t)^5 + \\
 & 1661518a'(t)^3a^{(3)}(t)^3a(t)^5 + 83486a'(t)a^{(3)}(t)^3a(t)^5 + 13383328a'(t)^2a''(t)^2a^{(3)}(t)^2a(t)^5 + \\
 & 222092a''(t)^2a^{(3)}(t)^2a(t)^5 + 342883a'(t)^4a^{(4)}(t)^2a(t)^5 + 36218a'(t)^2a^{(4)}(t)^2a(t)^5 + \\
 & 7922361a'(t)a''(t)^4a^{(3)}(t)a(t)^5 + 6367314a'(t)^2a''(t)^3a^{(4)}(t)a(t)^5 + 109330a''(t)^3a^{(4)}(t)a(t)^5 +
 \end{aligned}$$

$$\begin{aligned}
& +7065862a'(t)^3 a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 360386a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + \\
& 1918386a'(t)^3 a''(t)^2 a^{(5)}(t)a(t)^5 + 98592a'(t)a''(t)^2 a^{(5)}(t)a(t)^5 + 524802a'(t)^4 a^{(3)}(t)a^{(5)}(t)a(t)^5 + \\
& 55146a'(t)^2 a^{(3)}(t)a^{(5)}(t)a(t)^5 + 226014a'(t)^4 a''(t)a^{(6)}(t)a(t)^5 + 23712a'(t)^2 a''(t)a^{(6)}(t)a(t)^5 + \\
& 8283a'(t)^5 a^{(7)}(t)a(t)^5 + 1482a'(t)^3 a^{(7)}(t)a(t)^5 - 7346958a'(t)^2 a''(t)^5 a(t)^4 - 72761a''(t)^5 a(t)^4 - \\
& 11745252a'(t)^4 a''(t)a^{(3)}(t)^2 a(t)^4 - 725712a'(t)^2 a''(t)a^{(3)}(t)^2 a(t)^4 - 27707028a'(t)^3 a''(t)^3 a^{(3)}(t)a(t)^4 - \\
& 819520a'(t)a''(t)^3 a^{(3)}(t)a(t)^4 - 8247105a'(t)^4 a''(t)^2 a^{(4)}(t)a(t)^4 - 520260a'(t)^2 a''(t)^2 a^{(4)}(t)a(t)^4 - \\
& 1848228a'(t)^5 a^{(3)}(t)a^{(4)}(t)a(t)^4 - 205296a'(t)^3 a^{(3)}(t)a^{(4)}(t)a(t)^4 - 973482a'(t)^5 a''(t)a^{(5)}(t)a(t)^4 - \\
& 110136a'(t)^3 a''(t)a^{(5)}(t)a(t)^4 - 36723a'(t)^6 a^{(6)}(t)a(t)^4 - 6747a'(t)^4 a^{(6)}(t)a(t)^4 + 17816751a'(t)^4 a''(t)^4 a(t)^3 + \\
& 721058a'(t)^2 a''(t)^4 a(t)^3 + 2352624a'(t)^6 a^{(3)}(t)^2 a(t)^3 + 274170a'(t)^4 a^{(3)}(t)^2 a(t)^3 + \\
& 24583191a'(t)^5 a''(t)^2 a^{(3)}(t)a(t)^3 + 1771146a'(t)^3 a''(t)^2 a^{(3)}(t)a(t)^3 + 3256248a'(t)^6 a''(t)a^{(4)}(t)a(t)^3 + \\
& 389376a'(t)^4 a''(t)a^{(4)}(t)a(t)^3 + 135300a'(t)^7 a^{(5)}(t)a(t)^3 + 25350a'(t)^5 a^{(5)}(t)a(t)^3 - 15430357a'(t)^6 a''(t)^3 a(t)^2 - \\
& 1252745a'(t)^4 a''(t)^3 a(t)^2 - 7747848a'(t)^7 a''(t)a^{(3)}(t)a(t)^2 - 967590a'(t)^5 a''(t)a^{(3)}(t)a(t)^2 - \\
& 385200a'(t)^8 a^{(4)}(t)a(t)^2 - 73125a'(t)^6 a^{(4)}(t)a(t)^2 + 5645124a'(t)^8 a''(t)^2 a(t) + 741195a'(t)^6 a''(t)^2 a(t) + \\
& 749700a'(t)^9 a^{(3)}(t)a(t) + 143325a'(t)^7 a^{(3)}(t)a(t) - 749700a'(t)^{10} a''(t) - 143325a'(t)^8 a''(t))
\end{aligned}$$

## Check on round sphere $S^4$

For  $a(t) = \sin(t)$  we have

$$a_{12}(\text{sphere}) = \frac{10331 \sin^3(t)}{8648640}.$$

Hence

$$\int_0^\pi a_{12}(\text{spher}) dt = \frac{4}{3} \frac{10331}{8648640} = \frac{10331}{6486480}.$$

Which agrees with the direct computation done in Connes-Chamseddine.

## Rationality of heat coefficients

**Theorem:** The terms  $a_{2n}$  in the expansion of the spectral action for the Robertson-Walker metric with scale factor  $a(t)$  is of the form

$$\frac{1}{a(t)^{2n-3}} Q_{2n} \left( a(t), a'(t), \dots, a^{(2n)}(t) \right),$$

where  $Q_{2n}$  is a polynomial with *rational* coefficients.



By direct computation in Hopf coordinates, we found the vector fields which respectively form bases for left and right invariant vector fields on  $SU(2)$ :

$$X_1^L = \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2},$$

$$X_2^L = \sin(\phi_1 + \phi_2) \frac{\partial}{\partial \eta} + \cot(\eta) \cos(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_1} - \tan(\eta) \cos(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_2},$$

$$X_3^L = \cos(\phi_1 + \phi_2) \frac{\partial}{\partial \eta} - \cot(\eta) \sin(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_1} + \tan(\eta) \sin(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_2},$$

and  $X_1^R, X_2^R, X_3^R$ . One checks that these vector fields are Killing vector fields for the Robertson-Walker metrics on the four dimensional space.