

Very Basic Noncommutative Geometry

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1 Introduction

One of the major advances of science in the 20th century was the discovery of a mathematical formulation of quantum mechanics by Heisenberg in 1925 [41]. From a mathematical point of view, transition from classical mechanics to quantum mechanics amounts to, among other things, passing from the *commutative algebra of classical observables* to the *noncommutative algebra of quantum mechanical observables*. Recall that in classical mechanics an observable (e.g. energy, position, momentum, etc.) is a function on a manifold called the phase space of the system. Immediately after Heisenberg's work, ensuing papers by Dirac [32] and Born-Heisenberg-Jordan [5], made it clear that a quantum mechanical observable is a (selfadjoint) operator on a Hilbert space called the state space of the system. Thus the commutative algebra of functions on a space is replaced by the noncommutative algebra of operators on a Hilbert space.

A little more than fifty years after these developments, Alain Connes realized that a similar procedure can in fact be applied to areas of mathematics where the classical notions of space (e.g. measure space, locally compact space, or a smooth space) loses its applicability and pertinence and can be replaced by a new idea of space, represented by a noncommutative algebra.

Connes' theory, which is generally known as *noncommutative geometry*, is a rapidly growing new area of mathematics that interacts with and contributes to many disciplines in mathematics and physics. For a recent survey see Connes' article [16]. Examples of such interactions and contributions include: theory of operator algebras, index theory of elliptic operators, algebraic and differential topology, number theory, standard model of elementary particles, quantum Hall effect, renormalization in quantum field theory, and string theory. (For a description of these relations in more details see the report below.) To understand the basic ideas of noncommutative geometry one should perhaps first come to grips with the idea of a *noncommutative space*.

The inadequacy of the classical notions of space manifests itself for example when one deals with highly singular "bad quotients"; spaces such as the quotient of a nice space by the ergodic action of a group or the space of leaves of a foliation. In all these examples the quotient space is typically ill behaved even as a topological space. For example it may fail to be even Hausdorff, or have enough open sets, let alone being a reasonably smooth space. The unitary dual of a noncompact (Lie) group, except when the group is abelian

or almost abelian, is another example of an ill behaved space.

One of Connes' key observations is that in all these situations one can attach a noncommutative algebra, through a *noncommutative quotient construction*, that captures most of the information. Examples of this noncommutative quotient construction include crossed product by action of a group, or by action of a groupoid. In general the noncommutative quotient is the groupoid algebra of a topological groupoid.

Noncommutative geometry has as its limiting case the classical geometry, but geometry expressed in algebraic terms. Thus to understand its relation with classical geometry one should first understand one of the most important ideas of mathematics which can be expressed as a *duality* between commutative algebra and geometry. This is by no means a new observation or a new trend. To the contrary, this duality has always existed and been utilized in mathematics and its applications. The earliest example is perhaps the use of numbers in counting! It is, however, the case that throughout the history each new generation of mathematicians find new ways of formulating this principle and at the same time broaden its scope. Just to mention a few highlights of this rich history we mention Descartes (analytic geometry), Hilbert (affine varieties and commutative algebras), Gelfand-Naimark (locally compact spaces and commutative C^* -algebras), Grothendieck (Schemes and topos theory), and Connes (noncommutative geometry).

A key idea here is the well-known relation between a space and the commutative algebra of functions on that space. More precisely there is a duality between certain categories of geometric spaces and categories of algebras representing those spaces. Noncommutative geometry builds on, and vastly extends, this fundamental duality between geometry and commutative algebras.

For example, by a celebrated theorem of Gelfand and Naimark [35] one knows that the category of locally compact Hausdorff spaces is equivalent to the dual of the category of commutative C^* -algebras. Thus one can think of not necessarily commutative C^* -algebras as the dual of a category of *noncommutative locally compact spaces*. What makes this a successful proposal is first of all a rich supply of examples and secondly the possibility of extending many of the topological and geometric invariants to this new class of spaces. Let us briefly recall a few other examples from a long list of results in mathematics that put in duality certain categories of geometric objects with a corresponding category of algebraic objects.

To wit, Hilbert's Nullstellensatz states that the category of algebraic vari-

eties over an algebraically closed field is equivalent to the dual of the category of finitely generated commutative algebras without nilpotent elements (so called reduced algebras). This is a perfect analogue of the Gelfand-Naimark theorem in the world of commutative algebras.

Similarly, the Serre-Swan theorem states that the category of vector bundles over a compact Hausdorff space (resp. affine algebraic variety) X is equivalent to the category of finitely generated projective modules over the algebra of continuous functions (resp. regular functions) on X .

Thus a pervasive idea in noncommutative geometry is to treat (certain classes) of noncommutative algebras as noncommutative spaces and try to extend tools of geometry, topology, and analysis to this new setting. It should be emphasized, however, that, as a rule, this extension is never straightforward and always involve surprises and new phenomena. For example the theory of the flow of weights and the corresponding modular automorphism group in von Neumann algebras has no counterpart in classical measure theory, though the theory of von Neumann algebras is generally regarded as noncommutative measure theory. Similarly the extension of de Rham homology for manifolds to cyclic cohomology for noncommutative algebras was not straightforward and needed some highly nontrivial considerations.

Of all the topological invariants for spaces, topological K -theory has the most straightforward extension to the noncommutative realm. Recall that topological K -theory classifies vector bundles on a topological space. Using the above mentioned Serre-Swan theorem, it is natural to define, for a not necessarily commutative ring A , $K_0(A)$ as the group defined by the semi-group of isomorphism classes of finite projective A -modules. The definition of $K_1(A)$ follows the same pattern as in the commutative case, provided A is a Banach algebra and the main theorem of topological K -theory, the Bott periodicity theorem, extends to all Banach algebras.

The situation is much less clear for K -homology, the theory dual to K -theory. By the work of Atiyah, Brown-Douglas-Fillmore, and Kasparov, one can say, roughly speaking, that K -homology cycles on a space X are represented by abstract elliptic operators on X and while K -theory classifies vector bundles on X , K -homology classifies the abstract elliptic operators on X . The pairing between K -theory and K -homology takes the form $\langle [D], [E] \rangle =$ the Fredholm index of the elliptic operator D with coefficients in the vector bundle E . Now one good thing about this way of formulating K -homology is that it almost immediately extends to noncommutative C^* -algebras. The two theories are unified in a single theory called KK -theory

due to G. Kasparov.

Cyclic cohomology was discovered by Connes in 1981 [11, 13] as the right noncommutative analogue of de Rham homology of currents and as a target space for noncommutative Chern character maps from both K -theory and K -homology. One of the main motivations of Connes seems to be transverse index theory on foliated spaces. Cyclic cohomology can be used to identify the K -theoretic index of transversally elliptic operators which lie in the K -theory of the noncommutative algebra of the foliation. The formalism of cyclic cohomology and Chern-Connes character maps form an indispensable part of noncommutative geometry. In a different direction, cyclic homology also appeared in the 1983 work of Tsygan [60] and was used, independently, also by Loday and Quillen [54] in their study of the Lie algebra homology of the Lie algebra of stable matrices over an associative algebra. We won't pursue this aspect of cyclic homology in these notes.

A very interesting recent development in cyclic cohomology theory is the *Hopf-cyclic cohomology* of Hopf algebras and Hopf module (co)algebras in general. Motivated by the original work of Connes and Moscovici [18, 19] this theory is now extended and elaborated on by several authors [1, 2, 38, 39, 49, 50, 51, 52]. There are also very interesting relations between cocycles for Hopf-cyclic cohomology theory of the Connes-Moscovici Hopf algebra \mathcal{H}_1 and operations on spaces of modular forms and modular Hecke algebras [20, 21], and spaces of \mathbb{Q} -lattices [23]. We will say nothing about these developments in these notes. Neither we shall discuss the approach of Cuntz and Quillen to cyclic cohomology theory and their celebrated proof of excision property for periodic (bivariant) cyclic cohomology [25, 30, 27, 28, 29].

The following “dictionary” illustrates noncommutative analogues of some of the classical theories and concepts originally conceived for spaces. In these notes we deal only with a few items of this dictionary. For a much fuller account and explanations, as well as applications of noncommutative geometry, the reader should consult Connes' beautiful book [15].

commutative	noncommutative
measure space	von Neumann algebra
locally compact space	C^* - algebra
vector bundle	finite projective module
complex variable	operator on a Hilbert space
real variable	selfadjoint operator
infinitesimal	compact operator
range of a function	spectrum of an operator
K -theory	K -theory
vector field	derivation
integral	trace
closed de Rham current	cyclic cocycle
de Rham complex	Hochschild homology
de Rham cohomology	cyclic homology
Chern character	Chern-Connes character
Chern-Weil theory	noncommutative Chern-Weil theory
elliptic operator	K -cycle
spin Riemannian manifold	spectral triple
index theorem	local index formula
group, Lie algebra	Hopf algebra, quantum group
symmetry	action of Hopf algebra

Noncommutative geometry is already a vast subject. These notes are just meant to be an introduction to a few aspects of this fascinating enterprise. To get a much better sense of the beauty and depth of the subject the reader should consult Connes' magnificent book [15] or his recent survey [16] and references therein. Meanwhile, to give a sense of the state of the subject at the present time, its relation with other fields of mathematics, and its most pressing issues, we reproduce here part of the text of the final report prepared by the organizers of a conference on noncommutative geometry in 2003 ¹:

“1. The Baum-Connes conjecture

¹BIRS Workshop on Noncommutative Geometry, Banff International Research Station, Banff, Alberta, Canada, April 2003, Organized by Alain Connes, Joachim Cuntz, George Elliott, Masoud Khalkhali, and Boris Tsygan. Full report available at: www.pims.math.ca/birs.

This conjecture, in its simplest form, is formulated for any locally compact topological group. There are more general Baum-Connes conjectures with coefficients for groups acting on C^ -algebras, for groupoid C^* -algebras, etc., that for the sake of brevity we don't consider here. In a nutshell the Baum-Connes conjecture predicts that the K -theory of the group C^* -algebra of a given topological group is isomorphic, via an explicit map called the Baum-Connes map, to an appropriately defined K -homology of the classifying space of the group. In other words invariants of groups defined through noncommutative geometric tools coincide with invariants defined through classical algebraic topology tools. The Novikov conjecture on the homotopy invariance of higher signatures of non-simply connected manifolds is a consequence of the Baum-Connes conjecture (the relevant group here is the fundamental group of the manifold). Major advances were made in this problem in the past seven years by Higson-Kasparov, Lafforgue, Nest-Echterhoff-Chabert, Yu, Puschnigg and others.*

2. Cyclic cohomology and KK-theory

A major discovery made by Alain Connes in 1981, and independently by Boris Tsygan in 1983, was the discovery of cyclic cohomology as the right noncommutative analogue of de Rham homology and a natural target for a Chern character map from K -theory and K -homology. Coupled with K -theory, K -homology and KK -theory, the formalism of cyclic cohomology fully extends many aspects of classical differential topology like Chern-Weil theory to noncommutative spaces. It is an indispensable tool in noncommutative geometry. In recent years Joachim Cuntz and Dan Quillen have formulated an alternative powerful new approach to cyclic homology theories which brings with it many new insights as well as a successful resolution of an old open problem in this area, namely establishing the excision property of periodic cyclic cohomology.

For applications of noncommutative geometry to problems of index theory, e.g. index theory on foliated spaces, it is necessary to extend the formalism of cyclic cohomology to a bivariant cyclic theory for topological algebras and to extend Connes's Chern character to a fully bivariant setting. The most general approach to this problem is due to Joachim Cuntz. In fact the approach of Cuntz made it possible to extend the domain (and definition) of KK -theory to very general categories of topological algebras (rather than just C^ -algebras). The fruitfulness of this idea manifests itself in the V. Lafforgue's proof of the*

Baum-Connes conjecture for groups with property T, where the extension of KK functor to Banach algebras plays an important role.

A new trend in cyclic cohomology theory is the study of the cyclic cohomology of Hopf algebras and quantum groups. Many noncommutative spaces, such as quantum spheres and quantum homogeneous spaces, admit a quantum group of symmetries. A remarkable discovery of Connes and Moscovici in the past few years is the fact that diverse structures, such as the space of leaves of a (codimension one) foliation or the space of modular forms, have a unified quantum symmetry. In their study of transversally elliptic operators on foliated manifolds Connes and Moscovici came up with a new noncommutative and non-cocommutative Hopf algebra denoted by \mathcal{H}_n (the Connes-Moscovici Hopf algebra). \mathcal{H}_n acts on the transverse foliation algebra of codimension n foliations and thus appears as the quantized symmetries of a foliation. They noticed that if one extends the noncommutative Chern-Weil theory of Connes from group and Lie algebra actions to actions of Hopf algebras, then the characteristic classes defined via the local index formula are in the image of this new characteristic map. This extension of Chern-Weil theory involved the introduction of cyclic cohomology for Hopf algebras.

3. Index theory and noncommutative geometry

The index theorem of Atiyah and Singer and its various generalizations and ramifications are at the core of noncommutative geometry and its applications. A modern abstract index theorem in the noncommutative setting is the local index formula of Connes and Moscovici. A key ingredient of such an abstract index formula is the idea of an spectral triple due to Connes. Broadly speaking, and neglecting the parity, a spectral triple (A, H, D) consists of an algebra A acting by bounded operators on the Hilbert space H and a self-adjoint operator D on H . This data must satisfy certain regularity properties which constitute an abstraction of basic elliptic estimates for elliptic PDE's acting on sections of vector bundles on compact manifolds. The local index formula replaces the old non-local Chern-Connes cocycle by a new Chern character form $Ch(A, H, D)$ of the given spectral triple in the cyclic complex of the algebra A . It is a local formula in the sense that the cochain $Ch(A, H, D)$ depends, in the classical case, only on the germ of the heat kernel of D along the diagonal and in particular is independent of smooth perturbations. This makes the formula extremely attractive for practical calculations. The challenge now is to apply this formula to diverse situations

beyond the cases considered so far, namely transversally elliptic operators on foliations (Connes and Moscovici) and the Dirac operator on quantum SU_2 (Connes).

4. Noncommutative geometry and number theory

Current applications and connections of noncommutative geometry to number theory can be divided into four categories. (1) The work of Bost and Connes, where they construct a noncommutative dynamical system (B, σ_t) with partition function the Riemann zeta function $\zeta(\beta)$, where β is the inverse temperature. They show that at the pole $\beta = 1$ there is an spontaneous symmetry breaking. The symmetry group of this system is the group of idèles which is isomorphic to the Galois group $\text{Gal}(Q^{\text{ab}}/Q)$. This gives a natural interpretation of the zeta function as the partition function of a quantum statistical mechanical system. In particular the class field theory isomorphism appears very naturally in this context. This approach has been extended to the Dedekind zeta function of an arbitrary number field by Cohen, Harari-Leichtnam, and Arledge-Raeburn-Laca. All these results concern abelian extensions of number fields and their generalization to non-abelian extensions is still lacking. (2) The work of Connes on the Riemann hypothesis. It starts by producing a conjectural trace formula which refines the Arthur-Selberg trace formula. The main result of this theory states that this trace formula is valid if and only if the Riemann hypothesis is satisfied by all L -functions with Grössencharakter on the given number field k . (3) The work of Connes and Moscovici on quantum symmetries of the modular Hecke algebras $\mathcal{A}(\Gamma)$ where they show that this algebra admits a natural action of the transverse Hopf algebra \mathcal{H}_1 . Here Γ is a congruence subgroup of $SL(2, Z)$ and the algebra $\mathcal{A}(\Gamma)$ is the crossed product of the algebra of modular forms of level Γ by the action of the Hecke operators. The action of the generators X, Y and δ_n of \mathcal{H}_1 corresponds to the Ramanujan operator, to the weight or number operator, and to the action of certain group cocycles on $GL^+(2, Q)$, respectively. What is very surprising is that the same Hopf algebra \mathcal{H}_1 also acts naturally on the (noncommutative) transverse space of codimension one foliations. (4) Relations with arithmetic algebraic geometry and Arakelov theory. This is currently being pursued by Consani, Deninger, Manin, Marcolli and others.

5. Deformation quantization and quantum geometry

The noncommutative algebras that appear in noncommutative geometry usually are obtained either as the result of a process called noncommutative quotient construction or by deformation quantization of some algebra of functions on a classical space. These two constructions are not mutually exclusive. The starting point of deformation quantization is an algebra of functions on a Poisson manifold where the Poisson structure gives the infinitesimal direction of quantization. The existence of deformation quantizations for all Poisson manifolds was finally settled by M. Kontsevich in 1997 after a series of partial results for symplectic manifolds. The algebra of pseudodifferential operators on a manifold is a deformation quantization of the algebra of classical symbols on the cosphere bundle of the manifold. This simple observation is the beginning of an approach to the proof of the index theorem, and its many generalizations by Elliott-Natsume-Nest and Nest-Tsygan, using cyclic cohomology theory. The same can be said about Connes's groupoid approach to index theorems. In a different direction, quantum geometry also consists of the study of noncommutative metric spaces and noncommutative complex structures."

Let us now briefly describe the contents of these notes. In Section 2 we describe some of the fundamental algebra-geometry correspondences at work in mathematics. The most basic ones for noncommutative geometry are the Gelfand-Naimark and the Serre-Swan theorems. In Section 3 we describe the noncommutative quotient construction and give several examples. This is one of the most universal methods of constructing noncommutative spaces directly related to classical geometric examples. Section 4 is devoted to cyclic cohomology and its various definitions. In Section 5 we define the Chern-Connes character map, or the noncommutative Chern character map, from K -theory to cyclic cohomology. In an effort to make these notes as self contained as possible, we have added three appendices covering very basic material on C^* -algebras, projective modules, and category theory language.

These notes are partly based on series of lectures I gave at the Fields Institute in Toronto, Canada, in Fall 2002 and at the Institute for Advanced Studies in Physics and Mathematics (IPM), Tehran, Iran, in Spring 2004. I also used part of these notes in my lectures at the *second annual spring institute and workshop on noncommutative geometry* in Spring 2004, Vanderbilt University, USA. It is a great pleasure to thank the organizers of this event, Alain Connes (director), to whom I owe much more than I can adequately express, Dietmar Bisch, Bruce Hughes, Gennady Kasparov, and Guoliang

Yu. I would also like to thank Reza Khosrovshahi the director of the mathematics division of IPM in Tehran whose encouragement and support was instrumental in bringing these notes to existence.

2 Some examples of geometry-algebra correspondence

We give several examples of geometry-commutative algebra correspondences. They all put into correspondence, or duality, certain categories of geometric objects with a category of algebraic objects. Presumably, the more one knows about these relations the better one is prepared to pursue noncommutative geometry.

2.1 Locally compact spaces and commutative C^* -algebras

In functional analysis the celebrated *Gelfand-Naimark Theorem* [35] states that the category of locally compact Hausdorff spaces is anti-equivalent to the category of commutative C^* -algebras:

$$\{\text{locally compact Hausdorff spaces}\} \simeq \{\text{commutative } C^*\text{-algebras}\}^{op}.$$

Let \mathcal{S} be the category whose objects are locally compact Hausdorff spaces and whose morphisms are continuous and *proper* maps. (Recall that a map $f : X \rightarrow Y$ is called proper if for any compact $K \subset Y$, $f^{-1}(K)$ is compact; of course, if X is compact and f is continuous, then f is proper).

Let \mathcal{C} be the category whose objects are commutative C^* -algebras and whose morphisms are *proper* $*$ -homomorphisms. (A $*$ -homomorphism $f : A \rightarrow B$ is called proper if for any approximate identity (e_i) in A , $f(e_i)$ is an approximate identity in B . See Appendix A for definitions.)

Define two contravariant functors

$$C_0 : \mathcal{C} \rightarrow \mathcal{S}, \quad \Omega : \mathcal{S} \rightarrow \mathcal{C},$$

as follows. For a locally compact Hausdorff space X , let $C_0(X)$ denote the algebra of complex valued continuous functions on X that “vanish at ∞ ”. This means for any $\epsilon > 0$ there is a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ for $x \notin K$:

$$C_0(X) = \{f : X \rightarrow \mathbb{C}, f \text{ is continuous and } f(\infty) = 0\}.$$

Under pointwise addition and scalar multiplication $C_0(X)$ is obviously an algebra over the field of complex numbers \mathbb{C} . Endowed with the sup-norm

$$\|f\| = \|f\|_\infty = \sup\{|f(x)|; x \in X\},$$

and $*$ -operation

$$f \mapsto f^*, f^*(x) = \bar{f}(x),$$

one checks that $C_0(X)$ is a commutative C^* -algebra. If $f : X \rightarrow Y$ is a continuous and proper map, let

$$C_0(f) = f^* : C_0(Y) \longrightarrow C_0(X), \quad f^*(g) = g \circ f,$$

be the pullback of f . It is a proper $*$ -homomorphism of C^* -algebras. We have thus defined the functor C_0 .

To define Ω , called the *functor of points* or the *spectrum functor*, let A be a commutative C^* -algebra. Let

$$\Omega(A) = \text{set of characters of } A = \text{Hom}_{\mathbb{C}}(A, \mathbb{C}),$$

where a *character* is simply a nonzero algebra map $A \rightarrow \mathbb{C}$. (it turns out that they are also $*$ -morphisms). $\Omega(A)$ is a locally compact Hausdorff space under the topology of pointwise convergence. Given a proper morphism of C^* -algebras $f : A \rightarrow B$, let

$$\Omega(f) : \Omega(B) \rightarrow \Omega(A), \quad \Omega(f) = f^*,$$

where $f^*(\varphi) = \varphi \circ f$. It can be shown that $\Omega(f)$ is a proper and continuous map.

To show that C_0 and Ω are equivalences of categories, quasi-inverse to each other, one shows that for any locally compact Hausdorff space X and any commutative C^* -algebra A , there are natural isomorphisms

$$X \xrightarrow{\sim} \Omega(C_0(X)), \quad x \mapsto e_x,$$

$$A \xrightarrow{\sim} C_0(\Omega(A)), \quad a \mapsto \hat{a}.$$

Here e_x is the *evaluation at x* map defined by $e_x(f) = f(x)$, and $a \mapsto \hat{a}$ is the celebrated *Gelfand transform* defined by $\hat{a}(\varphi) = \varphi(a)$. The first isomorphism is much easier to establish and does not require the theory of Banach algebras. The second isomorphism is what is proved by Gelfand and Naimark in 1943

[35] using Gelfand’s theory of commutative Banach algebras. We sketch a proof of this result in Appendix A.

Under the Gelfand-Naimark correspondence compact Hausdorff spaces correspond to unital C^* -algebras. We therefore have a duality, or equivalence of categories

$$\{\mathbf{compact\ Hausdorff\ spaces}\} \simeq \{\mathbf{commutative\ unital\ } C^*\text{-algebras}\}^{op}.$$

Based on Gelfand-Naimark theorem, we can think of the dual of the category of not necessarily commutative C^* -algebras as the category of *noncommutative locally compact Hausdorff spaces*. Various operations and concepts for spaces can be paraphrased in terms of algebras of functions on spaces and can then be immediately generalized to noncommutative spaces. This is the easy part of noncommutative geometry! Here is a dictionary suggested by the Gelfand-Naimark theorem:

space	algebra
compact	unital
1-point compactification	unitization
Stone-Cech compactification	multiplier algebra
closed subspace; inclusion	closed ideal; quotient algebra
surjection	injection
injection	surjection
homeomorphism	automorphism
Borel measure	positive functional
probability measure	state
disjoint union	direct sum
cartesian product	minimal tensor product

2.2 Vector bundles and finite projective modules

Swan’s Theorem [59] states that the category of complex vector bundles on a compact Hausdorff space X is equivalent to the category of finite (i.e. finitely generated) projective modules over the algebra $C(X)$ of continuous functions on X :

$$\{\mathbf{vector\ bundles\ on\ } X\} \simeq \{\mathbf{finite\ projective\ } C(X)\text{-modules}\}.$$

There are similar results for real and quaternionic vector bundles [59]. This result was motivated and in fact is the topological counterpart of an analogous result, due to Serre, which characterizes algebraic vector bundles over an

affine algebraic variety as finite projective modules over the coordinate ring of the variety. Swan's theorem sometimes is called the Serre-Swan theorem.

Recall that a right module P over a unital algebra A is called *projective* if there exists a right A -module Q such that

$$P \oplus Q \simeq \bigoplus_I A,$$

is a free module. Equivalently, P is projective if every module surjection $P \rightarrow Q \rightarrow 0$ splits as a right A -module map. P is called *finite* if there exists a surjection $A^n \rightarrow P \rightarrow 0$ for some integer n .

We describe the Serre-Swan correspondence between vector bundles and finite projective modules. Given a vector bundle $p : E \rightarrow X$, let

$$P = \Gamma(E) = \{s : X \rightarrow E; ps = id_X\}$$

be the set of all continuous *global sections* of E . It is clear that under fiberwise scalar multiplication and addition, P is a $C(X)$ module. If $f : E \rightarrow F$ is a bundle map, we define a module map $\Gamma(f) : \Gamma(E) \rightarrow \Gamma(F)$ by $\Gamma(f)(s)(x) = f(s(x))$ for all $s \in \Gamma(E)$ and $x \in X$. We have thus defined a functor Γ , called the global section functor, from the category of vector bundles over X and continuous bundle maps to the category of $C(X)$ -modules and module maps.

Using compactness of X and a partition of unity one shows that there is a vector bundle F on X such that $E \oplus F \simeq X \times \mathbb{C}^n$ is a trivial bundle. Let Q be the space of global sections of F . We have

$$P \oplus Q \simeq A^n,$$

which shows that P is finite projective.

To show that all finite projective $C(X)$ -modules arise in this way we proceed as follows. Given a finite projective $C(X)$ -module P , let Q be a $C(X)$ -module such that $P \oplus Q \simeq A^n$, for some integer n . Let $e : A^n \rightarrow A^n$ be the right A -linear map corresponding to the projection into first coordinate: $(p, q) \mapsto (p, 0)$. It is obviously an idempotent in $M_n(C(X))$. One defines a vector bundle E as the image of this idempotent e :

$$E = \{(x, v); e(x)v = v, \text{ for all } x \in X, v \in \mathbb{C}^n\} \subset X \times \mathbb{C}^n.$$

Now it is easily shown that $\Gamma(E) \simeq P$. With some more work it is shown that the functor Γ is full and faithful and hence defines an equivalence of categories.

Based on the Serre-Swan theorem, one usually thinks of finite projective modules over noncommutative algebras as *noncommutative vector bundles*. We give a few examples starting with a commutative one.

Examples

1. The *Hopf line bundle* on the two sphere S^2 , also known as *magnetic monopole bundle*, can be defined in various ways. (It was discovered, independently, by Hopf and Dirac in 1931, motivated by very different considerations). Here is an approach that lends itself to noncommutative generalizations. Let $\sigma_1, \sigma_2, \sigma_3$, be three matrices in $M_2(\mathbb{C})$ that satisfy the *canonical anticommutation relations*:

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij},$$

for all $i, j = 1, 2, 3$. Here δ_{ij} is the Kronecker symbol. A canonical choice is the so called *Pauli spin matrices*:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define a function

$$F : S^2 \rightarrow M_2(\mathbb{C}), \quad F(x_1, x_2, x_3) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,$$

where x_1, x_2, x_3 are coordinate functions on S^2 so that $x_1^2 + x_2^2 + x_3^2 = 1$. Then $F^2(x) = I_{2 \times 2}$ for all $x \in S^2$ and therefore

$$e = \frac{1 + F}{2}$$

is an idempotent in $M_2(C(S^2))$. It thus defines a complex vector bundle on S^2 . We have,

$$e(x_1, x_2, x_3) = \frac{1}{2} \begin{pmatrix} 1 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & 1 - x_3 \end{pmatrix}.$$

Since

$$\text{rank } F(x) = \text{trace } F(x) = 1$$

for all $x \in S^2$, we have in fact a complex line bundle over S^2 . It can be shown that it is the line bundle associated to the Hopf fibration

$$S^1 \rightarrow S^3 \rightarrow S^2.$$

Incidentally, e induces a map $f : S^2 \rightarrow P^1(\mathbb{C})$, where $f(x)$ is the 1-dimensional subspace defined by the image of $F(x)$, which is 1-1 and onto. Our line bundle is just the pull back of the canonical line bundle over $P^1(\mathbb{C})$.

This example can be generalized to higher dimensional spheres. One can construct matrices $\sigma_1, \dots, \sigma_{2n+1}$ in $M_{2^n}(\mathbb{C})$ satisfying the *Clifford algebra relations* [45]

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij},$$

for all $i, j = 1, \dots, 2n + 1$. Define a matrix valued function F on the $2n$ dimensional sphere S^{2n} , $F \in M_{2^n}(C(S^{2n}))$ by

$$F = \sum_{i=1}^{2n+1} x_i \sigma_i.$$

Then $F^2(x) = 1$ for all $x \in S^{2n}$ and hence $e = \frac{1+F}{2}$ is an idempotent.

2. (Hopf line bundle on quantum spheres)

The Podleś quantum sphere S_q^2 is the $*$ -algebra generated over \mathbb{C} by the elements a, a^* and b subject to the relations

$$aa^* + q^{-4}b^2 = 1, \quad a^*a + b^2 = 1, \quad ab = q^{-2}ba, \quad a^*b = q^2ba^*.$$

The quantum analogue of the Dirac(or Hopf) monopole line bundle over S^2 is given by the following idempotent in $M_2(S_q^2)$ [6]

$$\mathbf{e}_q = \frac{1}{2} \begin{bmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{bmatrix}.$$

It can be directly checked that $\mathbf{e}_q^2 = \mathbf{e}_q$.

3. (Projective modules on the noncommutative torus)

Let us first recall the definition of the *smooth noncommutative torus* \mathcal{A}_θ [10]. Among several possible definitions the following is the most direct one. Let $\theta \in \mathbb{R}$ be a fixed parameter and let

$$\mathcal{A}_\theta = \left\{ \sum_{m,n \in \mathbb{Z}} a_{mn} U^m V^n; (a_{mn}) \in \mathcal{S}(\mathbb{Z}^2) \right\},$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of rapidly decreasing sequences $(a_{mn}) \in \mathbb{C}$ indexed by \mathbb{Z}^2 . The relation

$$VU = e^{2\pi i \theta} UV,$$

defines an algebra structure on \mathcal{A}_θ .

Let $E = \mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions on \mathbb{R} . It is easily checked that the following formulas define an \mathcal{A}_θ module structure on E :

$$(Uf)(x) = f(x + \theta), \quad (Vf)(x) = e^{-2\pi i\theta} f(x).$$

It can be shown that E is finite and projective [10].

This construction can be generalized [10, 24, 58]. Let n, m be integers with $m > 0$ and let $E_{n,m} = \mathcal{S}(\mathbb{R} \times \mathbb{Z}_m)$, where \mathbb{Z}_m is the cyclic group of order m . The following formulas define an \mathcal{A}_θ module structure on $E_{n,m}$:

$$\begin{aligned} (Uf)(x, j) &= f\left(x + \theta - \frac{n}{m}, j - 1\right), \\ (Vf)(x, j) &= e^{2\pi i(x - j\frac{n}{m})} f(x, j). \end{aligned}$$

It can be shown that when $n - m\theta \neq 0$, the module $E_{n,m}$ is finite and projective. In particular for irrational θ it is always finite and projective.

For more examples of noncommutative vector bundles see [22].

2.3 Affine varieties and finitely generated commutative reduced algebras

In commutative algebra, *Hilbert's Nullstellensatz* [9] states that the category of affine algebraic varieties over an algebraically closed field \mathbb{F} is anti-equivalent to the category of finitely generated commutative reduced unital \mathbb{F} algebras:

$$\{\text{affine algebraic varieties}\} \simeq \{\text{finitely generated commutative reduced algebras}\}^{op}.$$

Recall that an *affine algebraic variety* (sometimes called an *algebraic set*) over a field \mathbb{F} is a subset of an affine space \mathbb{F}^n which is the set of zeros of a set of polynomials in n variables over \mathbb{F} . A morphism between affine varieties $V \subset \mathbb{F}^n$ and $W \subset \mathbb{F}^m$ is a map $f : V \rightarrow W$ which is the restriction of a polynomial map $\mathbb{F}^n \rightarrow \mathbb{F}^m$. It is clear that affine varieties and morphisms between them form a category.

A *reduced algebra* is by definition an algebra with no *nilpotent elements*, i.e. if $x^n = 0$ for some n then $x = 0$.

The above correspondence associates to a variety $V \subset \mathbb{F}^n$ its *coordinate ring* $\mathbb{F}[V]$ defined by

$$\mathbb{F}[V] := \text{Hom}_{\text{Aff}}(V, \{pt\}) \simeq \mathbb{F}[x_1, \dots, x_n]/I,$$

where I is the *vanishing ideal* of V defined by

$$I = \{f \in \mathbb{F}[x_1, \dots, x_n]; f(x) = 0 \text{ for all } x \in V\}.$$

Obviously $\mathbb{F}[V]$ is a finitely generated commutative unital reduced algebra. Moreover, Given a morphism of varieties $f : V \rightarrow W$, its pull-back defines an algebra homomorphism $f^* : \mathbb{F}[W] \rightarrow \mathbb{F}[V]$. We have thus defined the contravariant *coordinate ring functor* from affine varieties to finitely generated reduced commutative unital algebras.

Given a finitely generated commutative unital algebra A with n generators we can write it as

$$A \simeq \mathbb{F}[x_1, \dots, x_n]/I,$$

where the ideal I is a *radical ideal* if and only if A is a reduced algebra. Let

$$V := \{x \in \mathbb{F}^n; f(x) = 0 \text{ for all } f \in I\},$$

denote the variety defined by the ideal I . The classical form of Nullstellensatz [40] states that if \mathbb{F} is algebraically closed and A is reduced then A can be recovered as the coordinate ring of the variety V :

$$\mathbb{F}[V] \simeq A = \mathbb{F}[x_1, \dots, x_n]/I.$$

This is the main step in showing that the coordinate ring functor is an anti-equivalence of categories. Showing that the functor is full and faithful is easier. In Section 6 we sketch a proof of this fact when \mathbb{F} is the field of complex numbers.

2.4 Affine schemes and commutative rings

The above correspondence between finitely generated reduced commutative algebras and affine varieties is not an ideal result. One is naturally interested in larger classes of algebras, like algebras with nilpotent elements as well as algebras over fields which are not algebraically closed or algebras over arbitrary rings; this last case is particularly important in number theory.

In general one wants to know what kind of geometric objects correspond to a commutative ring and how this correspondence goes. *Affine schemes* are exactly defined to address this question. We follow the exposition in [40].

Let A be a commutative unital ring. The *prime spectrum* (or simply the *spectrum*) of A is a pair $(\text{Spec}A, \mathcal{O}_A)$ where $\text{Spec}A$ is a topological space and \mathcal{O}_A is a sheaf of rings on $\text{Spec}A$ defined as follows. As a set $\text{Spec}A$ consists of all *prime ideals* of A (an ideal $I \subset A$ is called *prime* if for all a, b in A , $ab \in I$ implies that either $a \in I$, or $b \in I$). Given an ideal $I \subset A$, let $V(I) \subset \text{Spec}A$ be the set of all prime ideals which contain I . We can define a topology on $\text{Spec}A$, called the *Zariski topology*, by declaring sets of the type $V(I)$ to be closed (this makes sense since the easily established relations $V(IJ) = V(I) \cup V(J)$ and $V(\sum I_i) = \cap V(I_i)$ show that the intersection of a family of closed sets is closed and the union of two closed sets is closed as well). One checks that $\text{Spec}A$ is always compact but is not necessarily Hausdorff.

For each prime ideal $p \subset A$, let A_p denote the *localization* of A at p . For an open set $U \subset \text{Spec}A$, let $\mathcal{O}_A(U)$ be the set of all continuous sections $s : U \rightarrow \cup_{p \in U} A_p$. (By definition a section s is called continuous if locally around any point $p \in U$ it is of the form $\frac{f}{g}$, with $g \notin p$). One checks that \mathcal{O}_A is a sheaf of commutative rings on $\text{Spec}A$.

Now $(\text{Spec}A, \mathcal{O}_A)$ is a so called *ringed space* and $A \mapsto (\text{Spec}A, \mathcal{O}_A)$ is functor called the *spectrum functor*. A unital ring homomorphism $f : A \rightarrow B$ defines a continuous map $f^* : \text{Spec}B \rightarrow \text{Spec}A$ by $f^*(p) = f^{-1}(p)$ for all prime ideals $p \subset B$. (note that if I is a maximal ideal $f^{-1}(I)$ is not necessarily maximal. This is one of the reasons one considers, for arbitrary rings, the prime spectrum and not the maximal spectrum.)

An *affine scheme* is a ringed space (X, \mathcal{O}) such that X is homeomorphic to $\text{Spec}A$ for a commutative ring A and \mathcal{O} is isomorphic to \mathcal{O}_A . The spectrum functor defines an equivalence of categories

$$\{\mathbf{affine\ schemes}\} \simeq \{\mathbf{commutative\ rings}\}^{op}.$$

The inverse equivalence is given by the *global section functor* that sends an affine scheme to the ring of its global sections.

In the same vein categories of modules over a ring can be identified with categories of sheaves of modules over the spectrum of the ring. Let A be a commutative ring and let M be an A -module. We define a sheaf of modules \mathcal{M} over $\text{Spec}A$ as follows. For each prime ideal $p \subset A$, let M_p denote the

localization of M at p . For any open set $U \subset \text{Spec} A$ let $\mathcal{M}(U)$ denote the set of continuous sections $s : U \rightarrow \cup_p M_p$ (this means that s is locally a fraction $\frac{m}{f}$ with $m \in M$ and $f \in A_p$). One can recover M from \mathcal{M} by showing that $\tilde{M} \simeq \Gamma \mathcal{M}$ is the space of global sections of M . The functors $M \mapsto \mathcal{M}$ and $\mathcal{M} \mapsto \Gamma \mathcal{M}$ define equivalence of categories [40]:

$$\{\text{modules over } A\} \simeq \{\text{quasi-coherent sheaves on } \text{Spec } A\}.$$

2.5 Compact Riemann surfaces and algebraic function fields

It can be shown that the category of compact Riemann surfaces is anti-equivalent to the category of algebraic function fields:

$$\{\text{compact Riemann surfaces}\} \simeq \{\text{algebraic function fields}\}^{op}.$$

Recall that a *Riemann surface* is a complex manifold of complex dimension one. A morphism between Riemann surfaces X and Y is a holomorphic map $f : X \rightarrow Y$.

An *algebraic function field* is a finite extension of the field $\mathbb{C}(x)$ of rational functions in one variable. A morphism of function fields is simply an algebra map.

To a compact Riemann surface one associates the field $M(X)$ of meromorphic functions on X . For example the field of meromorphic functions on the Riemann sphere is the field of rational functions $\mathbb{C}(x)$. In the other direction, to a finite extension of $\mathbb{C}(x)$ one associates the compact Riemann surface of the algebraic function $p(z, w) = 0$. Here w is a generator of the field over $\mathbb{C}(x)$. This correspondence is essentially due to Riemann.

2.6 Sets and Boolean algebras

Perhaps the simplest notion of space, free of any extra structure, is the concept of a set. In a sense set theory can be regarded as the geometrization of logic. There is a duality between the category of sets and the category of complete atomic Boolean algebras (see, e.g., M. Barr's *Acyclic Models*, CRM Monograph Series, Vol 17, AMS publications, 2002):

$$\{\text{sets}\} \simeq \{\text{complete atomic Boolean algebras}\}^{op}.$$

Recall that a *Boolean algebra* is a unital ring B in which $x^2 = x$ for all x in B . A Boolean algebra is necessarily commutative as can be easily shown. One defines an order relation on B by declaring $x \leq y$ if there is an y' such that $x = yy'$. It can be checked that this is in fact a partial order relation on B . An *Atom* in a Boolean algebra is an element x such that $x > 0$ and there is no y with $0 < y < x$. A Boolean algebra is *atomic* if every element x is the supremum of all the atoms smaller than x . A Boolean algebra is *complete* if every subset has a supremum and infimum. A morphism of complete Boolean algebras is a unital ring map which preserves all infs and sups. (Of course any unital ring map between Boolean algebras preserves finite sups and infs).

Now, given a set S let

$$B = \mathbf{2}^S = \{f : S \longrightarrow \mathbf{2}\},$$

where $\mathbf{2} := \{0, 1\}$. Note that B is a complete atomic Boolean algebra. Any map $f : S \rightarrow T$ between sets defines a morphism of complete atomic Boolean algebras via pullback: $f^*(g) := g \circ f$, and $S \mapsto \mathbf{2}^S$ is a contravariant functor.

In the opposite direction, given a Boolean algebra B , one defines its *spectrum* $\Omega(B)$ by

$$\Omega(B) = \text{Hom}_{\text{Boolean}}(B, \mathbf{2}),$$

where we now think of $\mathbf{2}$ as a Boolean algebra with two elements. It can be shown that the two functors that we have defined are anti-equivalences of categories, quasi-inverse to each other. Thus once again we have a duality between a certain category of geometric objects, namely sets, and a category of commutative algebras, namely complete atomic Boolean algebras.

3 Noncommutative quotients

In this section we recall the method of noncommutative quotients as advanced by Connes in [15]. This is a technique that allows one to replace “bad quotients” by nice noncommutative spaces, represented by noncommutative algebras. In some cases, like noncommutative quotients for group actions, the noncommutative quotient can be defined as a crossed product algebra. In general, however, noncommutative quotients are defined as groupoid algebras. In Section 3.1 we recall the definition of a groupoid together with its various refinements like topological, smooth and étale groupoids. In Section 3.2 we define the groupoid algebra of a groupoid and give several examples.

An important concept is Morita equivalence of algebras. We treat both the purely algebraic theory as well as the notion of strong Morita equivalence for C^* -algebras in Section 3.3. Finally noncommutative quotients are defined in Section 3.4.

3.1 Groupoids

Definition 3.1. *A groupoid is a small category in which every morphism is an isomorphism.*

Let \mathcal{G} be groupoid. We denote the set of objects of \mathcal{G} by $\mathcal{G}^{(0)}$ and, by a small abuse of notation, the set of morphisms of \mathcal{G} by \mathcal{G} . Every morphism has a *source*, has a *target* and has an *inverse*. They define maps, denoted by s , t , and i , respectively,

$$\begin{aligned} s : \mathcal{G} &\longrightarrow \mathcal{G}^{(0)}, & t : \mathcal{G} &\longrightarrow \mathcal{G}^{(0)}, \\ i : \mathcal{G} &\longrightarrow \mathcal{G}. \end{aligned}$$

Composition $\gamma_1 \circ \gamma_2$ of morphisms γ_1 and γ_2 is only defined if $s(\gamma_1) = t(\gamma_2)$. Composition defines a map

$$\circ : \mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2); s(\gamma_1) = t(\gamma_2)\} \longrightarrow \mathcal{G}.$$

Examples

1. (Groups). Every group G defines at least two groupoids in a natural way:

1.a Define a category \mathcal{G} with one object $*$ and

$$Hom_{\mathcal{G}}(*, *) = G,$$

where the composition of morphisms is simply the group multiplication. This is obviously a groupoid.

2.b Define a category \mathcal{G} with

$$obj \mathcal{G} = G, \quad Hom_{\mathcal{G}}(s, t) = \{g \in G; gsg^{-1} = t\}.$$

Again, with composition defined by group multiplication, \mathcal{G} is a groupoid.

2. (Equivalence relations) Let \sim denote an equivalence relation on a set X . We define a groupoid \mathcal{G} , called the *graph of \sim* , as follows. Let

$$\begin{aligned} obj \mathcal{G} = X, \quad Hom_{\mathcal{G}}(x, y) &= * \quad \text{if } x \sim y, \\ &= \emptyset \quad \text{otherwise.} \end{aligned}$$

Note that the set of morphisms of \mathcal{G} is identified with the graph of the relation \sim in the usual sense:

$$\mathcal{G} = \{(x, y); x \sim y\} \subset X \times X.$$

Two extreme cases of this graph construction are particularly important. When the equivalence relation reduces to equality, i.e., $x \sim y$ iff $x = y$, we have

$$\mathcal{G} = \Delta(X) = \{(x, x); x \in X\}.$$

On the other extreme when $x \sim y$ for all x and y , we obtain the *groupoid of pairs* where

$$\mathcal{G} = X \times X.$$

3. (Group actions). Example 1) can be generalized as follows. Let

$$G \times X \longrightarrow X, \quad (g, x) \mapsto gx,$$

denote the action of a group G on a set X . We define a groupoid $\mathcal{G} = X \rtimes G$, called the *transformation groupoid* of the action, as follows. Let $\text{obj } \mathcal{G} = X$, and

$$\text{Hom}_{\mathcal{G}}(x, y) = \{g \in G; gx = y\}.$$

Composition of morphisms is defined via group multiplication. It is easily checked that \mathcal{G} is a groupoid. Its set of morphisms can be identified as

$$\mathcal{G} \simeq X \times G,$$

where the composition of morphisms is given by

$$(gx, h) \circ (x, g) = (x, hg).$$

Note that Example 1.a corresponds to the action of a group on a point and example 1.b corresponds to the action of a group on itself via conjugation.

As we shall see, one can not get very far with just discrete groupoids. To get really interesting examples like the groupoids associated to continuous actions of topological groups and to foliations, one needs to consider topological as well as smooth groupoids, much in the same way as one studies topological and Lie groups.

A *topological groupoid* is a groupoid such that its set of morphisms \mathcal{G} , and set of objects $\mathcal{G}^{(0)}$ are topological spaces, and its composition, source, target and inversion maps are continuous.

A special class of topological groupoids, called étale or r-discrete groupoids, are particularly convenient to work with. An *étale groupoid* is a topological groupoid such that its set of morphisms \mathcal{G} is a locally compact topological space and the fibers of the target map $t : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$

$$\mathcal{G}^x = t^{-1}(x), \quad x \in \mathcal{G}^{(0)},$$

are discrete.

A *Lie groupoid* is a groupoid such that \mathcal{G} and $\mathcal{G}^{(0)}$ are smooth manifolds, the inclusion $\mathcal{G}^{(0)} \rightarrow \mathcal{G}$ as well as the maps s, t, i and the composition map \circ are smooth, and s and t are submersions. This last condition will guarantee that the domain of the composition map $\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2); s(\gamma_1) = t(\gamma_2)\}$ is a smooth manifold.

Examples:

1. Let G be a discrete group acting by homeomorphisms on a locally compact Hausdorff space X . The transformation groupoid $X \rtimes G$ is naturally an étale groupoid. If G is a Lie group acting smoothly on a smooth manifold X , then the transformation groupoid $X \rtimes G$ is a Lie groupoid.

2. Let (V, \mathcal{F}) be a foliated manifold and let T be a complete transversal for the foliation. This means that T is transversal to the leaves of the foliation and each leaf has at least one intersection with T . One defines an (smooth) étale groupoid \mathcal{G} as follows. The objects of \mathcal{G} is the transversal T with its smooth structure. For any two points x and y in T let $Hom_{\mathcal{G}}(x, y) = \emptyset$ if x and y are not in the same leaf. When they are in the same leaf, say L , let $Hom_{\mathcal{G}}(x, y)$ denote the set of all continuous paths in L connecting x and y modulo the equivalence relation defined by *holonomy*. It can be shown that \mathcal{G} is an smooth étale groupoid (cf. [15] for details and many examples).

3.2 Groupoid algebras

The notion of *groupoid algebra* of a groupoid is a generalization of the notion of *group algebra* (or *convolution algebra*) of a group and it reduces to group algebras for groupoids with one object. To define the groupoid algebra of a locally compact topological groupoid in general one needs the analogue of a Haar measure for groupoids called a Haar system. While we won't recall

its general definition here, we should mention that, unlike locally compact groups, an arbitrary locally compact groupoid need not have a Haar system [56]. For discrete groupoids as well as étale groupoids and Lie groupoids, however, the convolution product can be easily defined. We start by recalling the definition of the groupoid algebra of a discrete groupoid. As we shall see, in the discrete case the groupoid algebra can be easily described in terms of matrix algebras and group algebras.

Let \mathcal{G} be a discrete groupoid and let

$$\mathbb{C}\mathcal{G} = \bigoplus_{\gamma \in \mathcal{G}} \mathbb{C}\gamma,$$

denote the vector space generated by the set of morphisms of \mathcal{G} as its basis. The formula

$$\begin{aligned} \gamma_1\gamma_2 &= \gamma_1 \circ \gamma_2, & \text{if } \gamma_1 \circ \gamma_2 \text{ is defined,} \\ \gamma_1\gamma_2 &= 0, & \text{otherwise,} \end{aligned}$$

defines an associative product on $\mathbb{C}\mathcal{G}$. The resulting algebra is called the *groupoid algebra* of the groupoid \mathcal{G} . Note that $\mathbb{C}\mathcal{G}$ is unital if and only if the set $\mathcal{G}^{(0)}$ of objects of \mathcal{G} is finite. The unit then is given by

$$1 = \sum_{x \in \mathcal{G}^{(0)}} id_x.$$

An alternative description of the groupoid algebra $\mathbb{C}\mathcal{G}$ which is more appropriate for generalization to étale and topological groupoids is as follows. Note that

$$\mathbb{C}\mathcal{G} \simeq \{f : \mathcal{G} \rightarrow \mathbb{C}; f \text{ has finite support}\},$$

and the product is given by the *convolution product*

$$(fg)(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2).$$

Given an étale groupoid \mathcal{G} , Let

$$C_c(\mathcal{G}) = \{f : \mathcal{G} \rightarrow \mathbb{C}; f \text{ is continuous and has compact support}\}.$$

Under the above convolution product $C_c(\mathcal{G})$ is an algebra called the *convolution algebra* of \mathcal{G} . Note that for any γ the above sum is finite (why?) and the convolution of two functions with compact support has compact support.

When the groupoid is the transformation groupoid of a group action, the groupoid algebra reduces to a *crossed product algebra*. We recall its general definition. Let G be a discrete group, A be an algebra and let $Aut(A)$ denote the group of automorphisms of A . An *action* of G on A is a group homomorphism

$$\alpha : G \longrightarrow Aut(A).$$

Sometimes one refers to the triple (A, G, α) as a *noncommutative dynamical system*. We use the simplified notation $g(a) := \alpha(g)(a)$. The *crossed product* or *semidirect product algebra* $A \rtimes G$ is defined as follows. As a vector space

$$A \rtimes G = A \otimes \mathbb{C}G.$$

Its product is defined by

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh.$$

It is easily checked that endowed with the above product, $A \rtimes G$ is an associative algebra. It is unital if and only if A is unital and G acts by unital automorphisms.

One checks that $A \rtimes G$ is the universal algebra generated by subalgebras A and $\mathbb{C}G$ subject to the relation

$$gag^{-1} = g(a),$$

for all g in G and a in A .

We need a C^* -algebraic analogue of the above construction. In the above situation assume that A is a C^* -algebra and let $Aut(A)$ denote the group of C^* -automorphisms of A . We define a pre- C^* norm on the algebraic crossed product $A \rtimes G$ as follows. We choose a faithful representation

$$\pi : A \longrightarrow \mathcal{L}(H)$$

of A on a Hilbert space H and then define a faithful representation of the algebraic crossed product $A \rtimes G$ on the Hilbert space $l^2(G, H)$ by

$$(a \otimes g)(\psi)(h) = \pi(a)\psi(g^{-1}h).$$

The *reduced C^* -crossed product* $A \rtimes_r G$ is by definition the completion of the algebraic crossed product with respect to the norm induced from the above faithful representation. It can be shown that this definition is independent

of the choice of the faithful representation π [31].

Examples:

1. It is not difficult to determine the general form of the groupoid algebra of a discrete groupoid. Let

$$\mathcal{G}^0 = \bigcup_i \mathcal{G}_i^0$$

denote the decomposition of the set of objects of \mathcal{G} into its “connected components”. By definition two objects x and y belong to the same connected component if there is a morphism γ with source $(\gamma) = x$ and target $(\gamma) = y$. We have a direct sum decomposition of the groupoid algebra $\mathbb{C}\mathcal{G}$:

$$\mathbb{C}\mathcal{G} \simeq \bigoplus_i \mathbb{C}\mathcal{G}_i.$$

Thus suffices to consider only groupoids with a connected set of objects. Choose an object $x_0 \in \mathcal{G}^0$, and let

$$G = \text{Hom}_{\mathcal{G}}(x_0, x_0)$$

be the *isotropy group* of x_0 . Assume first that $G = \{1\}$ is the trivial group. For simplicity assume that \mathcal{G}^0 is a finite set with n elements. In other words our groupoid \mathcal{G} is the groupoid of pairs on a set of n elements. Recall that

$$\mathcal{G} = \{(i, j); i, j = 1, \dots, n\}$$

with composition given by

$$(l, k) \circ (j, i) = (l, i) \quad \text{if } k = j.$$

(Composition is not defined otherwise).

We claim that

$$\mathbb{C}\mathcal{G} \simeq M_n(\mathbb{C}).$$

Indeed it is easily checked that the map

$$(i, j) \mapsto E_{i,j},$$

where $E_{i,j}$ denote the matrix units defines an algebra isomorphism.

Remark 1. *As is emphasized by Connes in the opening section of [15], this is in fact the way Heisenberg discovered matrices in the context of quantum mechanics [41]. In other words noncommutative algebras appeared first in quantum mechanics as a groupoid algebra! We recommend the reader carefully examine the arguments of [41] and [15].*

In general, but still assuming that $\mathcal{G}^{(0)}$ is connected, it is easy to see that

$$\mathbb{C}\mathcal{G} \simeq \mathbb{C}G \otimes M_n(\mathbb{C}),$$

where $G = \text{Hom}_{\mathcal{G}}(x_0, x_0)$ is the isotropy group of \mathcal{G} .

2. We look at groupoid algebras of certain étale groupoids.

a. We start by an example from [15]: an étale groupoid defined by an equivalence relation. Let

$$X = [0, 1] \times \{1\} \cup [0, 1] \times \{2\}$$

denote the disjoint union of two copies of the interval $[0, 1]$. Let \sim denote the equivalence relation that identifies $(x, 1)$ in the first copy with $(x, 2)$ in the second copy for $0 < x < 1$. Let \mathcal{G} denote the corresponding groupoid with its topology inherited from $X \times X$. It is clear that \mathcal{G} is an étale groupoid. The elements of the groupoid algebra $C_c(\mathcal{G})$ can be identified as continuous matrix valued functions on $[0, 1]$ satisfying some boundary condition:

$$C_c(\mathcal{G}) = \{f : [0, 1] \rightarrow M_2(\mathbb{C}); \quad f(0) \text{ and } f(1) \text{ are diagonal}\}.$$

b. Let X be a locally compact Hausdorff space and assume a discrete group G acts on X by homeomorphisms. The induced action on $A = C_c(X)$, the algebra of continuous \mathbb{C} -valued functions on X with compact support, is defined by

$$(gf)(x) = f(g^{-1}x)$$

for all $f \in A$, $g \in G$, and $x \in X$. Let $\mathcal{G} = X \rtimes G$ denote the transformation groupoid defined by the action of G on X .

Exercise: Show that we have an algebra isomorphism

$$C_c(X \rtimes G) \simeq C_c(X) \rtimes G.$$

The groupoid C^* -algebra $C_r^*(\mathcal{G})$, though we have not defined it as such, turns out to be isomorphic with the reduced crossed product algebra $C_0(X) \rtimes_r$

G .

3. Let X be a locally compact Hausdorff space and let \mathcal{G} denote the groupoid of pairs on X . Its groupoid C^* -algebra $C^*\mathcal{G}$ is defined as follows. Let μ be a positive Borel measure on X . We define a convolution product and an $*$ -operation on the space of morphisms of \mathcal{G} , $C_c(X \times X)$, by

$$\begin{aligned} f * g(x, y) &= \int_X f(x, z)g(z, y)d\mu(z), \\ f^*(x, y) &= \overline{f(y, x)}. \end{aligned}$$

These operations turn $C_c(X \times X)$ into an $*$ -algebra. Convolution product defines a canonical $*$ -representation of this algebra on $L^2(X, \mu)$ by

$$C_c(X \times X) \longrightarrow \mathcal{L}(L^2(X, \mu)), \quad f \mapsto f * -.$$

The integral operator associated to a continuous function with compact support is a compact operator and it can be shown that the completion of the image of this map is the space of compact operators on $L^2(X, \mu)$. Thus we have

$$C^*\mathcal{G} \simeq \mathcal{K}(L^2(X, \mu)).$$

In the other extreme, for the equivalence relation defined by equality the groupoid C^* -algebra is given by

$$C^*\mathcal{G} \simeq C_0(X).$$

These two examples are continuous C^* -analogues of our discrete example 1 above.

3.3 Morita equivalence

3.3.1 Algebraic theory

In this section algebra means an associative unital algebra over a commutative ground ring k . All modules are assumed to be unitary in the sense that the unit of the algebra acts as identity operator on the module. Let A be an algebra. We denote by \mathcal{M}_A the category of right A -modules.

Algebras A and B are called *Morita equivalent* if there is an equivalence of categories

$$\mathcal{M}_A \simeq \mathcal{M}_B.$$

In general there are many ways to define a functor $F : \mathcal{M}_A \rightarrow \mathcal{M}_B$. By a simple observation of Eilenberg-Watts, however, if F preserves finite limits and colimits, then there exists a unique $A - B$ bimodule X such that

$$F(M) = M \otimes_A X, \quad \text{for all } M \in \mathcal{M}_A.$$

Composition of functors obtained in this way simply correspond to the balanced tensor product of the defining bimodules. It is therefore clear that A and B are Morita equivalent if and only if there exists an $A - B$ bimodule X and a $B - A$ bimodule Y such that we have isomorphisms of bimodules

$$X \otimes_B Y \simeq A, \quad Y \otimes_A X \simeq B,$$

where the A -bimodule structure on A is defined by $a(b)c = abc$, and similarly for B . Such bimodules are called *invertible (or equivalence) bimodules*.

Given an $A - B$ bimodule X , we define algebra homomorphisms

$$A \longrightarrow \text{End}_B(X), \quad B \longrightarrow \text{End}_A(X),$$

$$a \mapsto L_a, \quad b \mapsto R_b,$$

where L_a is the operator of left multiplication by a and R_b is the operator of right multiplication by b .

In general it is rather hard to characterize the invertible bimodules. The following theorem is one of the main results of Morita:

Theorem 3.1. *An $A - B$ bimodule X is invertible if and only if X is finite and projective both as a left A -module and as a right B -module and the natural maps*

$$A \rightarrow \text{End}_B(X), \quad B \rightarrow \text{End}_A(X),$$

are algebra isomorphisms.

Example. Any unital algebra A is Morita equivalent to the algebra $M_n(A)$ of $n \times n$ matrices over A . The $(A, M_n(A))$ equivalence bimodule is $X = A^n$ with obvious left A -action and right $M_n(A)$ -action. This example can be generalized as follows.

Example. Let P be a finite projective left A -module and let

$$B = \text{End}_A(P).$$

Then the algebras A and B are Morita equivalent. The equivalence $A - B$ bimodule is given by $X = P$ with obvious $A - B$ bimodule structure. As a special case, we obtain the following geometric example.

Example Let X be a compact Hausdorff space and E a complex vector bundle on X . Then the algebras $A = C(X)$ of continuous functions on X and $B = \Gamma(\text{End}(E))$ of global sections of the endomorphisms bundle of E are Morita equivalent. In fact, in view of Swan's theorem this is a special case of the last example with $P = \Gamma(E)$ the global sections of E . There are analogous results for real as well as quaternionic vector bundles. If X happens to be a smooth manifold we can let A to be the algebra of smooth functions on X and B be the algebra of smooth sections of $\text{End}(E)$.

Given a category \mathcal{C} we can consider the category $\text{Fun}(\mathcal{C})$ whose objects are functors from $\mathcal{C} \rightarrow \mathcal{C}$ and whose morphisms are natural transformations between functors. The *center* of a category \mathcal{C} is by definition the set of natural transformations from the identity functor to itself:

$$\mathcal{Z}(\mathcal{C}) := \text{Hom}_{\text{Fun}(\mathcal{C})}(\text{Id}, \text{Id}).$$

Equivalent categories obviously have isomorphic centers.

Let $\mathcal{Z}(A) = \{a \in A; ab = ba \text{ for all } b \in A\}$ denote the center of an algebra A . It is easily seen that for $\mathcal{C} = \mathcal{M}_A$ the natural map

$$\mathcal{Z}(A) \rightarrow \mathcal{Z}(\mathcal{C}), \quad a \mapsto R_a,$$

where $R_a(m) = ma$ for any module M and any $m \in M$, is one to one and onto. It follows that Morita equivalent algebras have isomorphic centers:

$$A \stackrel{M.E.}{\sim} B \Rightarrow \mathcal{Z}(A) \simeq \mathcal{Z}(B).$$

In particular two commutative algebras are Morita equivalent if and only if they are isomorphic. We say that commutativity is not a *Morita invariant property*.

Exercise: Let A be a unital k -algebra. Show that there is a 1-1 correspondence between space of traces on A and $M_n(A)$. Extend this fact to arbitrary Morita equivalent algebras.

We will see in Section 4 that Morita equivalent algebras have isomorphic Hochschild and cyclic (co)homology groups. They have isomorphic algebraic K -theory as well.

3.3.2 Strong Morita equivalence

Extending the Morita theory to non-unital algebras and to topological algebras needs more work and is not an easy task. For (not necessarily unital) C^* -algebras we have Rieffel's notion of *strong Morita equivalence* that we recall below.

For C^* -algebras one is mostly interested in their $*$ -representations on a Hilbert space. Thus one must consider equivalence $A - B$ bimodules X such that if H is a Hilbert space and a right A -module, then $H \otimes_A X$ is a Hilbert space as well. This leads naturally to the concepts of Hilbert module and Hilbert bimodule that we recall below.

Let B be a not necessarily unital C^* -algebra. A right *Hilbert module* over B is a right B -module X endowed with a B -valued inner product such that X is complete with respect to its natural norm. More precisely, we have a sesquilinear map

$$X \times X \longrightarrow B, \quad (x, y) \mapsto \langle x, y \rangle,$$

such that for all x, y in X and b in B we have

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, yb \rangle = \langle x, y \rangle b, \quad \text{and} \quad \langle x, x \rangle \geq 0 \text{ for } x \neq 0.$$

It can be shown that $\|x\| := \|\langle x, x \rangle\|^{1/2}$ is a norm on X . We assume X is complete with respect to this norm.

Of course, for $B = \mathbb{C}$, a Hilbert B -module is just a Hilbert space. A very simple geometric example to keep in mind is the following. Let M be a compact Hausdorff space and let E be a complex vector bundle on M endowed with a Hermitian inner product. One defines a Hilbert module structure on the space $X = \Gamma(E)$ of continuous sections of E by

$$\langle s, t \rangle (m) = \langle s(m), t(m) \rangle_m$$

for continuous sections s and t and $m \in M$.

A morphism of Hilbert B -modules X and Y is a bounded B -module map $X \rightarrow Y$. Every bounded operator on a Hilbert space has an adjoint. This is not the case for Hilbert modules. (This is simply because, even purely algebraically, a submodule of a module need not have a complementary submodule). A bounded B -linear map $T : X \rightarrow X$ is called *adjointable* if there is a bounded B -linear map $T^* : X \rightarrow X$ such that for all x and y in X we have

$$\langle Tx, y \rangle = \langle x, T^*y \rangle .$$

Let $\mathcal{L}_B(X)$ denote the algebra of bounded adjointable B -module maps $X \rightarrow X$. It is a C^* -algebra.

Let A and B be C^* -algebras. A *Hilbert $A - B$ bimodule* consists of a right Hilbert B -module X and a C^* map

$$A \longrightarrow \mathcal{L}_B(X).$$

C^* -algebras A and B are called *strongly Morita equivalent* if there is a Hilbert $A - B$ bimodule X and a Hilbert $B - A$ bimodule Y such that we have isomorphisms of bimodules

$$X \otimes_B Y \simeq A, \quad Y \otimes_A X \simeq B.$$

(The tensor products are completions of algebraic tensor products with respect to their natural pre-Hilbert module structures.)

Two unital C^* -algebras are strongly Morita equivalent if and only if they are Morita equivalent as algebras. Any C^* -algebra A is Morita equivalent to its *stabilization* $A \otimes \mathcal{K}$, where \otimes is the C^* tensor product and \mathcal{K} is the algebra of compact operators on a Hilbert space. Strongly Morita equivalent algebras have naturally isomorphic topological K -theory. (cf. [37] for more details, and a proof of these statements or references to original sources.)

3.4 Noncommutative quotients

From a purely set theoretic point of view, all one needs to form a quotient space X/\sim is an equivalence relation \sim on a set X . The equivalence relation however is usually obtained from a much richer structure by forgetting part of this structure. For example, \sim may arise from an action of a group G on X where $x \sim y$ if and only if $gx = y$ for some g in G (*orbit equivalence*). Note that there may be, in general, many g with this property. That is x may be identifiable with y in more than one way. Of course when we form the equivalence relation this extra information is lost. The key idea in dealing with bad quotients in Connes' theory is to keep track of this extra information!

We call, rather vaguely, this extra structure the *quotient data*. Now Connes's dictum in forming noncommutative quotients can be summarized as follows:

quotient data \rightsquigarrow groupoid \rightsquigarrow groupoid algebra,

where the noncommutative quotient is defined to be the groupoid algebra itself.

Why is this a reasonable approach? The answer is that first of all, by a theorem of M. Rieffel (see below) when the classical quotient, defined by a group action, is a reasonable space, the algebra of continuous functions on the classical quotient is strongly Morita equivalent to the groupoid algebra. Now it is known that Morita equivalent algebras have isomorphic K -theory, Hochschild and cyclic (co)homology groups. Thus the topological invariants defined via noncommutative geometry are the same for the two constructions and no information is lost.

For bad quotients there is no reasonable space but we think of the noncommutative algebra defined as a groupoid algebra as representing a noncommutative quotient space. Thanks to noncommutative geometry, tools like K -theory, K -homology, cyclic cohomology and the local index formula, etc., can be applied to great advantage in the study of these noncommutative spaces.

Example 1.

a) We start with a simple example from [15]. Let $X = \{a, b\}$ be a set with two elements and define an equivalence relation on X that identifies a and b , $a \sim b$:

$$\begin{array}{ccc} a & & b \\ \bullet & \longleftrightarrow & \bullet \end{array}$$

The corresponding groupoid here is the groupoid of pairs on the set X . By Example 1 in Section 3.3 its groupoid algebra is the algebra of 2 by 2 matrices $M_2(\mathbb{C})$. The identification is given by

$$f_{aa}(a, a) + f_{ab}(a, b) + f_{ba}(b, a) + f_{bb}(b, b) \mapsto \begin{pmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{pmatrix}.$$

The algebra of functions on the classical quotient, on the other hand, is given by

$$\{f : X \rightarrow \mathbb{C}; f(a) = f(b)\} \simeq \mathbb{C}.$$

Thus the classical quotient and the noncommutative quotient are Morita equivalent.

$$M_2(\mathbb{C}) \xleftarrow{\text{noncommutative quotient}} \begin{array}{ccc} a & & b \\ \bullet & \longleftrightarrow & \bullet \end{array} \xrightarrow{\text{classical quotient}} \mathbb{C}$$

b) The above example can be generalized. For example let X be a finite set with n elements with the equivalence relation $x \sim y$ for all x, y in X . The corresponding groupoid \mathcal{G} is the groupoid of pairs and its groupoid algebra, representing the noncommutative quotient, is

$$\mathbb{C}\mathcal{G} \simeq M_n(\mathbb{C}).$$

The algebra of functions on the classical quotient is given by

$$\{f : X \rightarrow \mathbb{C}; f(a) = f(b) \text{ for all } a, b \text{ in } X\} \simeq \mathbb{C}.$$

Again the classical quotient is obviously Morita equivalent to the noncommutative quotient.

c) Let G be a group (not necessarily finite) acting on a finite set X . The algebra of functions on the classical quotient is

$$C(X/G) = \{f : X \rightarrow \mathbb{C}; f(x) = f(gx) \text{ for all } g \in G, x \in X\} \simeq \bigoplus_{\mathcal{O}} \mathbb{C},$$

where \mathcal{O} denotes the set of orbits of X under the action of G .

The noncommutative quotient, on the other hand, is defined to be the groupoid algebra of the transformation groupoid $\mathcal{G} = X \rtimes G$. Note that as we saw before this algebra is isomorphic to the crossed product algebra $C(X) \rtimes G$. From Section 3.3 we have,

$$\mathbb{C}\mathcal{G} \simeq C(X) \rtimes G \simeq \bigoplus_{i \in \mathcal{O}} G_i \otimes M_{n_i}(\mathbb{C}),$$

where G_i is the isotropy group of the i -th orbit, and n_i is the size of the i -th orbit. Comparing the classical quotient with the noncommutative quotient we see that:

i) If the action of G is free then $G_i = \{1\}$ for all orbits i and therefore the two algebras are Morita equivalent:

$$C(X/G) \simeq \bigoplus_{\mathcal{O}} \mathbb{C} \stackrel{M.E.}{\sim} \bigoplus_{i \in \mathcal{O}} M_{n_i}(\mathbb{C}) \simeq \mathbb{C}\mathcal{G}.$$

ii) The information about the isotropy groups is not lost in the noncommutative quotient construction, while the classical quotient totally neglects the

isotropy groups.

d) Let X be a locally compact Hausdorff space and consider the equivalence relation $x \sim y$ for all x and y in X . The corresponding groupoid is again the groupoid of pairs. It is a locally compact topological groupoid and its groupoid C^* -algebra as we saw in Section 3.3 is the algebra of compact operators $\mathcal{K}(L^2(X, \mu))$. This algebra is obviously strongly Morita equivalent to the classical quotient algebra \mathbb{C} .

Example 2. Let $\theta \in \mathbb{R}$ be a fixed real number. Consider the action of the group of integers $G = \mathbb{Z}$ on the unit circle $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ via *rotation by θ* :

$$(n, z) \mapsto e^{2\pi i n \theta} z.$$

For $\theta = \frac{p}{q}$ a rational number, the quotient space \mathbb{T}/\mathbb{Z} is a circle and the classical quotient algebra

$$“C(\mathbb{T}/\mathbb{Z})” := \{f \in C(\mathbb{T}); f(gz) = f(z), \text{ for all } g, z\} \simeq C(\mathbb{T}).$$

The noncommutative quotient $A_\theta = C(\mathbb{T}) \rtimes_r \mathbb{Z}$, for any θ , is the unital C^* -algebra generated by two unitaries U and V subject to the relation

$$VU = e^{2\pi i \theta} UV.$$

It can be shown that when θ is a rational number A_θ is isomorphic to the space of continuous sections of the endomorphism bundle $End(E)$ of a complex vector bundle E over the 2-torus \mathbb{T}^2 . Thus for θ rational, A_θ is Morita equivalent to $C(\mathbb{T}^2)$:

$$C(\mathbb{T}) \rtimes_r \mathbb{Z} \stackrel{M.E.}{\sim} C(\mathbb{T}^2).$$

If θ is an irrational number then each orbit is dense in \mathbb{T} and the quotient space \mathbb{T}/\mathbb{Z} has only two open set. It is an uncountable set with a trivial topology. In particular it is not Hausdorff. Obviously, a continuous function on the circle which is constant on each orbit is necessarily constant since orbits are dense. Therefore

$$“C(\mathbb{T}/\mathbb{Z})” \simeq \mathbb{C}.$$

The noncommutative quotient A_θ in this case is a simple C^* -algebra and is not Morita equivalent to $C(\mathbb{T}^2)$.

Let G be a discrete group acting by homeomorphisms on a locally compact Hausdorff space X . Recall that the action is called *free* if for all $g \neq e$, we have $gx \neq x$ for all $x \in X$. The action is called *proper* if the map $G \times X \rightarrow X, (g, x) \mapsto gx$ is a proper map in the sense that the inverse image of a compact set is compact. One shows that when the action is free and proper the orbit space X/G of a locally compact and Hausdorff space is again a locally compact and Hausdorff space. Similarly, if X is a smooth manifold and the action is free and proper then there exists a unique smooth structure on X/G such that the quotient map $X \rightarrow X/G$ is smooth.

The following result of M. Rieffel [57] clarifies the relation between the classical quotients and noncommutative quotients for group actions:

Theorem 3.2. *Assume G acts freely and properly on a locally compact Hausdorff space X . Then we have a strong Morita equivalence between the C^* -algebras $C_0(X/G)$ and $C_0(X) \rtimes_r G$.*

4 Cyclic cohomology

Cyclic cohomology was discovered by Alain Connes in 1981 [11]. One of Connes' main motivations came from index theory on foliated spaces. The K -theoretic index of a transversally elliptic operator on a foliated manifold is an element of the K -theory group of a noncommutative algebra, called the foliation algebra of the given foliated manifold. Connes realized that to identify this class it would be desirable to have a noncommutative analogue of the Chern character with values in a, as yet unknown, cohomology theory for noncommutative algebras that would play the role of de Rham homology of smooth manifolds.

Now to define a noncommutative de Rham theory for noncommutative algebras is a highly nontrivial matter. This is in sharp contrast with the situation in K -theory where extending the topological K -theory to Banach algebras is essentially a routine matter. Note that the usual algebraic formulation of de Rham theory starts with the module of Kaehler differentials and its exterior algebra which does not make sense for noncommutative algebras.

Instead the answer was found by Connes by analyzing the algebraic structures hidden in *traces of products of commutators*. These expressions are directly defined in terms of an elliptic operator and its parametrix and were shown, via an index formula, to give the index of the operator when paired with a K -theory class.

Let us read what Connes wrote in the Oberwolfach conference notebook after his talk, summarizing his discovery and how he arrived at it [11]:

“The transverse elliptic theory for foliations requires as a preliminary step a purely algebraic work, of computing for a noncommutative algebra \mathcal{A} the cohomology of the following complex: n -cochains are multilinear functions $\varphi(f^0, \dots, f^n)$ of $f^0, \dots, f^n \in \mathcal{A}$ where

$$\varphi(f^1, \dots, f^0) = (-1)^n \varphi(f^0, \dots, f^n)$$

and the boundary is

$$\begin{aligned} b\varphi(f^0, \dots, f^{n+1}) &= \varphi(f^0 f^1, \dots, f^{n+1}) - \varphi(f^0, f^1 f^2, \dots, f^{n+1}) + \dots \\ &\quad + (-1)^{n+1} \varphi(f^{n+1} f^0, \dots, f^n). \end{aligned}$$

The basic class associated to a transversally elliptic operator, for $\mathcal{A} =$ the algebra of the foliation is given by:

$$\varphi(f^0, \dots, f^n) = \text{Trace}(\varepsilon F[F, f^0][F, f^1] \cdots [F, f^n]), \quad f^i \in \mathcal{A}$$

where

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and Q is a parametrix of P . An operation

$$S : H^n(\mathcal{A}) \rightarrow H^{n+2}(\mathcal{A})$$

is constructed as well as a pairing

$$K(\mathcal{A}) \times H(\mathcal{A}) \rightarrow \mathbb{C}$$

where $K(\mathcal{A})$ is the algebraic K -theory of A . It gives the index of the operator from its associated class φ . Moreover $\langle e, \varphi \rangle = \langle e, S\varphi \rangle$ so that the important group to determine is the inductive limit $H_g = \varinjlim H^n(\mathcal{A})$ for the map S . Using the tools of homological algebra the groups $H^n(\mathcal{A}, \mathcal{A}^)$ of Hochschild cohomology with coefficients in the bimodule \mathcal{A}^* are easier to determine and the solution of the problem is obtained in two steps,*

1) the construction of a map

$$B : H^n(\mathcal{A}, \mathcal{A}^*) \rightarrow H^{n-1}(\mathcal{A})$$

and the proof of a long exact sequence

$$\dots \rightarrow H^n(\mathcal{A}, \mathcal{A}^*) \xrightarrow{B} H^{n-1}(\mathcal{A}) \xrightarrow{S} H^{n+1}(\mathcal{A}) \xrightarrow{I} H^{n+1}(\mathcal{A}, \mathcal{A}^*) \rightarrow \dots$$

where I is the obvious map from the cohomology of the above complex to the Hochschild cohomology.

2) The construction of a spectral sequence with E_2 term given by the cohomology of the degree -1 differential $I \circ B$ on the Hochschild groups $H^n(\mathcal{A}, \mathcal{A}^*)$ and which converges strongly to a graded group associated to the inductive limit.

This purely algebraic theory is then used. For $\mathcal{A} = C^\infty(V)$ one gets the de Rham homology of currents, and for the pseudo torus, i.e. the algebra of the Kronecker foliation, one finds that the Hochschild cohomology depends on the Diophantine nature of the rotation number while the above theory gives H_g^0 of dimension 1, H_g^1 of dimension 2, and H_g^2 of dimension 1 as expected but from some remarkable cancellations”.

In a different direction, cyclic homology also appeared in the 1983 work of Tsygan [60] and was used also, independently, by Loday and Quillen [54]. The Loday-Quillen-Tsygan theorem states that the cyclic homology of an algebra A is the primitive part (in the sense of Hopf algebras) of the Lie algebra homology of the Lie algebra $gl(A)$ of stable matrices. Equivalently, the Lie algebra homology of $gl(A)$ is isomorphic with the exterior algebra over the cyclic homology of A with dimension shifted by 1:

$$H_\bullet^{Lie}(gl(A)) \simeq \wedge(HC_\bullet(A)[-1]).$$

We won't pursue this connection in these notes.

In section 4.1 we recall basic notions of Hochschild (co)homology theory and give several computations. Theorems of Connes [13] (resp. Hochschild-Kostant-Rosenberg [44]) on the Hochschild cohomology of the algebra of smooth functions on a manifold (resp. algebra of regular functions on a smooth affine variety) are among the most important results of this theory.

In Section 4.2 we define cyclic cohomology via Connes' cyclic complex. The easiest approach, perhaps, to introduce the map B is to introduce first a bicomplex called cyclic bicomplex. This leads to a new definition of cyclic (co)homology, a definition of the operator B , and a proof of the long exact sequence of Connes, relating Hochschild, and cyclic cohomology groups. A third definition of cyclic cohomology is via Connes's (b, B) -bicomplex. The

equivalence of these three definitions is established by explicit maps. Finally we recall Connes' computation of the cyclic (co)homology of the algebra of smooth functions on a manifold, and Burghelea's result on the cyclic homology of group rings.

4.1 Hochschild (co)homology

Hochschild cohomology of associative algebras was defined by G. Hochschild through an explicit complex in [43]. This complex is a generalization of the standard complex for group cohomology. One of the original motivations was to give a cohomological criterion for separability of algebras as well as a classification of (simple types) of algebra extensions in terms of second cohomology. Once it was realized, by Cartan and Eilenberg [8], that Hochschild cohomology is an example of their newly discovered theory of derived functors, tools of homological algebra like resolutions became available.

The Hochschild-Kostant-Rosenberg theorem [44] and its smooth version by Connes[13] identifies the Hochschild homology of the algebra of regular functions on a smooth affine variety or smooth functions on a manifold with differential forms and is among the most important results of this theory. Because of this result one usually thinks of the Hochschild homology of an algebra A with coefficients in A as a noncommutative analogue of differential forms on A .

As we shall see later in this section Hochschild (co)homology is related to cyclic (co)homology through Connes' long exact sequence. For this reason computing the Hochschild (co)homology is often the first step in computing the cyclic (co)homology of a given algebra.

Let A be an algebra and let M be an A -bimodule. Thus M is a left and right A -module and the two actions are compatible in the sense that $a(mb) = (am)b$, for all a, b in A and m in M . The *Hochschild cochain complex of A with coefficients in M* , denoted $(C^\bullet(A, M), \delta)$, is defined as

$$C^0(A, M) = M, \quad C^n(A, M) = \text{Hom}(A^{\otimes n}, M), \quad n \geq 1,$$

$$(\delta m)(a) = ma - am,$$

$$\begin{aligned} (\delta f)(a_1, \dots, a_{n+1}) &= a_1 f(a_2, \dots, a_{n+1}) + \sum_{i=1}^n (-1)^{i+1} f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1}, \end{aligned}$$

where $m \in M = C^0(A, M)$, and $f \in C^n(A, M)$, $n \geq 1$.

One checks that

$$\delta^2 = 0.$$

The cohomology of the complex $(C^\bullet(A, M), \delta)$ is by definition the *Hochschild cohomology* of A with coefficients in M and will be denoted by $H^\bullet(A, M)$.

Among all bimodules M over an algebra A , the following two bimodules play an important role:

1) $M=A$, with bimodule structure $a(b)c = abc$, for all a, b, c in A . The Hochschild complex $C^\bullet(A, A)$ is also known as the *deformation complex*, or *Gerstenhaber complex* of A . It plays an important role in deformation theory of associative algebras pioneered by Gerstenhaber [36]. For example it is easy to see that $H^2(A, A)$ is the space of *infinitesimal deformations* of A and $H^3(A, A)$ is the *space of obstructions* for deformations of A .

2) $M = A^* = Hom(A, k)$ with bimodule structure defined by

$$(afb)(c) = f(bca),$$

for all a, b, c in A , and f in A^* . This bimodule is relevant to cyclic cohomology. Indeed as we shall see the Hochschild groups $H^\bullet(A, A^*)$ and the cyclic cohomology groups $HC^\bullet(A)$ enter into a long exact sequence (Connes's long sequence). Using the identification

$$Hom(A^{\otimes n}, A^*) \simeq Hom(A^{\otimes(n+1)}, k), \quad f \mapsto \varphi,$$

$$\varphi(a_0, a_1, \dots, a_n) = f(a_1, \dots, a_n)(a_0),$$

the Hochschild differential δ is transformed into the differential b given by

$$\begin{aligned} b\varphi(a_0, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \dots, a_n). \end{aligned}$$

Thus for $n = 0, 1, 2$ we have the following formulas for b :

$$\begin{aligned} b\varphi(a_0, a_1) &= \varphi(a_0 a_1) - \varphi(a_1 a_0), \\ b\varphi(a_0, a_1, a_2) &= \varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, a_1), \\ b\varphi(a_0, a_1, a_2, a_3) &= \varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3) \\ &\quad + \varphi(a_0, a_1, a_2 a_3) - \varphi(a_3 a_0, a_1, a_2). \end{aligned}$$

We give a few examples of Hochschild cohomology in low dimensions.

Examples

1. $n = 0$. It is clear that

$$H^0(A, M) = \{m \in M; ma = am \text{ for all } a \in A\}.$$

In particular for $M = A^*$,

$$H^0(A, A^*) = \{f : A \rightarrow k; f(ab) = f(ba) \text{ for all } a, b \in A\},$$

is the space of traces on A .

Exercise: For $A = k[x, \frac{d}{dx}]$, the algebra of differential operators with polynomial coefficients, show that $H^0(A, A^*) = 0$.

2. $n = 1$. A Hochschild 1-cocycle $f \in C^1(A, M)$ is simply a *derivation*, i.e. a linear map $f : A \rightarrow M$ such that

$$f(ab) = af(b) + f(a)b,$$

for all a, b in A . A cocycle is a *coboundary* if and only if the corresponding derivation is *inner*, that is there exists m in M such that $f(a) = ma - am$ for all a in A . Therefore

$$H^1(A, M) = \frac{\text{derivations}}{\text{inner derivations}}$$

Sometimes this is called the space of *outer derivations* of A to M .

Exercise: 1) Show that any derivation on the algebra $C(X)$ of continuous functions on a compact Hausdorff space X is zero. (Hint: If $f = g^2$ and $g(x) = 0$ then $f'(x) = 0$.)

2) Show that any derivation on the matrix algebra $M_n(k)$ is inner. (This was proved by Dirac in [32] where derivations are called *quantum differentials*.)

3) Show that any derivation on the *Weyl algebra* $A = k[x, \frac{d}{dx}]$ is inner as well.

3. $n = 2$. We show, following Hochschild [43], that $H^2(A, M)$ classifies *abelian extensions* of A by M . Let A be a unital algebra over a field k . By definition, an abelian extension is an exact sequence of algebras

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0,$$

such that B is unital, M has trivial multiplication ($M^2 = 0$), and the induced A -bimodule structure on M coincides with the original bimodule structure. Let $E(A, M)$ denote the set of isomorphism classes of such extensions. We define a natural bijection

$$E(A, M) \simeq H^2(A, M)$$

as follows. Given an extension as above, let $s : A \rightarrow B$ be a linear splitting for the projection $B \rightarrow A$, and let $f : A \otimes A \rightarrow M$ be its *curvature* defined by,

$$f(a, b) = s(ab) - s(a)s(b),$$

for all a, b in A . One can easily check that f is a Hochschild 2-cocycle and its class is independent of the choice of splitting s . In the other direction, given a 2-cochain $f : A \otimes A \rightarrow M$, we try to define a multiplication on $B = A \oplus M$ via

$$(a, m)(a', m') = (aa', am' + ma' + f(a, a')).$$

It can be checked that this defines an associative multiplication if and only if f is a 2-cocycle. The extension associated to a 2-cocycle f is the extension

$$0 \longrightarrow M \longrightarrow A \oplus M \longrightarrow A \longrightarrow 0.$$

It can be checked that these two maps are bijective and inverse to each other.

We show that Hochschild cohomology is a derived functor. Let A^{op} denote the *opposite algebra* of A , where $A^{op} = A$ and the new multiplication is defined by $a.b := ba$. There is a one to one correspondence between A -bimodules and left $A \otimes A^{op}$ -modules defined by

$$(a \otimes b^{op})m = amb.$$

Define a functor from the category of left $A \otimes A^{op}$ modules to k -modules by

$$M \mapsto Hom_{A \otimes A^{op}}(A, M) = \{m \in M; ma = am \text{ for all } a \in A\} = H^0(A, M).$$

To show that Hochschild cohomology is the derived functor of the functor $Hom_{A \otimes A^{op}}(A, -)$, we introduce the *bar resolution* of A . It is defined by

$$0 \longleftarrow A \xleftarrow{b'} B_1(A) \xleftarrow{b'} B_2(A) \cdots,$$

where $B_n(A) = A \otimes A^{op} \otimes A^{\otimes n}$ is the free left $A \otimes A^{op}$ module generated by $A^{\otimes n}$. The differential b' is defined by

$$\begin{aligned} b'(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) &= aa_1 \otimes b \otimes a_2 \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (a \otimes b \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \cdots \otimes a_n) \\ &\quad + (-1)^n (a \otimes a_n b \otimes a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

Define the operators $s : B_n(A) \rightarrow B_{n+1}(A)$, $n \geq 0$, by

$$s(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes b \otimes a \otimes a_1 \otimes \cdots \otimes a_n.$$

One checks that

$$b's + sb' = id$$

which shows that $(B_\bullet(A), b')$ is acyclic. Thus $(B_\bullet(A), b')$ is a projective resolution of A as an A -bimodule. Now, for any A -bimodule M we have

$$Hom_{A \otimes A^{op}}(B_\bullet(A), M) \simeq (C^\bullet(A, M), \delta),$$

which shows that Hochschild cohomology is a derived functor.

One can therefore use resolutions to compute Hochschild cohomology groups. Here are a few exercises

1. Let

$$A = T(V) = k \oplus V \oplus V^{\otimes 2} \oplus \cdots,$$

be the tensor algebra of a vector space V . Show that

$$0 \longleftarrow T(V) \xleftarrow{\delta} T(V) \otimes T(V) \xleftarrow{\delta} T(V) \otimes V \otimes T(V) \longleftarrow 0,$$

$$\delta(x \otimes y) = xy, \quad \delta(x \otimes v \otimes y) = xv \otimes y - x \otimes vy,$$

is a free resolution of $T(V)$. Conclude that A has Hochschild cohomological dimension 1 in the sense that $H^n(A, M) = 0$ for all M and all $n \geq 2$. Compute H^0 and H^1 [53].

2. Let $A = k[x_1, \cdots, x_n]$ be the polynomial algebra in n variables over a field k of characteristic zero. Let V be an n dimensional vector space over k . Define a resolution of the form

$$0 \longleftarrow A \longleftarrow A \otimes A \longleftarrow A \otimes V \otimes A \longleftarrow \cdots \longleftarrow A \otimes \wedge^i V \otimes A \cdots \longleftarrow A \otimes \wedge^n V \otimes A \longleftarrow 0$$

by tensoring resolutions in 1) above for one dimensional vector spaces.

Conclude that for any symmetric A -bimodule M ,

$$H^i(A, M) \simeq M \otimes \wedge^i V, \quad i = 0, 1, \dots$$

Before proceeding further let us recall the definition of the *Hochschild homology* of an algebra with coefficients in a bimodule M . The *Hochschild complex of A with coefficients in M* , $(C_\bullet(A, M), \delta)$, is defined by

$$C_0(A, M) = M, \quad \text{and} \quad C_n(A, M) = M \otimes A^{\otimes n}, \quad n = 1, 2, \dots$$

and the *Hochschild boundary* $\delta : C_n(A, M) \longrightarrow C_{n-1}(A, M)$ is defined by

$$\begin{aligned} \delta(m \otimes a_1 \otimes \dots \otimes a_n) &= ma_1 \otimes a_1 \cdots a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes a_i a_{i+1} \cdots a_n \\ &\quad + (-1)^n a_n m \otimes a_1 \otimes \dots \otimes a_n. \end{aligned}$$

The Hochschild homology of A with coefficients in M is, by definition, the homology of the complex $(C_\bullet(A, M), \delta)$. We denote this homology by $H_\bullet(A, M)$. It is clear that

$$H_0(A, M) = M/[A, M],$$

where $[A, M]$ is the subspace of M spanned by commutators $am - ma$ for a in A and m in M .

The following facts are easily established:

1. Hochschild homology $H_\bullet(A, M)$ is the derived functor of the functor

$$A \otimes A^{op} - Mod \longrightarrow k - Mod, \quad M \mapsto A \otimes_{A \otimes A^{op}} M,$$

i.e.

$$H_n(A, M) = Tor_n^{A \otimes A^{op}}(A, M).$$

For the proof one uses the bar resolution as before.

2. (Duality) Let $M^* = Hom(M, k)$. It is an A -bimodule via $(afb)(m) = f(bma)$. One checks that the natural isomorphism

$$Hom(A^{\otimes n}, M^*) \simeq Hom(M \otimes A^{\otimes n}, k), \quad n = 0, 1, \dots$$

is compatible with differentials. Thus if k is field of characteristic zero, we have

$$H^\bullet(A, M^*) \simeq (H_\bullet(A, M))^*.$$

From now on we denote by $HH^n(A)$ the Hochschild group $H^n(A, A^*)$ and by $HH_n(A)$ the Hochschild group $H_n(A, A)$.

For applications of Hochschild and cyclic (co)homology to noncommutative geometry, it is crucial to consider topological algebras, topological bimodules and continuous chains and cochains on them. For example while the algebraic Hochschild groups of the algebra of smooth functions on a smooth manifold are not known, its topological Hochschild (co)homology is computed by Connes as we recall below. We will give only a brief outline of the definitions and refer the reader to [13, 15] for more details.

Let A be a locally convex topological algebra and M be a locally convex topological A -bimodule. Thus A is a locally convex topological vector space and the multiplication map $A \times A \rightarrow A$ is continuous. Similarly M is a locally convex topological vector space such that both module maps $A \times M \rightarrow M$ and $M \times A \rightarrow M$ are continuous. In the definition of continuous Hochschild homology one uses the *projective tensor product* $M \hat{\otimes} A^{\hat{\otimes} n}$ of locally convex spaces. The algebraic Hochschild boundary, being continuous, naturally extends to topological completions.

For cohomology one should use *jointly continuous* multilinear maps

$$\varphi : A \times \cdots \times A \rightarrow M.$$

With these provisions, the rest of the algebraic formalism remains the same and carries over to the topological set up. In using projective resolutions, one should use only those topological resolutions that admit a continuous linear splitting. This guarantees that the comparison theorem for projective resolutions remain true in the continuous setting.

We give a few examples of Hochschild (co)homology computations. In particular we shall see that group homology and Lie algebra homology are instances of Hochschild homology. We start by recalling the classical results of Connes [13] and Hochschild-Kostant-Rosenberg [44] on the Hochschild homology of smooth commutative algebras.

Example (Commutative Algebras)

Let A be a commutative unital algebra over a ring k . We recall the definition of the *algebraic de Rham complex* of A . The module of 1-forms over A , denoted by $\Omega^1 A$, is defined to be a left A -module $\Omega^1 A$ with a universal derivation

$$d : A \longrightarrow \Omega^1 A.$$

This means that any other derivation $\delta : A \rightarrow M$ into a left A -module M , uniquely factorizes through d . One usually defines $\Omega^1 A = I/I^2$ where I is the kernel of the multiplication map $A \otimes A \rightarrow A$. Note that since A is commutative this map is an algebra homomorphism. d is defined by

$$d(a) = a \otimes 1 - 1 \otimes a \pmod{I^2}.$$

One defines the space of n -forms on A as the n -th exterior power of the A -module $\Omega^1 A$:

$$\Omega^n A := \wedge_A^n \Omega^1 A.$$

There is a unique extension of d to a graded derivation

$$d : \Omega^\bullet A \longrightarrow \Omega^{\bullet+1} A.$$

It satisfies the relation $d^2 = 0$. The *algebraic de Rham cohomology* of A is defined to be the cohomology of the complex $(\Omega^\bullet A, d)$.

Let M be a symmetric A -bimodule. We compare the complex $(M \otimes_A \Omega^\bullet A, 0)$ with the Hochschild complex of A with coefficients in M . Consider the *antisymmetrization map*

$$\begin{aligned} \varepsilon_n : M \otimes_A \Omega^n A &\longrightarrow M \otimes A^{\otimes n}, \quad n = 0, 1, 2, \dots, \\ \varepsilon_n(m \otimes da_1 \wedge \dots \wedge da_n) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) m \otimes a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)}, \end{aligned}$$

where S_n denotes the symmetric group on n letters. We also have a map

$$\mu_n : M \otimes A^{\otimes n} \longrightarrow M \otimes_A \Omega^n A, \quad n = 0, 1, \dots$$

$$\mu_n(m \otimes a_1 \otimes \dots \otimes a_n) = m \otimes da_1 \wedge \dots \wedge da_n.$$

One checks that both maps are morphisms of complexes, i.e.

$$\delta \circ \varepsilon_n = 0, \quad \mu_n \circ \delta = 0.$$

Moreover, one has

$$\mu_n \circ \varepsilon_n = n! Id_n.$$

It follows that if k is a field of characteristic zero then the antisymmetrization map

$$\varepsilon_n : M \otimes_A \Omega^n A \longrightarrow H_n(A, M),$$

is an inclusion. For $M = A$ we obtain a natural inclusion

$$\Omega^n A \longrightarrow HH_n(A).$$

The celebrated Hochschild-Kostant-Rosenberg theorem [44] states that if A is the algebra of regular functions on an smooth affine variety the above map is an isomorphism.

Let M be a smooth closed manifold and let $A = C^\infty(M)$ be the algebra of smooth complex valued functions on M . It is a locally convex (in fact, Frechet) topological algebra. Fixing a finite atlas on M , one defines a family of seminorms

$$p_n(f) = \sup\{|\partial^I(f)|; |I| \leq n\},$$

where the supremum is over all coordinate charts. It is easily seen that the induced topology is independent of the choice of atlas. In [13], using an explicit resolution, Connes shows that the canonical map

$$HH_n^{cont}(A) \rightarrow \Omega^n M, \quad f_0 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \cdots df_n,$$

is an isomorphism. In fact the original, equivalent, formulation of Connes in [13] is for continuous Hochschild cohomology $HH^n(A)$ which is shown to be isomorphic to the continuous dual of $\Omega^n M$ (space of *n*-dimensional *de Rham currents*).

Example (Group Algebras)

It is clear from the original definitions that group (co)homology is an example of Hochschild (co)homology. Let G be a group and let M be a left G -module over the ground ring k . Recall that the standard complex for computing group cohomology [53] is the complex

$$M \xrightarrow{\delta} C^1(G, M) \xrightarrow{\delta} C^2(G, M) \xrightarrow{\delta} \cdots,$$

where

$$C^n(G, M) = \{f : G^n \longrightarrow M\}.$$

The differential δ is defined by

$$(\delta m)(g) = gm - m,$$

$$\begin{aligned} \delta f(g_1, \cdots, g_{n+1}) &= g_1 f(g_2, \cdots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) \\ &\quad + (-1)^{n+1} f(g_1, g_2, \cdots, g_n). \end{aligned}$$

Let $A = kG$ denote the group algebra of the group G over k . Then M is a kG -bimodule via the two actions

$$g.m = g(m), \quad m.g = m,$$

for all g in G and m in M . It is clear that for all n ,

$$C^n(kG, M) \simeq C^n(G, M),$$

and the two differentials are the same. It follows that the cohomology of G with coefficients in M coincides with the Hochschild cohomology of kG with coefficients in M .

Example (Enveloping Algebras).

We show that Lie algebra (co)homology is an example of Hochschild (co)homology. Let \mathfrak{g} be a Lie algebra and M be a \mathfrak{g} -module. This simply means that we have a Lie algebra morphism

$$\mathfrak{g} \longrightarrow \text{End}_k(M).$$

The *Lie algebra homology* of \mathfrak{g} with coefficients in M is the homology of the *Chevalley-Eilenberg complex* defined by

$$M \longleftarrow M \otimes \wedge^1 \mathfrak{g} \longleftarrow M \otimes \wedge^2 \mathfrak{g} \longleftarrow M \cdots,$$

where the differential is defined by

$$\delta(m \otimes X) = X(m),$$

$$\begin{aligned} \delta(m \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_n) &= \sum_{i < j} (-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \cdots \wedge \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_n \\ &+ \sum_i (-1)^i X_i(m) \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n. \end{aligned}$$

One checks that $\delta^2 = 0$.

Let $U(\mathfrak{g})$ denote the enveloping algebra of \mathfrak{g} . Given a \mathfrak{g} module M we define a $U(\mathfrak{g})$ -bimodule $M' = M$ with left and right $U(\mathfrak{g})$ -actions defined by

$$X \cdot m = X(m), \quad m \cdot X = 0.$$

Define a map

$$\begin{aligned}\varepsilon_n : C_n^{Lie}(\mathfrak{g}, M) &\longrightarrow C_n(U(\mathfrak{g}), M'), \\ \varepsilon_n(m \otimes X_1 \wedge \cdots \wedge X_n) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) m \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}.\end{aligned}$$

One checks that ε is a chain map (prove this!). We claim that it is a quasi-isomorphism, i.e. it induces isomorphism on homology. To prove this, we define a filtration on $(C_\bullet(U(\mathfrak{g}), M), \delta)$ using the Poincare-Birkhoff-Witt filtration on $U(\mathfrak{g})$. The associated E^1 term is the de Rham complex of the symmetric algebra $S(\mathfrak{g})$. The induced map is the antisymmetrization map

$$\varepsilon_n : M \otimes \wedge^n \mathfrak{g} \rightarrow M \otimes S(\mathfrak{g})^{\otimes n}.$$

By Hochschild-Kostant-Rosenberg's theorem, this map is a quasi-isomorphism hence the original map is a quasi-isomorphism.

Example (Morita Invariance of Hochschild (Co)Homology)

Let A and B be unital Morita equivalent k -algebras. Let X be an equivalence $A - B$ bimodule and Y its inverse bimodule. Let M be an $A - A$ bimodule and $N = Y \otimes_A M \otimes_A X$ the corresponding B -bimodule. Morita invariance of Hochschild homology states that there is a natural isomorphism

$$H_n(A, M) \simeq H_n(B, N),$$

for all $n \geq 0$ [53]. We sketch a proof of this result for the special case where $B = M_k(A)$ is the algebra of k by k matrices over A .

Let M be an A -bimodule, and let $M_k(M)$ be the space of k by k matrices with coefficients in M . It is a bimodule over $M_k(A)$. The *generalized trace map* is defined by

$$Tr : C_n(M_k(A), M_k(M)) \longrightarrow C_n(A, M),$$

$$\begin{aligned}Tr(\alpha_0 \otimes m_0 \otimes \alpha_1 \otimes a_1 \otimes \cdots \otimes \alpha_n \otimes a_n) &= \\ tr(\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n) m_0 \otimes a_1 \otimes \cdots \otimes a_n,\end{aligned}$$

where $\alpha_i \in M_k(k)$, $a_i \in A$, $m_0 \in M$, and $tr : M_k(k) \longrightarrow k$ is the standard trace of matrices.

Exercise:

1. Show that Tr is a chain map.
2. Let $i : A \rightarrow M_k(A)$ be the map that sends a in A to the matrix with only one non-zero component in the upper left corner equal to a . There is a similar map $M \rightarrow M_k(M)$. Define a map

$$i_* : C_n(A, M) \longrightarrow C_n(M_k(A), M_k(M)),$$

$$i_*(m \otimes a_1 \otimes \cdots \otimes a_n) = i(m) \otimes i(a_1) \otimes \cdots \otimes i(a_n).$$

Show that

$$Tr \circ i_* = id.$$

It is however not true that $i_* \circ Tr = id$. There is instead a homotopy between $i_* \circ Tr$ and id . The homotopy is given in [53] and we won't reproduce it here.

As a special case of the Morita invariance theorem, we have an isomorphism of Hochschild homology groups

$$HH_n(A) = HH_n(M_k(A)),$$

for all n and k .

We need to know, for example when defining the noncommutative Chern character map, that inner automorphisms act by identity on Hochschild homology and inner derivations act by zero. Let A be an algebra, let $u \in A$ be an invertible element and let $a \in A$ be any element. They induce the chain maps

$$\Theta : C_n(A) \rightarrow C_n(A) \quad a_0 \otimes \cdots \otimes a_n \mapsto ua_0u^{-1} \otimes \cdots \otimes ua_nu^{-1},$$

$$L_a : C_n(A) \rightarrow C_n(A) \quad a_0 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^n a_0 \otimes \cdots \otimes [a, a_i] \otimes \cdots \otimes a_n.$$

Lemma 4.1. Θ induces the identity map on Hochschild homology and L_a induces the zero map.

Proof. The maps [53], $h_i : A^{\otimes n+1} \rightarrow A^{\otimes n+2}$, $i = 0, \dots, n$

$$h_i(a_0 \otimes \cdots \otimes a_n) = (a_0u^{-1} \otimes ua_1u^{-1}, \dots, u \otimes a_{i+1} \cdots \otimes a_n)$$

define a homotopy

$$h = \sum_{i=0}^n (-1)^i h_i$$

between id and Θ .

For the second part one checks again that the maps $h_i : A^{\otimes n+1} \rightarrow A^{\otimes n+2}$, $i = 0, \dots, n$,

$$h_i(a_0 \otimes \cdots \otimes a_n) = (a_0 \otimes \cdots \otimes a_i \otimes a \cdots \otimes a_n),$$

define a homotopy between L_a and 0 [53].

□

4.2 Cyclic (co)homology

4.2.1 Connes' cyclic complex

Cyclic cohomology was first defined by Connes [11, 13] through a remarkable subcomplex of the Hochschild complex called the *cyclic complex*. Let k be a field of characteristic zero and let $(C^\bullet(A), b)$ denote the Hochschild complex of a k -algebra A with coefficients in the A -bimodule A^* . We have

$$C^n(A) = Hom(A^{\otimes(n+1)}, k), \quad n = 0, 1, \dots,$$

$$\begin{aligned} (bf)(a_0, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} f(a_{n+1} a_0, \dots, \dots, a_n). \end{aligned}$$

An n -cochain $f \in C^n(A)$ is called *cyclic* if

$$f(a_n, a_0, \dots, a_{n-1}) = (-1)^n f(a_0, a_1, \dots, a_n)$$

for all a_0, \dots, a_n in A . We denote the space of cyclic cochains on A by $C_\lambda^n(A)$.

Lemma 4.2. *The space of cyclic cochains is invariant under the action of b , i.e. for all n ,*

$$b C_\lambda^n(A) \subset C_\lambda^{n+1}(A).$$

Proof. Define the operators $\lambda : C^n(A) \rightarrow C^n(A)$ and $b' : C^n(A) \rightarrow C^{n+1}(A)$ by

$$\begin{aligned} (\lambda f)(a_0, \dots, a_n) &= (-1)^n f(a_n, a_0, \dots, a_{n-1}), \\ (b' f)(a_0, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}). \end{aligned}$$

One checks that

$$(1 - \lambda)b = b'(1 - \lambda).$$

Since

$$C_\lambda^n(A) = \text{Ker}(1 - \lambda),$$

the lemma is proved. \square

We therefore have a subcomplex of the Hochschild complex, called the *cyclic complex* of A :

$$C_\lambda^0(A) \xrightarrow{b} C_\lambda^1(A) \xrightarrow{b} C_\lambda^2(A) \xrightarrow{b} \dots$$

The cohomology of this complex is called the *cyclic cohomology* of A and will be denoted by $HC^n(A)$, $n = 0, 1, 2, \dots$. A cocycle for cyclic cohomology is called a *cyclic cocycle*. It satisfies the two conditions:

$$(1 - \lambda)f = 0, \quad \text{and} \quad bf = 0.$$

The inclusion of complexes

$$(C_\lambda^\bullet(A), b) \longrightarrow (C^\bullet(A), b),$$

induces a map I from the cyclic cohomology of A to the Hochschild cohomology of A with coefficients in the A -bimodule A^* :

$$I : HC^n(A) \longrightarrow HH^n(A), \quad n = 0, 1, 2, \dots$$

We shall see that this map is part of a long exact sequence relating Hochschild and cyclic cohomology, called Connes' long exact sequence. For the moment we mention that I need not be injective (see example below).

Examples

1. Let $A = k$ be a field of characteristic zero. We have

$$C_\lambda^{2n}(k) \simeq k, \quad C_\lambda^{2n+1}(k) = 0.$$

The cyclic complex reduces to

$$0 \longrightarrow k \longrightarrow 0 \longrightarrow k \longrightarrow \cdots .$$

It follows that for all $n \geq 0$,

$$HC^{2n}(k) = k, \quad HC^{2n+1}(k) = 0.$$

Since $HH^n(k) = 0$ for $n \geq 1$, we conclude that the map I need not be injective and the cyclic complex is not a retraction of the Hochschild complex.

2. It is clear that $HC^0(A) = HH^0(A)$ is the space of traces on A .

3. Let $A = C^\infty(M)$ be the algebra of smooth functions on a closed smooth manifold M of dimension n . One checks that

$$\varphi(f_0, f_1, \cdots, f_n) = \int_M f_0 df_1 \cdots df_n,$$

is a cyclic n -cocycle on A . In fact the Hochschild cocycle property $b\varphi = 0$ is a consequence of the graded commutativity of the algebra of differential forms and the cyclic property $(1 - \lambda)\varphi = 0$ follows from Stokes formula.

This example can be generalized in several directions. For example, Let V be an m -dimensional closed singular chain (a cycle) on M , e.g. V can be a closed m -dimensional submanifold of M . Then integration on V defines an m -dimensional cyclic cocycle on A :

$$\varphi(f_0, f_1, \cdots, f_m) = \int_V f_0 df_1 \cdots df_m.$$

We obtain a map

$$H_m(M, \mathbb{C}) \longrightarrow HC^m(A), \quad m = 0, 1, \cdots ,$$

from singular homology (or its equivalents) to cyclic cohomology.

More generally, let C be a *closed m -dimensional de Rham current* on M . Thus $C : \Omega^m M \rightarrow \mathbb{C}$ is a continuous linear functional on $\Omega^m M$ such that $dC(\omega) := C(d\omega) = 0$ for all $\omega \in \Omega^{m-1} M$. Then one checks that the cochain φ defined by

$$\varphi(f_0, f_1, \cdots, f_m) = \langle C, f_0 df_1 \cdots df_m \rangle,$$

is a cyclic cocycle on A .

A noncommutative generalization of this procedure involves the notion of a *cycle on an algebra* due to Connes [13] that we recall now. It gives a geometric and intuitively appealing presentation for cyclic cocycles. It also leads to a definition of cup product in cyclic cohomology and the S operator. Let

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \dots$$

be a differential graded algebra, where we assume that $\Omega^0 = A$ and the differential $d : \Omega^i \rightarrow \Omega^{i+1}$ increases the degree. d is a graded derivation in the sense that

$$d(\omega_1\omega_2) = d(\omega_1)\omega_2 + (-1)^{\deg(\omega_1)}\omega_1d(\omega_2), \quad \text{and} \quad d^2 = 0.$$

A *closed graded trace* of dimension n on Ω is a linear map

$$\int : \Omega^n \longrightarrow k$$

such that

$$\int d\omega = 0, \quad \text{and} \quad \int (\omega_1\omega_2 - (-1)^{\deg(\omega_1)\deg(\omega_2)}\omega_2\omega_1) = 0,$$

for all ω in Ω^{n-1} , ω_1 in Ω^i , ω_2 in Ω^j and $i + j = n$. A triple of the form (A, Ω, \int) is called a *cycle* over the algebra A .

Given a closed graded trace \int on A , one defines a cyclic n -cocycle on A by

$$\varphi(a_0, a_1, \dots, a_n) = \int a_0 da_1 \cdots da_n.$$

Exercise: Check that φ is a cyclic n -cocycle.

Conversely, one can show that any cyclic cocycle on A is obtained in this way. To do this we introduce the algebra $(\Omega A, d)$ of *noncommutative differential forms* on A as follows. ΩA is the universal (nonunital) differential graded algebra generated by A as a subalgebra. We have $\Omega^0 A = A$, and $\Omega^n A$ is linearly generated over k by expressions $a_0 da_1 \cdots da_n$ and $da_1 \cdots da_n$ for $a_i \in A$ (cf. [13] for details). The differential d is defined by

$$d(a_0 da_1 \cdots da_n) = da_0 da_1 \cdots da_n, \quad \text{and} \quad d(da_1 \cdots da_n) = 0.$$

Now it is easily checked that the relation

$$\varphi(a_0, a_1, \dots, a_n) = \int a_0 da_1 \cdots da_n,$$

defines a 1-1 correspondence

$$\{\text{cyclic } n\text{-cocycles on } A\} \simeq \{\text{closed graded traces on } \Omega^n A\}.$$

Exercise: Give a similar description for Hochschild cocycles $\varphi \in C^n(A, A^*)$.

3. (From group cocycles to cyclic cocycles)

Let G be a discrete group and let $c(g_1, \dots, g_n)$ be a group n -cocycle on G . Assume c is *normalized* in the sense that

$$c(g_1, \dots, g_n) = 0,$$

if $g_i = e$ for some i , or if $g_1 g_2 \cdots g_n = e$. One checks that

$$\begin{aligned} \varphi_c(g_0, \dots, g_n) &= c(g_1, \dots, g_n) \quad \text{if } g_0 g_1 \cdots g_n = e, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

is a cyclic n -cocycle on the group algebra kG [12, 15]. In this way one obtains a map

$$H^n(G, k) \longrightarrow HC^n(kG), \quad c \mapsto \varphi_c.$$

By a theorem of D. Burghelea [7] (see below), the cyclic cohomology group $HC^n(kG)$ decomposes over the conjugacy classes of G and the component corresponding to the conjugacy class of the identity is exactly the group cohomology $H^n(G, k)$.

4. (From Lie algebra cocycles to cyclic cocycles)

Let \mathfrak{g} be a Lie algebra acting by derivations on an algebra A . This means we have a Lie algebra map

$$\mathfrak{g} \rightarrow \text{Der}(A, A),$$

from \mathfrak{g} into the Lie algebra of derivations on A . Let $\tau : A \rightarrow k$ be an *invariant trace* on A . Thus τ is a trace on A and

$$\tau(X(a)) = 0 \quad \text{for all } X \in \mathfrak{g}, a \in A.$$

For each $n \geq 0$, define a map

$$\wedge^n \mathfrak{g} \longrightarrow C^n(A), \quad c \mapsto \varphi_c$$

$$\varphi_c(a_0, a_1, \dots, a_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \tau(a_0 X_{\sigma(1)}(a_1) \cdots X_{\sigma(n)}(a_n)),$$

where $c = X_1 \wedge \cdots \wedge X_n$.

Exercise: Check that 1) For any c , φ_c is a Hochschild cocycle, i.e. $b\varphi_c = 0$.

2) If c is a Lie algebra cycle (i.e. if $\delta(c) = 0$), the φ_c is a cyclic cocycle.

We therefore obtain, for each $n \geq 0$, a map

$$\chi_\tau : H_n^{\text{Lie}}(\mathfrak{g}, k) \longrightarrow HC^n(A), \quad c \mapsto \varphi_c,$$

from the Lie algebra homology of \mathfrak{g} with trivial coefficients to the cyclic cohomology of A [12].

In particular if \mathfrak{g} is abelian then $H_n^{\text{Lie}}(\mathfrak{g}) \simeq \wedge^n(\mathfrak{g})$ and we obtain a well defined map

$$\wedge^n(\mathfrak{g}) \longrightarrow HC^n(A), \quad n = 0, 1, \dots.$$

Here is an example of this construction, first appeared in [10]. Let $A = \mathcal{A}_\theta$ denote the ‘‘algebra of smooth functions’’ on the noncommutative torus. Let $X_1 = (1, 0)$, $X_2 = (0, 1)$. There is an action of the abelian Lie algebra \mathbb{R}^2 on \mathcal{A}_θ defined on generators of \mathcal{A}_θ by

$$\begin{aligned} X_1(U) &= U, & X_1(V) &= 0, \\ X_2(U) &= 0, & X_2(V) &= V. \end{aligned}$$

The induced derivations on \mathcal{A}_θ are given by

$$\begin{aligned} X_1\left(\sum a_{m,n} U^m V^n\right) &= \sum m a_{m,n} U^m V^n, \\ X_2\left(\sum a_{m,n} U^m V^n\right) &= \sum n a_{m,n} U^m V^n. \end{aligned}$$

It is easily checked that the trace τ on \mathcal{A}_θ defined by

$$\tau\left(\sum a_{m,n} U^m V^n\right) = a_{0,0},$$

is invariant under the above action of \mathbb{R}^2 . The generators of $H_\bullet^{\text{Lie}}(\mathbb{R}^2, \mathbb{R})$ are: $1, X_1, X_2, X_1 \wedge X_2$.

We therefore obtain the following 0-dimensional, 1-dimensional and 2-dimensional cyclic cocycles on \mathcal{A}_θ :

$$\begin{aligned}\varphi_0(a_0) &= \tau(a_0), \\ \varphi_1(a_0, a_1) &= \tau(a_0 X_1(a_1)), \quad \varphi'_1(a_0, a_1) = \tau(a_0 X_2(a_1)), \\ \varphi_2(a_0, a_1, a_2) &= \tau(a_0(X_1(a_1)X_2(a_2) - X_2(a_1)X_1(a_2))).\end{aligned}$$

It is shown by Connes [13] that these classes generate the continuous periodic cyclic cohomology of \mathcal{A}_θ .

5. (Cup product and the S -operation on cyclic cohomology)

Let (A, Ω, f) be an m -dimensional cycle on an algebra A and (B, Ω', f') and n -dimensional cycle on an algebra B . Let $\Omega \otimes \Omega'$ denote the (graded) tensor product of the differential graded algebras Ω and Ω' . By definition, we have

$$(\Omega \otimes \Omega')_n = \bigoplus_{i+j=n} \Omega_i \otimes \Omega'_j,$$

$$d(\omega \otimes \omega') = (d\omega) \otimes \omega' + (-1)^{\deg(\omega)} \omega \otimes (d\omega'),$$

$$\int'' \omega \otimes \omega' = \int \omega \int' \omega', \quad \text{if } \deg(\omega) = m, \deg(\omega') = n.$$

It is easily checked that \int'' is a closed graded trace of dimension $m + n$ on $\Omega \otimes \Omega'$.

Using the universal property of noncommutative differential forms, applied to the identity map $A \otimes B \longrightarrow \Omega_0 \otimes \Omega'_0$, one obtains a morphism of differential graded algebras

$$(\Omega(A \otimes B), d) \longrightarrow (\Omega \otimes \Omega', d).$$

We therefore obtain a closed graded trace of dimension $m + n$ on $(\Omega(A \otimes B), d)$. In [13], it is shown that the resulting map, called *cup product* in cyclic cohomology,

$$\# : HC^m(A) \otimes HC^n(B) \rightarrow HC^{m+n}(A \otimes B)$$

is well defined.

Exercise: By following the steps in the definition of the cup product, find an “explicit” formula for $\varphi\#\psi$, when φ is an m -dimensional cyclic cocycle and ψ is an n -dimensional cyclic cocycle [13].

Let β denote the generator of $HC^2(k)$ defined by $\beta(1, 1, 1) = 1$. Using the cup product with β we obtain the S -map

$$S : HC^n(A) \rightarrow HC^{n+2}(A), \quad \varphi \mapsto \varphi\#\beta.$$

Exercise: Find explicit formulas for $S\varphi$ when φ is a 0 or 1 dimensional cyclic cocycle.

In the next section we give a different approach to S via the cyclic bi-complex.

The *generalized trace map*

$$Tr : C_n^\lambda(A) \rightarrow C_n^\lambda(M_p(A)),$$

defined by

$$(Tr\varphi)(a_0 \otimes m_0, \dots, a_n \otimes m_n) = tr(m_0 \cdots m_n)\varphi(a_0, \dots, a_n),$$

can be shown to be an example of cup product as well [13]. Indeed we have

$$Tr(\varphi) = \varphi\#tr.$$

So far we studied the cyclic cohomology of algebras. There is a “dual” theory called *cyclic homology* that we introduce now. Let A be an algebra and for $n \geq 0$ let $C_n(A) = A^{\otimes(n+1)}$. For each $n \geq 0$, define the operators $b : C_n(A) \rightarrow C_{n-1}(A)$, $b' : C_n(A) \rightarrow C_{n-1}(A)$, and $\lambda : C_n(A) \rightarrow C_n(A)$ by

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes a_n) \\ &\quad + (-1)^n (a_n a_0 \otimes a_1 \cdots \otimes a_{n-1}), \\ b'(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes a_n), \\ \lambda(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_n \otimes a_0 \cdots \otimes a_{n-1}). \end{aligned}$$

The relation

$$(1 - \lambda)b' = b(1 - \lambda)$$

can be easily established. Let $(C_\bullet(A), b)$ denote the Hochschild complex of A with coefficients in the A -bimodule A and let

$$C_n^\lambda(A) := C_n(A)/\text{Im}(1 - \lambda).$$

The relation $(1 - \lambda)b' = b(1 - \lambda)$ shows that the operator b is well-defined on $C_\bullet^\lambda(A)$. The quotient complex

$$(C_\bullet^\lambda(A), b)$$

is called *cyclic complex* of A . Its homology, denoted by $HC_n(A)$, $n = 0, 1, \dots$, is called the *cyclic homology* of A .

Example For $n = 0$, $HC_0(A) = A/[A, A]$ is the *commutator quotient* of A . Here $[A, A]$ denotes the subspace of A generated by the commutators $ab - ba$, for a and b in A .

4.2.2 Connes' long exact sequence

Our goal in this section is to establish the long exact sequence of Connes relating Hochschild and cyclic homology groups. There is a similar sequence relating Hochschild and cyclic cohomology groups. Connes' original proof in [13] is based on the notion of *cobordism of cycles*. This leads to an operator $B : HH^n(A) \rightarrow HC^{n-1}(A)$. He then shows that the three operators $I : HC^n(A) \rightarrow HH^n(A)$, $S : HC^n(A) \rightarrow HC^{n+2}(A)$ and B fit into a long exact sequence. An alternative approach is based on the following bicomplex, called the *cyclic bicomplex* of A and denoted by $\mathcal{C}(A)$ [12, 54, 53]:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-\lambda} \dots \\ & \downarrow b & & \downarrow -b' & & \downarrow b & \\ & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-\lambda} \dots \\ & \downarrow b & & \downarrow -b' & & \downarrow b & \\ & A & \xleftarrow{1-\lambda} & A & \xleftarrow{N} & A & \xleftarrow{1-\lambda} \dots \end{array}$$

Here the operator $N : A^{\otimes(n+1)} \rightarrow A^{\otimes(n+1)}$ is defined by

$$N = 1 + \lambda + \lambda^2 + \dots + \lambda^n.$$

The relations

$$\begin{aligned} N(1 - \lambda) &= (1 - \lambda)N = 0, & Nb &= b'N, \\ (1 - \lambda)b' &= b(1 - \lambda), \end{aligned}$$

can be easily verified. Coupled with relations $b^2 = 0$, $b'^2 = 0$, it follows that $\mathcal{C}(A)$ is a bicomplex.

In particular we can consider its total complex $(Tot\mathcal{C}(A), \delta)$. We show that there is a quasi-isomorphism of complexes

$$(Tot\mathcal{C}(A), \delta) \stackrel{q.i.}{\simeq} (C^\lambda(A), b).$$

To this end, for $n \geq 0$, define a map

$$\begin{aligned} (Tot_n\mathcal{C}(A), \delta) &\longrightarrow (C_n^\lambda(A), b) \\ (x_0, x_1, \dots, x_n) &\mapsto [x_n], \end{aligned}$$

where $[x_n]$ denotes the class of $x_n \in A^{\otimes(n+1)}$ in $C_n^\lambda(A) = A^{\otimes(n+1)}/Im(1 - \lambda)$. One checks that this is a morphism of complexes. Now assume that k is a field of characteristic zero. We show that the rows of $\mathcal{C}(A)$, i.e. the complexes

$$A^{\otimes n+1} \xleftarrow{1-\lambda} A^{\otimes n+1} \xleftarrow{N} A^{\otimes n+1} \dots$$

are exact for each $n \geq 0$. The relation

$$Ker(1 - \lambda) = ImN$$

is obvious. To show that $KerN = Im(1 - \lambda)$, define the operator

$$H = \frac{1}{n+1}(1 + 2\lambda + 3\lambda^2 + \dots + (n+1)\lambda^n) : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}.$$

We have

$$(1 - \lambda)H = N + 1,$$

which shows that $KerN = Im(1 - \lambda)$. It follows that the spectral sequence converging to the total homology of $\mathcal{C}(A)$ collapses and the total homology is the homology of its E^2 -term. But the E^2 -term is exactly the cyclic complex $(C_\bullet^\lambda(A), b)$ of A . Alternatively, one can simply apply a ‘‘tic-tac-toe’’ argument to finish the proof.

We can utilize the 2-periodicity of the cyclic bicomplex $\mathcal{C}(A)$ to define a short exact sequence of complexes

$$0 \longrightarrow Tot'\mathcal{C}(A) \longrightarrow Tot\mathcal{C}(A) \longrightarrow Tot\mathcal{C}(A) \longrightarrow 0,$$

where $Tot'\mathcal{C}(A)$ denotes the total complex of the first two columns and $Tot\mathcal{C}(A)[2]$ is the shifted by two 2 complex. The last map is defined by truncation:

$$(x_0, x_1, \dots, x_n) \mapsto (x_0, \dots, x_{n-2}).$$

The kernel of this map is the total complex of the first two columns of $\mathcal{C}(A)$:

$$Tot'_n\mathcal{C}(A) = A^{\otimes n} \oplus A^{\otimes(n-1)}.$$

Now when A is unital the b' -complex is acyclic. To prove this we define an operator $s : A^{\otimes n} \longrightarrow A^{\otimes(n+1)}$,

$$s(a_0 \otimes \dots \otimes a_{n-1}) = 1 \otimes a_0 \otimes \dots \otimes a_{n-1}.$$

One checks that

$$b's + sb' = id,$$

which shows that the b' -complex is acyclic. We conclude that the complex $Tot'_\bullet\mathcal{C}(A)$ is homotopy equivalent to the Hochschild complex $(C_\bullet(A), b)$. Therefore the long exact sequence associated to the above short exact sequence is of the form:

$$\dots \longrightarrow HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} HC_{n-1}(A) \longrightarrow \dots$$

This exact sequence was first obtained by Connes in 1981 [11, 13] in its cohomological form:

$$\dots \longleftarrow HC^n(A) \xleftarrow{S} HC^{n-2}(A) \xleftarrow{B} HH^{n-1}(A) \xleftarrow{I} HC^{n-1}(A) \longleftarrow \dots$$

The *periodicity operator*

$$S : HC_n(A) \longrightarrow HC_{n-2}(A)$$

is induced by the truncation map

$$(x_0, x_1, \dots, x_n) \mapsto (x_0, \dots, x_{n-2}).$$

The operator B is the *connecting homomorphism* of the long exact sequence. It can therefore be represented on the level of chains by the formula

$$B = N_S(1 - \lambda) : C_n(A) \rightarrow C_{n+1}(A).$$

Exercise: Recall how the connecting homomorphism is defined for the long exact sequence associated to a short exact sequence of complexes and derive the above formula for B .

Remark 2. We defined, using the cup product with the generator of $HC^2(k)$, an operation $S : HC^n(A) \rightarrow HC^{n+2}(A)$, for $n = 0, 1, \dots$. It can be shown that this definition coincides with the cohomological form of the above definition of S .

Typical applications of Connes' long exact sequence involve extracting information on cyclic homology from Hochschild homology. We list some of them:

1. Let $f : A \rightarrow A'$ be an algebra homomorphism and suppose that the induced maps on Hochschild groups

$$f_* : HH_n(A) \rightarrow HH_n(A'),$$

are isomorphisms for all $n \geq 0$. Then

$$f_* : HC_n(A) \rightarrow HC_n(A')$$

is an isomorphism for all $n \geq 0$. This simply follows by comparing the *SBI* sequences for A and B and applying the “five lemma”. In particular, it follows that inner automorphisms act as identity on cyclic homology, while inner derivations act like zero on cyclic homology.

2. (Morita invariance of cyclic homology). Let A and B be Morita equivalent unital algebras. The Morita invariance property of cyclic homology states that there is a natural isomorphism

$$HC_n(A) \simeq HC_n(B), \quad n = 0, 1, \dots$$

For a proof of this fact in general see [53]. In the special case where $B = M_k(A)$ a simple proof can be given as follows. Indeed, by Morita invariance

property of Hochschild homology, we know that the inclusion $i : A \rightarrow M_k(A)$ induces isomorphism on Hochschild groups and therefore on cyclic groups by 1) above.

The bicomplex $\mathcal{C}(A)$ can be extended to the left. We obtain a bicomplex, in the upper half plane,

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 \dots & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-\lambda} & \dots \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & \\
 \dots & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-\lambda} & \dots \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & \\
 \dots & \xleftarrow{N} & A & \xleftarrow{1-\lambda} & A & \xleftarrow{N} & A & \xleftarrow{1-\lambda} & \dots
 \end{array}$$

The total homology (where direct products instead of direct sums is used in the definition of the total complex) of this bicomplex is by definition the *periodic cyclic homology* of A and is denoted by $HP_{\bullet}(A)$. Note that because of the 2-periodicity of this bicomplex $HP_{\bullet}(A)$ is 2-periodic.

Exercise: Show that the resulting homology is trivial if instead of direct products we use direct sums.

Similarly one defines the *periodic cyclic cohomology* $HP^{\bullet}(A)$, where this time one uses direct sums for the definition of the total complex. Since cohomology and direct limits commute, these periodic groups are indeed direct limits of cyclic cohomology groups under the S -map:

$$HP^n(A) = \varinjlim HC^{m+2k}(A).$$

One checks, using the relations

$$b's + sb' = 1, \quad bN = Nb', \quad (1 - \lambda)b = b'(1 - \lambda)$$

that

$$B^2 = 0, \quad bB + Bb = 0.$$

Indeed, we have

$$\begin{aligned}
B^2 &= Ns(1-\lambda)Ns(1-\lambda) = 0, \\
bB + Bb &= bNs(1-\lambda) + Ns(1-\lambda)b = \\
&= Nb's(1-\lambda) + Nsb'(1-\lambda) = \\
&= N(1)(1-\lambda) = 0.
\end{aligned}$$

The relations $b^2 = B^2 = bB + Bb = 0$ suggest the following new bi-complex for unital algebras, called *Connes' (b, B) -bicomplex*. It was first defined in [11, 13]. As we shall see, it leads to a third definition of cyclic (co)homology. Let A be a unital algebra. The (b, B) -bicomplex of A is the following bicomplex:

$$\begin{array}{ccccc}
& \vdots & & \vdots & & \vdots \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
\downarrow b & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array}$$

Lemma 4.3. *The complexes $Tot\mathcal{B}(A)$ and $Tot\mathcal{C}(A)$ are homotopy equivalent.*

Proof. We define explicit chain maps between these complexes and show that they are chain homotopic via explicit homotopies. Define

$$\begin{aligned}
I : Tot\mathcal{B}(A) &\rightarrow Tot\mathcal{C}(A), & I &= id + sN \\
J : Tot\mathcal{C}(A) &\rightarrow Tot\mathcal{B}(A), & J &= id + Ns.
\end{aligned}$$

One checks that I and J are chain maps.

Now consider the operators

$$\begin{aligned}
g : Tot\mathcal{B}(A) &\rightarrow Tot\mathcal{B}(A), & g &= B_0s^2N \\
h : Tot\mathcal{C}(A) &\rightarrow Tot\mathcal{C}(A), & h &= s,
\end{aligned}$$

where $B_0 = (1-\lambda)s$.

We have, by direct computation:

$$\begin{aligned} I \circ J &= id + h\delta + \delta h, \\ I \circ J &= id + g\delta' + \delta'g, \end{aligned}$$

where δ (resp. δ') denotes the differential of $Tot\mathcal{C}(A)$ (resp. $Tot\mathcal{B}(A)$). (cf. [48] for details). The operator I appears in [54]. The operator h and the second homotopy formula was first defined in [48]. \square

It is clear that the above result extends to periodic cyclic (co)homology.

Examples

1. (Algebra of smooth functions). Let M be a closed smooth manifold, $A = C^\infty(M)$ denote the algebra of smooth complex valued functions on M , and let $(\Omega^\bullet M, d)$ denote the de Rham complex of M . We saw that, by a theorem of Connes, the map

$$\mu : C_n(A) \rightarrow \Omega^n M, \quad \mu(f_0 \otimes \cdots \otimes f_n) = \frac{1}{n!} f_0 df_1 \cdots df_n,$$

induces an isomorphism between the continuous Hochschild homology of A and differential forms on M :

$$HH_n(A) \simeq \Omega^n M.$$

To compute the continuous cyclic homology of A , we first show that under the map μ the operator B corresponds to the de Rham differential d . More precisely, for each integer $n \geq 0$ we have a commutative diagram:

$$\begin{array}{ccc} C_n(A) & \xrightarrow{\mu} & \Omega^n M \\ \downarrow B & & \downarrow d \\ C_{n+1}(A) & \xrightarrow{\mu} & \Omega^{n+1} M \end{array}$$

We have

$$\begin{aligned}
\mu B(f_0 \otimes \cdots \otimes f_n) &= \mu \sum_{i=0}^n (-1)^{ni} (1 \otimes f_i \otimes \cdots \otimes f_{i-1} - (-1)^n f_i \otimes \cdots \otimes f_{i-1} \otimes 1) \\
&= \frac{1}{(n+1)!} \sum_{i=0}^n (-1)^{ni} df_i \cdots df_{i-1} \\
&= \frac{1}{(n+1)!} (n+1) df_0 \cdots df_n \\
&= d\mu(f_0 \otimes \cdots \otimes f_n).
\end{aligned}$$

It follows that μ defines a morphism of bicomplexes

$$\mathcal{B}(A) \longrightarrow \Omega(A),$$

Where $\Omega(A)$ is the bicomplex

$$\begin{array}{ccccc}
& \vdots & & \vdots & & \vdots \\
\Omega^2 M & \xleftarrow{d} & \Omega^1 M & \xleftarrow{d} & \Omega^0 M \\
\downarrow 0 & & \downarrow 0 & & \\
\Omega^1 M & \xleftarrow{d} & \Omega^0 M & & \\
\downarrow 0 & & & & \\
\Omega^0 M & & & &
\end{array}$$

Since μ induces isomorphisms on row homologies, it induces isomorphisms on total homologies as well. Thus we have [13]:

$$HC_n(A) \simeq \Omega^n M / \text{Im}d \oplus H_{dR}^{n-2}(M) \oplus \cdots \oplus H_{dR}^k(M),$$

where $k=0$ if n is even and $k = 1$ if n is odd.

Using the same map μ acting between the corresponding periodic complexes, one concludes that the periodic cyclic homology of A is given by

$$HP_k(A) \simeq \bigoplus_i H_{dR}^{2i+k}(M), \quad k = 0, 1.$$

2. (Group algebras). Let kG denote the group algebra of a discrete group G over a field k of characteristic zero. By a theorem of Burghelea

[7], Hochschild and cyclic homology groups of kG decompose over the set of conjugacy classes of G where each summand is the group homology (with trivial coefficients) of a group associated to a conjugacy class. We recall this result.

Let \widehat{G} denote the set of conjugacy classes of G , G' be the set of conjugacy classes of elements of finite order, and let G'' denote the set of conjugacy classes of elements of infinite order. For an element $g \in G$, let $C_g = \{h \in G; hg = gh\}$ denote the centralizer of g , and let $W_g = C_g / \langle g \rangle$, where $\langle g \rangle$ is the group generated by g . Note that these groups depend, up to isomorphism, only on the conjugacy class of g . We denote the group homology with trivial coefficients of a group K by $H_*(K)$.

The Hochschild homology of kG is given by [7, 15, 53]:

$$HH_n(kG) \simeq \bigoplus_{g \in \widehat{G}} H_n(C_g).$$

There is a similar, but more complicated, decomposition for the cyclic homology of kG :

$$HC_n(kG) \simeq \bigoplus_{g \in G'} \left(\bigoplus_{i \geq 0} H_{n-2i}(W_g) \right) \bigoplus_{g \in G''} H_n(W_g).$$

In particular, the Hochschild group has $H_n(G)$ as a direct summand, while the cyclic homology group has $\bigoplus_i H_{n-2i}(G)$ as a direct summand (corresponding to the conjugacy class of the identity element of G).

5 Chern-Connes character

Recall that the classical *Chern character* is a natural transformation from K -theory to ordinary cohomology theory with rational coefficients [55]. More precisely for each compact Hausdorff space X we have a natural homomorphism

$$Ch : K^0(X) \longrightarrow \bigoplus_{i \geq 0} H^{2i}(X, \mathbb{Q}),$$

where K^0 (resp. H) denote the K -theory (resp. Čech cohomology with rational coefficients). It satisfies certain axioms and these axioms completely characterize Ch . But we won't recall these axioms here since they are not very useful for finding the noncommutative analogue of Ch . It is furthermore

known that Ch is a rational isomorphism in the sense that upon tensoring it with \mathbb{Q} we obtain an isomorphism

$$Ch_{\mathbb{Q}} : K^0(X) \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_{i \geq 0} H^{2i}(X, \mathbb{Q}).$$

When X is a smooth manifold there is an alternative construction of Ch , called the *Chern-Weil construction*, that uses the differential geometric notions of connection and curvature on vector bundles [55]. It goes as follows. Let E be a complex vector bundle on X and let ∇ be a connection on E . Thus

$$\nabla : C^\infty(E) \longrightarrow C^\infty(E) \otimes_A \Omega^1 X$$

is a \mathbb{C} -linear map satisfying the Leibniz condition

$$\nabla(fs) = f\nabla(s) + s \otimes df,$$

for all smooth sections s of E and smooth functions f on X . Let

$$\hat{\nabla} : C^\infty(E) \otimes_A \Omega^\bullet X \longrightarrow C^\infty(E) \otimes_A \Omega^{\bullet+1} X,$$

denote the natural extension of ∇ satisfying a graded Leibniz property. It can be easily shown that the *curvature operator* $\hat{\nabla}^2$ is an $\Omega^\bullet X$ -linear map. Thus it is completely determined by its restriction to $C^\infty(E)$. This gives us the *curvature form*

$$R \in C^\infty(\text{End}(E)) \otimes \Omega^2 X$$

of ∇ . Let

$$\text{Tr} : C^\infty(\text{End}(E)) \otimes_A \Omega^{ev} X \rightarrow \Omega^{ev} X,$$

denote the canonical trace. The Chern character of E is then defined to be the class of the non-homogeneous even form

$$Ch(E) = \text{Tr}(e^R).$$

(We have omitted the normalization factor of $\frac{1}{2\pi i}$ to be multiplied by R .) One shows that $Ch(E)$ is a closed form and its cohomology class is independent of the choice of connection.

In [10, 13, 15], Connes shows that this Chern-Weil theory admits a vast generalization. For example, for an algebra A and each integer $n \geq 0$ there are natural maps, called *Chern-Connes character maps*,

$$Ch_0^{2n} : K_0(A) \longrightarrow HC_{2n}(A),$$

$$Ch_1^{2n+1} : K_1(A) \longrightarrow HC_{2n+1}(A),$$

compatible with S -operation.

Alternatively the noncommutative Chern character can be defined as a pairing between cyclic cohomology groups and K -theory called *Chern-Connes pairing*:

$$HC^{2n}(A) \otimes K_0(A) \longrightarrow k,$$

$$HC^{2n+1}(A) \otimes K_1(A) \longrightarrow k.$$

These pairings are shown to be compatible with the periodicity operator S in the sense that

$$\langle [\varphi], [e] \rangle = \langle S[\varphi], [e] \rangle,$$

and thus induce a pairing between periodic cyclic cohomology and K -theory.

We start by recalling the definition of these pairings. Let $\varphi(a_0, \dots, a_{2n})$ be an *even cyclic cocycle* on an algebra A . For each integer $k \geq 1$, let

$$\tilde{\varphi} = tr \# \varphi \in C_\lambda^{2n}(M_k(A))$$

denote the extension of φ to the algebra of $k \times k$ matrices on A . Note that $\tilde{\varphi}$ is a cyclic cocycle as well and is given by the formula

$$\tilde{\varphi}(m_0 \otimes a_0, \dots, m_{2n} \otimes a_{2n}) = tr(m_0 \cdots m_{2n}) \varphi(a_0, \dots, a_{2n}).$$

Let $e \in M_k(A)$ be an idempotent. Define a bilinear map by the formula

$$\langle [\varphi], [e] \rangle = \tilde{\varphi}(e, \dots, e).$$

Let us first check that the value of the pairing depends only on the cyclic cohomology class of φ in $HC^{2n}(A)$. Suffices to assume $k = 1$ (why?). Let $\varphi = b\psi$ with $\psi \in C_\lambda^{2n-1}(A)$, be a coboundary. Then we have

$$\begin{aligned} \varphi(e, \dots, e) &= b\psi(e, \dots, e) \\ &= \psi(ee, e, \dots, e) - \psi(e, ee, \dots, e) + \cdots + (-1)^{2n} \psi(ee, e, \dots, e) \\ &= \psi(e, \dots, e) \\ &= 0, \end{aligned}$$

where the last relation follows from the cyclic property of ψ .

To verify that the value of $\langle [\varphi], [e] \rangle$, for fixed φ , only depends on the class of $[e] \in K_0(A)$ we have to check that for $u \in GL_k(A)$ an invertible

matrix, we have $\langle [\varphi], [e] \rangle = \langle [\varphi], [ueu^{-1}] \rangle$. Again suffices to show this for $k = 1$. But this is exactly the fact, proved in the last section, that inner automorphisms act by identity on cyclic cohomology.

The formulas in the *odd case* are as follows. Given an invertible matrix $u \in M_k(A)$ and an odd cyclic cocycle $\varphi(a_0, \dots, a_{2n+1})$ on A , we have

$$\langle [\varphi], [u] \rangle = \tilde{\varphi}(u^{-1} - 1, u - 1, \dots, u^{-1} - 1, u - 1).$$

Exercise: Show that the above formula defines a pairing $K_1(A) \otimes HC^{2n+1}(A) \rightarrow k$.

There are also formulas for Chern-Connes pairings when the cyclic cocycle is in the (b, B) or cyclic bicomplex; but we won't recall them here (cf. [15], [53]).

There is an alternative “infinitesimal proof” of the well-definement of these pairings which works for Banach (or certain classes of topological) algebras where elements of $K_0(A)$ can be defined as smooth homotopy classes of idempotents [15]:

Lemma 5.1. *Let $e_t, 0 \leq t \leq 1$, be a smooth family of idempotents in a Banach algebra A . There exists a smooth family $x_t, 0 \leq t \leq 1$ of elements of A such that*

$$\dot{e}_t := \frac{d}{dt}(e_t) = [x_t, e_t], \quad \text{for } 0 \leq t \leq 1.$$

Proof. Let

$$x_t = [\dot{e}_t, e_t] = \dot{e}_t e_t - e_t \dot{e}_t.$$

Differentiating the idempotent condition $e_t^2 = e_t$ with respect to t we obtain

$$\frac{d}{dt}(e_t^2) = \dot{e}_t e_t + e_t \dot{e}_t = \dot{e}_t.$$

Multiplying this last relation on the left by e_t we obtain

$$e_t \dot{e}_t e_t = 0.$$

Now we have

$$[x_t, e_t] = [\dot{e}_t e_t - e_t \dot{e}_t, e_t] = \dot{e}_t e_t + e_t \dot{e}_t = \dot{e}_t.$$

□

It follows that if $\tau : A \rightarrow \mathbb{C}$ is a trace (= a cyclic zero cocycle), then

$$\frac{d}{dt} \langle \tau, e_t \rangle = \frac{d}{dt} \tau(e_t) = \tau(\dot{e}_t) = \tau([x_t, e_t]) = 0.$$

So that the value of the pairing, for a fixed τ , depends only on the homotopy class of the idempotent. This shows that the pairing

$$\{ \text{traces on } A \} \times K_0(A) \longrightarrow \mathbb{C}$$

is well-defined.

This is generalized in

Lemma 5.2. *Let $\varphi(a_0, \dots, a_{2n})$ be a cyclic $2n$ -cocycle on A and let e_t be a smooth family of idempotents in A . Then the number*

$$\langle [\varphi], [e_t] \rangle = \varphi(e_t, \dots, e_t)$$

is constant in t .

Proof. Differentiating with respect to t and using the above Lemma, we obtain

$$\begin{aligned} \frac{d}{dt} \varphi(e_t, \dots, e_t) &= \varphi(\dot{e}_t, \dots, e_t) + \varphi(e_t, \dot{e}_t, \dots, e_t) \\ &\quad \dots + \varphi(e_t, \dots, e_t, \dot{e}_t) \\ &= \sum_{i=0}^{2n} \varphi(e_t, \dots, [x_t, e_t], \dots, e_t) \\ &= L_{x_t} \varphi(e_t, \dots, e_t). \end{aligned}$$

We saw that inner derivations act trivially on Hochschild and cyclic cohomology. This means that for each t there is a cyclic cocchain ψ_t such that the Lie derivative $L_{x_t} \varphi = b\psi_t$. We then have

$$\frac{d}{dt} \varphi(e_t, \dots, e_t) = (b\psi_t)(e_t, \dots, e_t) = 0.$$

□

Exercise: Repeat the above proof in the odd case.

The formulas for the even and odd *Chern-Connes character maps*

$$Ch_0^{2n} : K_0(A) \longrightarrow HC_{2n}(A),$$

$$Ch_1^{2n+1} : K_1(A) \longrightarrow HC_{2n+1}(A),$$

are as follows. In the even case, given an idempotent $e = (e_{ij}) \in M_k(A)$, we have

$$Ch_0^{2n}(e) = Tr(\underbrace{e \otimes e \cdots \otimes e}_{2n+1}) = \sum_{i_0, i_1, \dots, i_{2n}} e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_{2n} i_0}.$$

In low dimensions we have

$$\begin{aligned} Ch_0^0(e) &= \sum_{i=1}^k e_{ii}, \\ Ch_0^2(e) &= \sum_{i_0=1}^k \sum_{i_1=1}^k \sum_{i_2=1}^k e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes e_{i_2 i_0}. \end{aligned}$$

In the odd case, given an invertible matrix $u \in M_k(A)$, we have

$$Ch_1^{2n+1}([u]) = Tr(\underbrace{(u^{-1} - 1) \otimes (u - 1) \otimes \cdots \otimes (u^{-1} - 1) \otimes (u - 1)}_{2n+2}).$$

Examples:

1. For $n = 0$, $HC^0(A)$ is the space of traces on A . Therefore the Chern-Connes pairing reduces to the map

$$\{\text{traces on } A\} \times K_0(A) \longrightarrow k,$$

$$\langle \tau, [e] \rangle = \sum_{i=1}^n \tau(e_{ii}),$$

where $e = [e_{ij}] \in M_n(A)$ is an idempotent. The induced function on $K_0(A)$ is called the *dimension function* and denoted by dim_τ . Here is a slightly different approach to this dimension function.

Let E be a finite projective right A -module. A trace τ on A induces a trace on the endomorphism algebra of E ,

$$Tr : End_A(E) \longrightarrow k$$

as follows. First assume that $E = A^n$ is a free module. Then $End_A(E) \simeq M_n(A)$ and our trace map is defined by

$$Tr(a_{i,j}) = \sum a_{ii}.$$

It is easy to check that the above map is a trace. In general, there is an A -module F such that $E \oplus F \simeq A^n$ is a free module and $End_A(E)$ embeds in $M_n(A)$. One can check that the induced trace on $End_A(E)$ is independent of the choice of splitting.

Exercise: Since E is finite and projective, we have $End_A(E) \simeq E \otimes_A E^*$. The induced trace is simply the canonical pairing between E and E^* .

Definition 5.1. *The dimension function associated to a trace τ on A is the additive map*

$$dim_\tau : K_0(A) \longrightarrow k,$$

induced by the map

$$dim_\tau(E) = Tr(id_E),$$

for any finite projective A -module E .

It is clear that if E is a vector bundle on a connected topological space X and $\tau(f) = f(x_0)$, where $x_0 \in X$ is a fixed point, then $dim_\tau(E)$ is the rank of the vector bundle E and is an integer. One of the striking features of noncommutative geometry is the existence of noncommutative vector bundles with non integral dimensions. A beautiful example of this phenomenon is shown by *Rieffel's idempotent* $e \in \mathcal{A}_\theta$ with $\tau(e) = \theta$, where τ is the canonical trace on the noncommutative torus [15].

2. Let $A = C^\infty(S^1)$ denote the algebra of smooth complex valued functions on the circle. One knows that $K_1(A) \simeq K^1(S^1) \simeq \mathbb{Z}$ and $u(z) = z$ is a generator of this group. Let

$$\varphi(f_0, f_1) = \int_{S^1} f_0 df_1$$

denote the cyclic cocycle on A representing the fundamental class of S^1 in de Rham homology. We have

$$\langle [\varphi], [u] \rangle = \varphi(u, u^{-1}) = \int_{S^1} u du^{-1} = -2\pi i.$$

Alternatively the Chern character

$$Ch_1^1([u]) = u \otimes u^{-1} \in HC_1(A) \simeq H_{dR}^1(S^1),$$

is the class of the differential form $\omega = z^{-1}dz$, representing the fundamental class of S^1 in de Rham cohomology.

3. Let $A = C^\infty(S^2)$ and let $e \in M_2(A)$ denote the idempotent representing the Hopf line bundle on S^2 :

$$e = \frac{1}{2} \begin{pmatrix} 1 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & 1 - x_3 \end{pmatrix}.$$

Let us check that under the map

$$HC_2(A) \rightarrow \Omega^2 S^2, \quad a_0 \otimes a_1 \otimes a_2 \mapsto a_0 da_1 da_2,$$

the Chern-Connes character of e corresponds to the fundamental class of S^2 . We have

$$Ch_0^2(e) = Tr(e \otimes e \otimes e) \mapsto Tr(edede) = \frac{1}{8} Tr \begin{pmatrix} 1 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & 1 - x_3 \end{pmatrix} \begin{pmatrix} dx_3 & dx_1 + idx_2 \\ dx_1 - idx_2 & -dx_3 \end{pmatrix} \begin{pmatrix} dx_3 & dx_1 + idx_2 \\ dx_1 - idx_2 & -dx_3 \end{pmatrix}.$$

Performing the computation one obtains

$$Ch_0^2(e) \mapsto \frac{-i}{2} (x_1 dx_2 dx_3 - x_2 dx_1 dx_3 + x_3 dx_1 dx_2).$$

One can then integrate this 2-form on the two sphere S^2 . The result is $-2\pi i$. In particular the class of e in $K_0(A)$ is non-trivial, a fact which can not be proved using just $Ch_0^0(e) = Tr(e)$.

4. For smooth commutative algebras, the noncommutative Chern character reduces to the classical Chern character. We verify this only in the even case.

Let M be a smooth closed manifold and let E be a complex vector bundle on M . Let $e \in C^\infty(M, M_n(\mathbb{C}))$ be an idempotent representing the vector bundle E . One can check that the following formula defines a connection on E , called the Levi-Civita or Grassmanian connection:

$$\nabla(eV) = edV,$$

where $V : M \rightarrow \mathbb{C}^n$ is a smooth function and eV represents an arbitrary smooth section of E . Computing the curvature form we obtain

$$R(eV) = \hat{\nabla}^2(eV) = ededV = edede.eV,$$

which shows that the curvature form is the “matrix valued 2-form”

$$R = edede.$$

From $e^2 = e$, one easily obtains $ede.e = 0$. This implies that

$$R^n = (edede)^n = e \underbrace{dede \cdots dede}_{2n}.$$

Under the canonical map

$$HC_{2n}(A) \rightarrow H_{dR}^{2n}(M), \quad a_0 \otimes \cdots \otimes a_{2n} \mapsto \frac{1}{(2n)!} a_0 da_1 \cdots da_{2n},$$

we have

$$Ch_n^0(e) := Tr(e \otimes \cdots \otimes e) \mapsto \frac{1}{(2n)!} Tr(edede \cdots de).$$

The classical Chern-Weil formula for $Ch(E)$ is

$$Ch(E) = Tr(e^R) = Tr\left(\sum_{n \geq 0} \frac{R^n}{n!}\right).$$

So that its n th component is given by

$$Tr \frac{R^n}{n!} = \frac{1}{n!} Tr((edede)^n) = \frac{1}{n!} Tr(ede \cdots de).$$

A Banach and C^* -algebras

By an *algebra* in this book we mean an *associative algebra* over a commutative unital ground ring k . An algebra is called *unital* if there is a (necessarily unique) element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$. It is called *commutative* if $ab = ba$ for all $a, b \in A$.

Now let $k = \mathbb{R}$ or \mathbb{C} be the field of real or complex numbers. A *norm* on a real or complex algebra A is a map

$$\| \cdot \| : A \rightarrow \mathbb{R},$$

such that for all a, b in A and λ in k we have:

- 1) $\|a\| \geq 0$, and $\|a\| = 0$ iff $a = 0$,
- 2) $\|a + b\| \leq \|a\| + \|b\|$,
- 3) $\|\lambda a\| = |\lambda| \|a\|$,
- 4) $\|ab\| \leq \|a\| \|b\|$.

If A is unital, we assume that $\|1\| = 1$. An algebra endowed with a norm is called a *normed algebra*.

A *Banach algebra* is a normed algebra which is *complete*. Recall that a normed vector space A is called complete if any Cauchy sequence in A is convergent. One of the main consequences of completeness is that absolutely convergent series are convergent, i.e. if $\sum_{n=1}^{\infty} \|a_n\|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent in A . In particular the geometric series $\sum_{n=1}^{\infty} a^n$ is convergent if $\|a\| < 1$. From this it easily follows that the group of invertible elements in a unital Banach algebra A is an open subset of A .

An **-algebra* is a complex algebra endowed with an *-operation, i.e. a map

$$* : A \rightarrow A, \quad a \mapsto a^*,$$

which is *anti-linear* and *involution*:

1. $(a + b)^* = a^* + b^*$, $(\lambda a)^* = \bar{\lambda} a^*$,
2. $(ab)^* = b^* a^*$,
3. $(a^*)^* = a$,

for all a, b in A and λ in \mathbb{C} .

A *Banach $*$ -algebra* is a complex Banach algebra endowed with an $*$ -operation such that for all $a \in A$, $\|a^*\| = \|a\|$. In particular for all a in A we have

$$\|a^*a\| \leq \|a^*\| \|a\| = \|a\|^2.$$

Definition A.1. A *C^* -algebra* is a Banach $*$ -algebra A such that for all $a \in A$,

$$\|aa^*\| = \|a\|^2.$$

We refer to this last identity as the *C^* -identity*.

For reasons that will become clear later in this section, C^* -algebras occupy a very special place among all Banach algebras. This should be compared with the role played by Hilbert spaces among all Banach spaces. In fact, as we shall see there is an intimate relationship between Hilbert spaces and C^* -algebras thanks to the GNS construction and the Gelfand-Naimark embedding theorem.

A *morphism* of C^* -algebras is an algebra homomorphism

$$f : A \longrightarrow B$$

between C^* -algebras A and B such that f preserves the $*$ structure, i.e.

$$f(a^*) = f(a)^*, \quad \text{for all } a \in A.$$

It can be shown that morphisms of C^* -algebras are *contractive* in the sense that for all $a \in A$,

$$\|f(a)\| \leq \|a\|.$$

In particular they are automatically continuous. It follows from this fact that the norm of a C^* -algebra is unique in the sense that if $(A, \|\cdot\|_1)$ and $(A, \|\cdot\|_2)$ are both C^* -algebras then

$$\|a\|_1 = \|a\|_2,$$

for all $a \in A$. Note also that a morphism f of C^* -algebras is an *isomorphism* if and only if f is one to one and onto. Isomorphisms of C^* -algebras are necessarily *isometric*.

In sharp distinction from C^* -algebras, one can have different Banach algebra norms on the same algebra. For example on the algebra of $n \times n$

matrices one can have different Banach algebra norms and only one of them is a C^* -norm.

Examples:

1. (commutative C^* -algebras). Let X be a locally compact Hausdorff space. We associate to X several classes of algebras of functions on X which are C^* -algebras.

1.a Let

$$C_0(X) = \{f : X \rightarrow \mathbb{C}; f \text{ is continuous and } f \text{ vanishes at } \infty\} .$$

By definition, f vanishes at ∞ if for all $\epsilon > 0$ there exists a compact subset $K \subset X$ so that $|f(x)| < \epsilon$ for all $x \in X \setminus K$. Under pointwise addition and scalar multiplication $C_0(X)$ is obviously an algebra over the field of complex numbers \mathbb{C} . Endowed with the sup-norm

$$\|f\| = \|f\|_\infty = \sup\{|f(x)|; x \in X\},$$

and $*$ -operation

$$f \mapsto f^*, f^*(x) = \bar{f}(x),$$

one checks that $C_0(X)$ is a commutative C^* -algebra. It is unital if and only if X is compact. If X is compact, we simply write $C(X)$ instead of $C_0(X)$.

By a theorem of Gelfand and Naimark, any commutative C^* -algebra is of the type $C_0(X)$ for some locally compact Hausdorff space X (see below).

1.b Let

$$C_b(X) = \{f : X \rightarrow \mathbb{C}; f \text{ is continuous and bounded}\}.$$

Then with the same operations as above, $C_b(X)$ is a unital C^* -algebra. Note that $C_0(X) \subset C_b(X)$ is an *essential ideal* in $C_b(X)$. (An ideal I in an algebra A is an essential ideal if for all a in A , $aI = 0 \Rightarrow a = 0$.)

2. (commutative Banach algebras). It is easy to give examples of Banach algebras which are not C^* -algebras. For an integer $n \geq 1$, let

$$C^n[0, 1] = \{f : [0, 1] \rightarrow \mathbb{C}; f \in C^n\},$$

be the space of functions with continuous n -th derivative. We denote the i -th derivative of f by $f^{(i)}$. With the norm

$$\|f\|_n = \sum_{i=0}^n \frac{\|f^{(i)}\|_\infty}{i!}$$

and the $*$ -operation $f^*(x) = \bar{f}(x)$, one checks that $C^n[0, 1]$ is a Banach $*$ -algebra. It is, however, not a C^* -algebra as one can easily show that the C^* -identity fails. Note that for all $f \in C^n[0, 1]$,

$$\|f\|_\infty \leq \|f\|_n .$$

3. (noncommutative C^* -algebras). By a theorem of Gelfand and Naimark recalled below, any C^* -algebra can be realized as a closed $*$ -subalgebra of the algebra of bounded operators on a complex Hilbert space. We start with the simplest examples: the algebra of complex n by n matrices.
- 3.a Let $A = M_n(\mathbb{C})$ be the algebra of n by n matrices over the field of complex numbers \mathbb{C} . With operator norm and the standard adjoint operation $T \mapsto T^*$, A is a C^* -algebra (see below).

A direct sum of matrix algebras

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

is a C^* -algebra as well. It can be shown that any finite dimensional C^* -algebra is unital and is a direct sum of matrix algebras as above [31]. In other words, finite dimensional C^* -algebras are semi-simple.

- 3.b The above example can be generalized as follows. Let H be a complex Hilbert space and let $A = \mathcal{L}(H)$ denote the set of bounded linear operators $H \rightarrow H$. For a bounded operator T , we define T^* to be the *adjoint* of T defined by

$$\langle Tx, y \rangle = \langle x, T^*y \rangle, \quad x, y \in H.$$

Under the usual algebraic operations of addition and multiplication of operators and the *operator norm*

$$\|T\| = \sup\{\|T(x)\|; \|x\| \leq 1\},$$

$\mathcal{L}(H)$ is a C^* -algebra.

It is clear that any subalgebra $A \subset \mathcal{L}(H)$ which is *self-adjoint* in the sense that

$$T \in A \Rightarrow T^* \in A,$$

and is *norm closed*, in the sense that

$$T_n \in A, \|T_n - T\| \rightarrow 0 \Rightarrow T \in A,$$

is a C^* -algebra.

- 3.c (group C^* -algebras). Let G be a discrete group and let $H = \ell^2 G$ denote the Hilbert space of square summable functions on G ;

$$H = \{f : G \rightarrow \mathbb{C}; \sum_{g \in G} |f(g)|^2 < \infty\}.$$

The *left regular representation* of G is the unitary representation π of G on H , defined by

$$(\pi g)f(h) = f(g^{-1}h).$$

It has a linear extension to an (injective) algebra homomorphism

$$\pi : \mathbb{C}G \longrightarrow \mathcal{L}(H),$$

from the group algebra of G to the algebra of bounded operators on H . Its image $\pi(\mathbb{C}G)$ is a $*$ -subalgebra of $\mathcal{L}(H)$.

The *reduced group C^* -algebra* of G , denoted by C_r^*G , is the norm closure of $\pi(\mathbb{C}G)$ in $\mathcal{L}(H)$. It is obviously a C^* -algebra.

There is second C^* -algebra associated to any discrete group G as follows. The (non-reduced) *group C^* -algebra* of G is the norm completion of the $*$ -algebra $\mathbb{C}G$ under the norm

$$\|f\| = \sup \{\|\pi(f)\|; \pi \text{ is a } * \text{-representation of } \mathbb{C}G\},$$

where by a $*$ -representation we mean a $*$ -representation on a Hilbert space. Note that $\|f\|$ is finite since for $f = \sum_{g \in G} a_g g$ (finite sum) and any $*$ -representation π we have

$$\|\pi(f)\| \leq \sum \| \pi(a_g g) \| \leq \sum |a_g| \| \pi(g) \| \leq \sum |a_g|.$$

By its very definition it is clear that there is a 1-1 correspondence between unitary representations of G and C^* representations of C^*G .

Since the identity map $id : (\mathbb{C}G, \| \cdot \|) \rightarrow (\mathbb{C}G, \| \cdot \|_r)$ is continuous, we obtain a surjective C^* -algebra homomorphism

$$C^*G \longrightarrow C_r^*G.$$

It is known that this map is an isomorphism if and only if G is an amenable group [?]. Abelian groups are amenable.

We give a few examples of reduced group C^* -algebras. Let G be an abelian group and $\hat{G} = Hom(G, \mathbb{T})$ the group of characters of G . It is a locally compact Hausdorff space. Moreover it is easily seen that \hat{G} is in fact homeomorphic with the space of characters, or the maximal ideal space, of the C^* -algebra C_r^*G . Thus the Gelfand transform defines an isomorphism of C^* -algebras

$$C^*G \simeq C_0(\hat{G}).$$

In general one should think of the group C^* -algebra of a group G as the “algebra of functions” on the noncommutative space representing the unitary dual of G . Note that, by the above paragraph, this is justified in the commutative case. In the noncommutative case, the unitary dual is a badly behaved space in general but the noncommutative dual is a perfectly legitimate noncommutative space (see the unitary dual of the infinite dihedral group in [15] and its noncommutative replacement).

Let G be a finite group. Since G is finite the group C^* -algebra coincides with the group algebra of G . From basic representation theory we know that the group algebra $\mathbb{C}G$ decomposes as a sum of matrix algebras

$$C^*G \simeq \mathbb{C}G \simeq \bigoplus M_{n_i}(\mathbb{C}),$$

where the summation is over the set of conjugacy classes of G .

It is generally believed that the classic paper of Gelfand and Naimark [35] is the birth place of the theory of C^* -algebras. The following two results on the structure of C^* -algebras are proved in this paper:

Theorem A.1 (Gelfand-Naimark [35]). *a) Let A be a commutative C^* -algebra and let $\Omega(A)$ denote the maximal ideal space of A . Then the Gelfand transform*

$$A \rightarrow C_0(\Omega(A)), \quad a \mapsto \hat{a},$$

is an isomorphism of C^ -algebras.*

b) Any C^ -algebra is isomorphic to a C^* -subalgebra of the algebra $\mathcal{L}(H)$ of bounded linear operators on some Hilbert space H .*

In the remainder of this appendix we sketch the proofs of statements a) and b) above. They are based on Gelfand's theory of commutative Banach algebras, and the Gelfand-Naimark-Segal (GNS) construction of representations of C^* -algebras from states, respectively.

A.1 Gelfand's theory of commutative Banach algebras

The whole theory is based on the notion of spectrum of an element of a Banach algebra and the fact that the spectrum is non-empty. The notion of spectrum can be defined for elements of an arbitrary algebra and it can be easily shown that for finitely generated complex algebras the spectrum is non-empty. As is shown in [9], this latter fact leads to an easy proof of Hilbert's Nullstellensatz. This makes the proofs of the two major duality theorems remarkably similar. We use this approach in this book.

Let A be a unital algebra over a field \mathbb{F} . The *spectrum* of an element $a \in A$ is defined by

$$\text{sp}(a) = \{\lambda \in \mathbb{F}; a - \lambda 1 \text{ is not invertible}\}.$$

We should think of the spectrum as the noncommutative analogue of the set of values of a function. This is justified in Example 1 below.

Examples:

1. Let $A = C(X)$ be the algebra of continuous complex valued functions on a compact space X . For any $f \in A$,

$$\text{sp}(f) = \{f(x); x \in X\},$$

is the range of f .

2. Let $A = M_n(\mathbb{F})$ be the algebra of $n \times n$ matrices with coefficients in \mathbb{F} . For any matrix $a \in A$

$$\text{sp}(a) = \{\lambda \in \mathbb{F}; \det(a - \lambda 1) = 0\},$$

is the set of eigenvalues of a .

Exercise:

- 1) Show that if a is nilpotent then $\text{sp}(a) = \{0\}$.
- 2) Show that

$$\text{sp}(ab) \setminus \{0\} = \text{sp}(ba) \setminus \{0\}.$$

- 3) Let $T : H \rightarrow H$ be a Fredholm operator on a Hilbert space H . Let Q be an operator such that $1 - PQ$ and $1 - QP$ are trace class operators. Show that

$$\text{Index}(T) = \text{Tr}(1 - PQ) - \text{Tr}(1 - QP).$$

In general, the spectrum may be empty. We give two general results that guarantee the spectrum is non-empty. They are at the foundation of Gelfand-Naimark theorem and Hilbert's Nullstellensatz. Part b) is in [9].

Theorem A.2. (a) (Gelfand) Let A be a unital Banach algebra over \mathbb{C} . Then for any $a \in A$, $\text{sp}(a) \neq \emptyset$.

(b) Let A be a unital algebra over \mathbb{C} . Assume $\dim_{\mathbb{C}} A$ is countable. Then for any $a \in A$, $\text{sp}(a) \neq \emptyset$. Furthermore, an element a is nilpotent if and only if $\text{sp}(a) = \{0\}$.

Proof. We sketch a proof of both statements. For a) assume the spectrum of an element a is empty. Then the function

$$R : \mathbb{C} \rightarrow A, \quad \lambda \mapsto (a - \lambda 1)^{-1},$$

is holomorphic (in an extended sense), non-constant, and bounded. This is easily shown to contradict the classical Liouville's theorem from complex analysis.

For b), again assume the spectrum of a is empty. Then it can be shown that the uncountable set

$$\{(a - \lambda 1)^{-1}; \lambda \in \mathbb{C}\}$$

is a linearly independent set. But this contradicts the fact that $\dim_{\mathbb{C}} A$ is countable.

For the second part of b), assume $\text{sp}(a) = \{0\}$. Since $\dim_{\mathbb{C}}(A)$ is countable, any element $a \in A$ satisfies a polynomial equation. Let

$$p(a) = a^k(a - \lambda_1) \cdots (a - \lambda_n) = 0$$

be the minimal polynomial of A . Then $n = 0$ since otherwise an element $a - \lambda_i$ is not invertible with $\lambda_i \neq 0$. But this contradicts our assumption that $\text{sp}(a) = \{0\}$. The other direction is true in general and is easy. \square

The first part of the following corollary is known as Gelfand-Mazur theorem.

Corollary A.1. *Let A be either a unital complex Banach algebra or a unital complex algebra such that $\dim_{\mathbb{C}} A$ is countable. If A is a division algebra, then $A \simeq \mathbb{C}$.*

Let A be an algebra. By a *character* of A we mean a non-zero algebra homomorphism

$$\varphi : A \rightarrow \mathbb{F}.$$

Note that if A is unital, then $\varphi(1) = 1$. We establish the link between characters and maximal ideals of A . For the following result A is either a commutative unital complex Banach algebra, or is a commutative unital algebra with $\dim_{\mathbb{C}} A$ countable.

Corollary A.2. *The relation $I = \ker \varphi$ defines a 1-1 correspondence between the set of maximal ideals of A and the set of characters of A .*

Before embarking on the proof of Gelfand-Naimark theorem, we sketch a proof of Hilbert's Nullstellensatz, following [9].

Let

$$A = \mathbb{C}[x_1, \dots, x_n]/I$$

be a finitely generated commutative reduced algebra. Recall that reduced means if $a^n = 0$ for some n then $a = 0$ (no nilpotent elements). Equivalently the ideal I is radical. Let

$$V = \{z \in \mathbb{C}^n; p(z) = 0, \text{ for all } p \text{ in } I\},$$

let $J(V)$ be the ideal of functions vanishing on V , and let

$$\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]/J(V)$$

be the algebra of regular functions on V . Since $I \subset J(V)$, we have an algebra homomorphism

$$\pi : A \longrightarrow \mathbb{C}[V].$$

One of the original forms of Hilbert's Nullstellensatz states that this map is an isomorphism. It is clearly surjective. For its injectivity, let $a \in A$ and let $\pi(a) = 0$, or equivalently $\pi(a) \in J(V)$. Since a vanishes on all points of V , it follows that a is in the intersection of all the maximal ideals of A . This shows that its spectrum $\text{sp}(A) = \{0\}$. By Theorem A.2 (b), it follows that a is nilpotent and since A is reduced, we have $a = 0$.

The rest of this section is devoted to sketch a proof of the Gelfand-Naimark theorem on the structure of commutative C^* -algebras. Let A be a unital Banach algebra. It is easy to see that any character of A is continuous of norm 1. To prove this, note that if this is not the case then there exists an $a \in A$, with $\|a\| < 1$ and $\varphi(a) = 1$. Let $b = \sum_{n \geq 1} a^n$. Then from $a + ab = b$, we have

$$\varphi(b) = \varphi(a) + \varphi(a)\varphi(b) = 1 + \varphi(b),$$

which is impossible. Therefore $\|\varphi\| \leq 1$, and since $\varphi(1) = 1$, $\|\varphi\| = 1$.

Let A be a complex Banach algebra and let $\Omega(A)$ denote the set of characters of A . Thus if A is unital, then $\Omega(A) = \text{set of maximal ideals of } A$. It is clear that a pointwise limit of characters is again a character. Thus $\Omega(A)$ is a closed subset of the unit ball of the dual space A^* . Since the latter space is a compact Hausdorff space in the weak* topology, we conclude that $\Omega(A)$ is also a compact Hausdorff space.

If A is not unital, let $A^+ = A \oplus \mathbb{C}$ be the unitization of A . It is clear that $\Omega(A) = \Omega(A^+) \setminus \{\varphi_0\}$, where φ_0 is the trivial character $\varphi_0(a) = 0$ for all $a \in A$. Since $\Omega(A^+)$ is compact, we conclude that $\Omega(A)$ is a locally compact Hausdorff space. We have thus proved the lemma:

Lemma A.1. *Let A be a Banach algebra. Then $\Omega(A)$ is a locally compact Hausdorff space. $\Omega(A)$ is compact if and only if A is unital.*

Let $f : A \rightarrow B$ be a continuous homomorphism of commutative unital Banach algebras. Define a map $\Omega(f) = f^* : \Omega(B) \rightarrow \Omega(A)$ by

$$f^*(\varphi) = \varphi \circ f.$$

It is clear that $f^*\varphi$ is multiplicative, linear and continuous. Thus we have defined the *spectrum functor* Ω from the category of commutative unital complex Banach algebras to the category of compact Hausdorff spaces.

Next we define the *Gelfand transform*. Let A be a commutative Banach algebra. The Gelfand transform is the map Γ defined by

$$\begin{aligned}\Gamma : A &\rightarrow C_0(\Omega(A)), \\ a &\mapsto \hat{a}, \quad \hat{a}(\varphi) = \varphi(a).\end{aligned}$$

This map is obviously an algebra homomorphism. It is also clear that $\|\Gamma\| \leq 1$, i.e. Γ is contractive.

The spectrum of an element is easily seen to be a closed and bounded subset of \mathbb{C} (this follows from: $\|1 - a\| < 1 \Rightarrow a$ is invertible with inverse $a^{-1} = \sum_{n \geq 0} (1 - a)^n$.) Let

$$r(a) = \max \{|\lambda|; \lambda \in \text{sp}(a)\},$$

denote the *spectral radius* of an element $a \in A$. Note that $r(a) \leq \|a\|$.

Now Corollary 6.2 tells us that

$$\text{sp}(a) = \{\varphi(a); \varphi \in \Omega(A)\}.$$

The following result is then immediate:

Proposition A.1. *The Gelfand transform $A \rightarrow C_0(\Omega(A))$ is a norm decreasing algebra homomorphism and its image separates points of $\Omega(A)$. Moreover, for all $a \in A$, $\|\hat{a}\|_\infty = r(a)$.*

The kernel of the Gelfand transform is called the *radical* of the Banach algebra A . It consists of elements a whose spectral radius $r(a) = 0$, or equivalently, $\text{sp}(a) = \{0\}$. Hence the radical contains all the nilpotent elements. But it may contain more. An element a is called *quasi-nilpotent* if $\text{sp}(a) = \{0\}$. A is said to be *semi-simple* if its radical is zero, i.e. the only quasi-nilpotent elements of A is 0.

Example. We give an example of a commutative Banach algebra for which the Gelfand transform is injective but not surjective. Let $H(D)$ be the space of continuous functions on the unit disk D which are holomorphic in the interior of the disk. With the sup-norm $\|f\| = \|f\|_\infty$ it is a Banach algebra.

It is, however, not a C^* -algebra (why?). Show that $\Omega(A) \simeq D$ and the Gelfand transform is an isometric embedding $H(D) \rightarrow C(D)$.

We are now ready to prove the first main theorem of Gelfand and Naimark in [35]: for commutative C^* -algebras Γ is an isometric $*$ -isomorphism. We need a few simple facts about C^* -algebras first.

Let $f : A \rightarrow B$ be a morphism of C^* -algebras. It is easily seen that $\text{sp}(f(a)) \subset \text{sp}(a)$. Hence, using the C^* -identity, we have

$$\|f(a)\|^2 = \|f(a)f(a^*)\| = \|f(aa^*)\| = r(a^*a) \leq \|a^*a\| = \|a\|^2.$$

Let $a \in A$ be a standpoint element ($a = a^*$). Then $\text{sp}(a) \subset \mathbb{R}$ is real. Indeed since e^{ia} is unitary its spectrum is located on the unit circle (why?). Hence for $\lambda \in \text{sp}(a)$, $e^{i\lambda}$ is located on the unit circle which shows that λ is real. From this it follows that if $f : A \rightarrow \mathbb{C}$ is an algebra homomorphism (a character), then $f(a^*) = \overline{f(a)}$, for all a .

Theorem A.3. *Let A be a commutative C^* -algebra. The Gelfand transform $A \rightarrow C_0(\Omega(A))$ is an isomorphism of C^* -algebras.*

Proof. We prove the unital case. The non-unital case follows with minor modifications [31]. What we have shown so far amounts to the fact that Γ is an isometric $*$ -algebra map whose image separates points of $\Omega(A)$. Thus $\Gamma(A)$ is a closed $*$ -subalgebra of $C(\Omega(A))$ that separates points of $\Omega(A)$. By Stone-Weierstrass theorem, $\Gamma(A) = C(\Omega(A))$. □

The above theorem is one of the landmarks of Gelfand's theory of commutative Banach algebras. While a complete classification of all commutative Banach algebras seems to be impossible, this result classifies all commutative C^* -algebras.

Care must be applied in dealing with non-compact spaces and non-unital algebras. First of all it is clear that if X is not compact then the pull-back $f^* : C_0(Y) \rightarrow C_0(X)$, $f^*(g) = g \circ f$ of a continuous map $f : X \rightarrow Y$ is well defined if and only if f is a proper map. Secondly, one notes that not all C^* -maps $C_0(Y) \rightarrow C_0(X)$ are obtained in this way. For example, for $X = (0, 1)$ an open interval and Y a single point, the zero map $0 : C_0(Y) \rightarrow C_0(X)$, which is always a C^* -morphism, is not the pull-back of any proper map. It turns out that one way to single out the appropriate class of morphisms is as follows. A morphism of C^* -algebras $\varphi : A \rightarrow B$ is called *proper* if for an

approximate unit (e_i) of A , $\varphi(e_i)$ is an approximate unit of B . Recall that [31, 37] an *approximate unit* of a C^* -algebra A is an increasing net of positive elements $(e_i), i \in I$, of A such that for all $a \in A$, $e_i a \rightarrow a$, or equivalently, $a e_i \rightarrow a$. Now it can be shown that any proper map $C_0(Y) \rightarrow C_0(X)$ is the pull-back of a proper map $X \rightarrow Y$. Similarly, in the other direction, if $\varphi : A \rightarrow B$ is a proper morphism of C^* -algebras, then $\Omega(\varphi) : \Omega(B) \rightarrow \Omega(A)$ is a proper continuous map. We refer to [37] and references therein for more details.

We are now half-way through showing that the functors C_0 and Ω are quasi-inverse to each other. But the proof of the other half is much simpler and is left to the reader.

Lemma A.2. *Let X be a locally compact Hausdorff space. Then the evaluation map*

$$\begin{aligned} X &\rightarrow \Omega(C_0(X)), & x &\mapsto \varphi_x, \\ \varphi_x(f) &= f(x), \end{aligned}$$

is a homeomorphism.

Example A.1. *We give a few elementary applications of the Gelfand-Naimark correspondence between commutative C^* -algebras and locally compact spaces.*

1. *(Idempotents and connectedness). Let A be a unital commutative C^* -algebra. Then $\Omega(A)$ is disconnected iff A has a non-trivial idempotent (i.e. an element $e \neq 0, 1$ such that $e^2 = e$).*
2. *(Ideals and closed subsets). Let X be a compact Hausdorff space. The Gelfand-Naimark duality shows that there is a 1-1 correspondence*

$$\{\text{closed subsets of } X\} \leftrightarrow \{\text{closed ideals of } C(X)\},$$

where to each closed subset $Y \subset X$, we associate the ideal

$$I = \{f \in C(X); f(y) = 0 \quad \forall y \in Y\}$$

of all functions vanishing on Y . We have natural isomorphisms

$$\begin{aligned} C_0(X \setminus Y) &\simeq I, \\ C_0(X)/I &\simeq C_0(Y), \\ C_0(X/Y) &\simeq C_0(X \setminus Y)^+. \end{aligned}$$

3. (*Essential ideals and compactification*) Let X be a locally compact Hausdorff space. Recall that a Hausdorff compactification of X is a compact Hausdorff space Y where X is homeomorphic to a dense subset of Y . We consider X as a subspace of Y . Then X is open in Y and its boundary $Y \setminus X$ is compact. We have an exact sequence

$$0 \rightarrow C_0(X) \rightarrow C(Y) \rightarrow C(Y \setminus X) \rightarrow 0,$$

where $C_0(X)$ is an essential ideal of $C(Y)$. Conversely, show that any extension

$$0 \rightarrow C_0(X) \rightarrow A \rightarrow B \rightarrow 0,$$

Where A is a unital C^* -algebra and $C_0(X)$ is an essential ideal of A , defines a Hausdorff compactification of X . Thus, we have a 1-1 correspondence between Hausdorff compactifications of X and (isomorphism classes of) essential extensions of $C_0(X)$. In particular, the 1-point compactification and the Stone-Cech compactifications correspond to

$$0 \rightarrow C_0(X) \rightarrow C_0(X)^+ \rightarrow \mathbb{C} \rightarrow 0,$$

and

$$0 \rightarrow C_0(X) \rightarrow C_b(X) \rightarrow C(\beta X) \rightarrow 0$$

A.2 States and the GNS construction

Our goal in this section is to sketch a proof of the second main result of Gelfand and Naimark in [35] to the effect that any C^* -algebra can be embedded in the algebra of bounded operators on a Hilbert space. The main idea of the proof is an adaptation of the idea of *left regular representation* of algebras to the context of C^* -algebras, called Gelfand-Naimark-Segal (GNS) construction.

A *positive linear functional* on a C^* -algebra A is a \mathbb{C} -linear map $\varphi : A \rightarrow \mathbb{C}$ such that for all a in A ,

$$\varphi(a^*a) \geq 0.$$

A *state* on A is a positive linear functional φ with $\|\varphi\| = 1$. It can be shown that if A is unital then this last condition is equivalent to $\varphi(1) = 1$.

If φ_1 and φ_2 are states then for any $t \in [0, 1]$, $t\varphi_1 + (1-t)\varphi_2$ is a state as well. Thus the set of states of A , denoted by $\mathcal{S}(A)$, form a convex subset of

the unit ball of A^* . The extreme points of $\mathcal{S}(A)$ are called *pure states*.

Examples:

1. States are noncommutative analogues of probability measures. This idea is corroborated by the Riesz representation theorem: For a locally compact Hausdorff space X there is a 1-1 correspondence between states on $C_0(X)$ and Borel probability measures on X . To a probability measure μ is associated the state φ defined by

$$\varphi(f) = \int_X f d\mu.$$

φ is a pure state if and only if $\mu = \delta_x$ is a Dirac measure for a point $x \in X$.

2. Let $A = M_n(\mathbb{C})$ and $p \in A$ be a positive matrix with $\text{tr}(p) = 1$. (Such matrices, and their infinite dimensional analogues, are called *density matrices* in quantum statistical mechanics.) Then

$$\varphi(a) = \text{tr}(ap)$$

defines a state on A .

3. Let $\pi : A \rightarrow \mathcal{L}(H)$ be a *representation* of a unital C^* -algebra A on a Hilbert space H . This simply means that π is a morphism of unital C^* -algebras. Let $x \in H$ be a vector of length one. Then

$$\varphi(a) = \langle \pi(a)x, x \rangle$$

defines a state on A , called a *vector state*. In the following we show that, conversely, any state on A is a vector state with respect to a suitable representation called the GNS representation.

Let φ be a positive linear functional on A . Then

$$\langle a, b \rangle := \varphi(b^*a)$$

is linear in the first variable and anti-linear in the second variable. It is also semi-definite in the sense that $\langle a, a \rangle = \varphi(a^*a) \geq 0$ for all a in A . Thus it satisfies the *Cauchy-Schwartz inequality*: for all a, b

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

Let

$$N = \{a \in A; \varphi(a^*a) = 0\}.$$

It is easy to see, using the above Cauchy-Schwarz inequality, that N is a closed left ideal of A and the following positive definite inner product is well-defined on the quotient space A/N :

$$\langle x + N, y + N \rangle := \langle x, y \rangle .$$

Let H_φ denote the Hilbert space completion of A/N under the above inner product. The *left regular representation* $A \times A \rightarrow A, (a, b) \mapsto ab$ of A on itself induces a bounded linear map $A \times A/N \rightarrow A/N, (a, b + N) \mapsto ab + N$. We denote its unique extension to H_φ by

$$\pi_\varphi : A \longrightarrow \mathcal{L}(H_\varphi).$$

The representation (π_φ, H_φ) is called the GNS representation defined by the state φ . The state φ can be recovered from the representation (π_φ, H_φ) as a vector state as follows. Let $x = \pi_\varphi(1)$. Then for all a in A ,

$$\varphi(a) = \langle (\pi_\varphi a)(x), x \rangle .$$

The representation (π_φ, H_φ) may not be faithful. It can be shown that it is irreducible if and only if φ is a pure state [31]. To construct a faithful representation, and hence an embedding of A into the algebra of bounded operators on a Hilbert space, one first shows that there are enough pure states on A . The proof of the following result is based on Hahn-Banach and Krein-Milman theorems.

Lemma A.3. *For any selfadjoint element a of A , there exists a pure state φ on A such that $\varphi(a) = \|a\|$.*

Using the GNS representation associated to φ , we can then construct, for any $a \in A$, an irreducible representation π of A such that $\|\pi(a)\| = |\varphi(a)| = \|a\|$.

We can now prove the second theorem of Gelfand and Naimark.

Theorem A.4. *Every C^* -algebra is isomorphic to a C^* -subalgebra of the algebra of bounded operators on a Hilbert space.*

Proof. Let $\pi = \sum_{\varphi \in \mathcal{S}(A)} \pi_\varphi$ denote the direct sum of all GNS representations for all states of A . By the above remark π is faithful. \square

B Idempotents and finite projective modules

Let A be a unital algebra over a commutative ring k and let \mathcal{M}_A denote the category of right A -modules. We assume our modules M are *unitary* in the sense that the unit of the algebra acts as the identity of M . A morphism of this category is a right A -module map $f : M \rightarrow N$, i.e. $f(ma) = f(m)a$, for all a in A and m in M .

A *free* module, indexed by a set I , is a module of the type

$$M = A^I = \bigoplus_I A,$$

where the action of A is by component-wise right multiplication. Equivalently, M is free if and only if there are elements $m_i \in M, i \in I$, such that any $m \in M$ can be uniquely expressed as a finite sum $m = \sum_i m_i a_i$. A module M is called *finite* (= *finitely generated*) if there are elements m_1, m_2, \dots, m_k in M such that every element of $m \in M$ can be expressed as $m = m_1 a_1 + \dots + m_k a_k$, for some $a_i \in A$. Equivalently M is finite if there is a surjective A -module map $A^k \rightarrow M$ for some integer k .

Free modules correspond to trivial vector bundles. To obtain a more interesting class of modules we consider the class of *projective modules*. A module P is called projective if it is a direct summand of a free module. That is there exists a module Q such that

$$P \oplus Q \simeq A^I.$$

A module is said to be *finite projective* (= *finitely generated projective*), if it is both finitely generated and projective.

Lemma B.1. *Let P be an A -module. The following conditions on P are equivalent:*

1. P is projective.
2. Any surjection

$$M \xrightarrow{f} P \rightarrow 0,$$

splits in the category of A -modules.

3. For all A -modules N and M and morphisms f, g with g surjective in the following diagram, there exists a morphism \tilde{f} such that the diagram commutes:

$$\begin{array}{ccccc}
 & & P & & \\
 & \nearrow \exists \tilde{f} & \downarrow f & & \\
 N & \xrightarrow{g} & M & \longrightarrow & 0
 \end{array}$$

We say that \tilde{f} is a lifting of f along g .

4. The functor

$$\text{Hom}_A(P, -) : \mathcal{M}_A \rightarrow \mathcal{M}_k$$

is exact in the sense that for any short exact sequence of A -modules

$$0 \rightarrow R \rightarrow S \rightarrow T \rightarrow 0,$$

the sequence of k -modules

$$0 \rightarrow \text{Hom}_A(P, R) \rightarrow \text{Hom}_A(P, S) \rightarrow \text{Hom}_A(P, T) \rightarrow 0$$

is exact.

Example B.1. 1. Free modules are projective.

2. If A is a division ring, then any A -module is free, hence projective.
 3. $M = \mathbb{Z}/n\mathbb{Z}$, $n \geq 2$, is not projective as a \mathbb{Z} -module.
 4. A direct sum $P = \bigoplus_i P_i$ of modules is projective iff each summand P_i is projective.
 5. (idempotents) Let

$$e \in M_n(A) = \text{End}_A(A^n), \quad e^2 = e,$$

be an idempotent. Multiplication by e defines a right A -module map

$$P_e : A^n \rightarrow A^n, \quad x \mapsto ex.$$

Let $P = \text{Im}(P_e)$ and $Q = \text{Ker}(P_e)$ be the image and kernel of P_e . Then, using the idempotent condition $e^2 = e$, we obtain a direct sum decomposition

$$P \oplus Q = A^n,$$

which shows that both P and Q are projective modules. Moreover they are obviously finitely generated. It follows that both P and Q are finite projective modules.

Conversely, given any finite projective module P , let Q be a module such that $P \oplus Q \simeq A^n$ for some integer n . Let $e : A^n \rightarrow A^n$ be the right A -module map that corresponds to the projection map

$$(p, q) \mapsto (p, 0).$$

Then it is easily seen that we have an isomorphism of A -modules

$$P \simeq P_e.$$

This shows that any finite projective module is obtained from an idempotent in some matrix algebra over A .

The idempotent $e \in M_n(A)$ associated to a finite projective A -module P depends of course on the choice of the splitting $P \oplus Q \simeq A^n$. Let $P \oplus Q' \simeq A^m$ be another splitting and $f \in M_m(A)$ the corresponding idempotent. Define the operators $u \in \text{Hom}_A(A^m, A^n)$, $v \in \text{Hom}_A(A^n, A^m)$ as compositions

$$\begin{aligned} u : A^m &\xrightarrow{\sim} P \oplus Q \longrightarrow P \longrightarrow P \oplus Q' \xrightarrow{\sim} A^n, \\ v : A^n &\xrightarrow{\sim} P \oplus Q' \longrightarrow P \longrightarrow P \oplus Q \xrightarrow{\sim} A^m. \end{aligned}$$

We have

$$uv = e, \quad vu = f.$$

In general, two idempotents satisfying the above relations are called Murray-von Neumann equivalent. Conversely it is easily seen that Murray-von Neumann equivalent idempotents define isomorphic finite projective modules.

Defining finite projective modules through idempotents is certainly very convenient since both finiteness and projectivity of the module are automatic in this case. In some cases however this is not very useful. For example, modules over quantum tori are directly defined and checking directly that they are finite and projective is difficult. In this case the following method,

due to Rieffel [58], is very useful. (I am grateful to Henrique Bursztyn for bringing this method to my attention).

Let A and B be unital algebras over a ground ring k . Let X be an (A, B) -bimodule endowed with k -bilinear pairings (“algebra-valued” inner products):

$$\langle -, - \rangle_A: X \times X \longrightarrow A,$$

$$\langle -, - \rangle_B: X \times X \longrightarrow B,$$

satisfying the “associativity” condition

$$\langle x, y \rangle_A z = x \langle y, z \rangle_B \quad \text{for all } x, y, z \text{ in } X,$$

and the “fullness” conditions:

$$\langle X, X \rangle_A = A \quad \text{and} \quad \langle X, X \rangle_B = B.$$

We claim that X is a finite projective left A -module. Let 1_B be the unit of B . By fullness of $\langle -, - \rangle_B$, we can find x_i, y_i in $X, i = 1, \dots, k$ such that

$$1_B = \sum_{i=1}^k \langle x_i, y_i \rangle_B.$$

Let $e_i, i = 1, \dots, k$, be a basis for A^k . Define the map

$$P: A^k \longrightarrow X, \quad P(e_i) = y_i.$$

We claim that this map splits and hence X is finite and projective. Consider the map

$$I: X \longrightarrow A^k, \quad I(x) = \sum_i \langle x, x_i \rangle_A e_i.$$

We have

$$\begin{aligned} P \circ I(x) &= \sum_i \langle x, x_i \rangle_A y_i = \sum_i x \langle x_i, y_i \rangle_B \quad (\text{by associativity}) \\ &= x \quad (\text{since } \sum_i \langle x_i, y_i \rangle_B = 1_B). \end{aligned}$$

C Equivalence of categories

There are at least two ways to compare categories: isomorphism and equivalence. *Isomorphism* of categories is a very strong requirement and is hardly useful. *Equivalence* of categories, on the other hand, is a much more flexible concept and is very useful.

Categories A and B are said to be *equivalent* if there is a functor $F : A \rightarrow B$ and a functor $G : B \rightarrow A$, called a *quasi-inverse* of F , such that

$$F \circ G \simeq 1_B, \quad G \circ F \simeq 1_A,$$

where \simeq means isomorphism, or natural equivalence, of functors. This means, for every $X \in \text{obj}A$, $Y \in \text{obj}B$,

$$FG(Y) \sim Y, \quad \text{and} \quad GF(X) \sim X,$$

where \sim denotes natural isomorphism of objects.

If $F \circ G = 1_B$ and $G \circ F = 1_A$ (equality of functors), then we say that categories A and B are *isomorphic*, and F is an isomorphism.

Categories A and B are said to be *antiequivalent* if the opposite category A^{op} is equivalent to B .

Note that a functor $F : A \rightarrow B$ is an isomorphism if and only if $F : \text{obj}A \rightarrow \text{obj}B$ is 1-1, onto and F is *full and faithful* in the sense that for all $X, Y \in \text{obj}A$,

$$F : \text{Hom}_A(X, Y) \rightarrow \text{Hom}_B(F(X), F(Y))$$

is 1-1 (faithful) and onto (full).

It is easy to see that an equivalence $F : A \rightarrow B$ is full and faithful, but it may not be 1-1, or onto on the class of objects. As a result an equivalence may have many quasi-inverses. The following concept clarifies the situation with objects of equivalent categories.

A subcategory A' of a category A is called *skeletal* if 1) the embedding $A' \rightarrow A$ is full, i.e.

$$\text{Hom}_{A'}(X, Y) = \text{Hom}_A(X, Y)$$

for all $X, Y \in \text{obj}A'$ and 2) for any object $X \in \text{obj}A$, there is a unique object $X' \in \text{obj}A'$ isomorphic to X . Any skeleton of A is equivalent to A and it is not difficult to see that two categories A and B are equivalent if and only if they have isomorphic skeletal subcategories A' and B' .

In some examples, like the Gelfand-Naimark theorem, there is a canonical choice for a quasi-inverse for a given equivalence functor F ($F = C_0$ and

$G = \Omega$). There are instances, however, like the Serre-Swan theorem, where there is no canonical choice for a quasi-inverse. The following proposition gives a necessary and sufficient condition for a functor F to be an equivalence of categories. We leave its simple proof to the reader.

Proposition C.1. *A functor $F : A \rightarrow B$ is an equivalence of categories if and only if*

- a) *F is full and faithful, and*
- b) *Any object $Y \in \text{obj}B$ is isomorphic to an object of the form $F(X)$, for some $X \in \text{obj}A$.*

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