Hofmann-Streicher Universes via Coalgebra

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Lifting Grothendieck Universes to Presheaves

Adapted from Hofmann and Streicher (1997)

Let $\mathcal{U} \in Cat$ be a Grothendieck universe.

Given any $\mathcal{U}$-small category C, the category $Set^{C^{op}}$ can be enhanced into a model of dependent type theory.

At any $P \in Set^{C^{op}}$, we set

$$\mathrm{Ty}(P) :\equiv \mathcal{U}^{(\int P)^{op}} \quad .$$

Comprehension is given by the composite of

$$\mathrm{Ty}(P) \equiv \mathcal{U}^{(\int P)^{op}} \rightarrowtail Set^{(\int P)^{op}} \simeq Set^{C^{op}}/P \quad .$$

Lifting Grothendieck Universes to Presheaves (cont'd)

Adapted from Hofmann and Streicher (1997)

We would like a **type classifier** for $Set^{C^{op}}$, *i.e.* a category $\mho$ in $Set^{C^{op}}$ such that

$$Cat(Set^{C^{op}})(P, \mho) \cong \mathrm{Ty}(P) \quad ,$$

naturally in $P \in Set^{C^{op}}$.

Lifting Grothendieck Universes to Presheaves (cont'd)

Adapted from Hofmann and Streicher (1997)

We can arrive at a definition for $\mho$ by transcendental deduction.

If $\mho \in Cat(Set^{C^{op}})(\cong Cat^{C^{op}})$ exists, we must have, for each $I \in C$,

$$\begin{aligned} \mho(I) &\cong Cat^{C^{op}}(\mathrm{y}\, I, \mho) \\ &\cong Cat(Set^{C^{op}})(\mathrm{y}\, I, \mho) \\ &\cong \mathrm{Ty}(\mathrm{y}\, I) \\ &\equiv \mathcal{U}^{(\int \mathrm{y}\, I)^{op}} \\ &\cong \mathcal{U}^{(C/I)^{op}} \quad . \end{aligned}$$

So, we set

$$\mho(I) :\equiv \mathcal{U}^{(C/I)^{op}} \quad ,$$

for each $I \in C$.

Lifting Grothendieck Universes to Sheaves?

Hofmann and Streicher also raise the problem of sheaf models. We can formulate the problem as follows.

Given any $\mathcal{U}$-small site (C, J), the category $\mathrm{Sh}(C, J)$ can be enhanced into a model of dependent type theory.

At any $S \in \mathrm{Sh}(C, J)$, we set

$$\mathrm{Ty}(S) :\equiv \mathrm{Sh}_{<\mathcal{U}}(\int S, J_S) \quad .$$

Comprehension is given by the composite of

$$\mathrm{Ty}(S) \equiv \mathrm{Sh}_{<\mathcal{U}}(\int S, J_S) \rightarrowtail \mathrm{Sh}(\int S, J_S) \simeq \mathrm{Sh}(C, J)/S \quad .$$

Lifting Grothendieck Universes to Sheaves? (cont'd)

We would like a type classifier for $\mathsf{Sh}(C, J)$, *i.e.* a category $\mho$ in $\mathsf{Sh}(C, J)$ such that

$$Cat(\mathsf{Sh}(C, J))(S, \mho) \cong \mathsf{Ty}(S) \quad ,$$

naturally in $S \in \mathsf{Sh}(C, J)$.

Lifting Grothendieck Universes to Sheaves? (cont'd)

We can arrive at a (putative) definition for $\mho$ by transcendental deduction. If $\mho \in \mathit{Cat}(\mathrm{Sh}(C, J))(\hookrightarrow \mathit{Cat}(\mathit{Set}^{C^{op}}) \cong \mathit{Cat}^{C^{op}})$ exists, we must have, for each $I \in C$,

$$\begin{aligned}\mho(I) &\cong \mathit{Cat}^{C^{op}}(\mathrm{y}\, I, \mho)\\ &\cong \mathit{Cat}(\mathit{Set}^{C^{op}})(\mathrm{y}\, I, \mho)\\ &= \mathit{Cat}(\mathit{Set}^{C^{op}})(\mathrm{y}\, I, i\mho)\\ &\cong \mathit{Cat}(\mathrm{Sh}(C, J))(a\,\mathrm{y}\, I, \mho)\\ &\cong \mathrm{Ty}(a\,\mathrm{y}\, I)\\ &\equiv \mathrm{Sh}_{<\mathcal{U}}(\int a\,\mathrm{y}\, I, J_{a\,\mathrm{y}\, I}) \quad .\end{aligned}$$

So, we try setting $\mho(I) :\equiv \mathrm{Sh}_{<\mathcal{U}}(\int a\,\mathrm{y}\, I, J_{a\,\mathrm{y}\, I})$, for each $I \in C$.

Problem

However, our putative type classifier $\mho$ is not a sheaf in general! It may only be a stack, in which amalgamations are unique up to isomorphism.

In the case of $\mathrm{Sh}(X)$ on a $\mathcal{U}$-small topological space X, this was already essentially noticed by the time of SGA1 (Grothendieck 1960).

Note that, in this case, we have, for each $U \in \mathcal{O}(X)$,

$$\begin{aligned}
\mho(U) &\equiv \mathrm{Sh}_{<\mathcal{U}}(\int a\,\mathrm{y}\,U, J_{a\,\mathrm{y}\,U}) \\
&\cong \mathrm{Sh}_{<\mathcal{U}}(\int \mathrm{y}\,U, J_{\mathrm{y}\,U}) \\
&\cong \mathrm{Sh}_{<\mathcal{U}}(\mathcal{O}(X)/U, J/U) \\
&\ldots \\
&= \mathrm{Sh}_{<\mathcal{U}}(U) \quad .
\end{aligned}$$

Lifting Grothendieck Universes to Coalgebras

Motivation

The forgetful morphism of models

$$Set^{C^{op}} \to Set^{|C|}$$

is strictly comonadic and Cartesian.

We will use this to recover the Hofmann-Streicher universe by 'lifting' the type classifier of $Set^{|C|}$ (*i.e.* the family constant at $\mathcal{U}$) across the forgetful morphism to one in $Set^{C^{op}}$.

This emulates the known 'lifting' of the subobject classifier of $Set^{|C|}$ (*i.e.* the family constant at 2) across the forgetful morphism to one in $Set^{C^{op}}$.

Lifting Grothendieck Universes to Coalgebras (cont'd)

Let

$$\Box : \mathcal{E} \to \mathcal{E}$$

be a comonad of models of dependent type theory.

Then, for any coalgebra $(\Gamma, \gamma) \in \mathcal{E}^{\Box}$, we have an induced comonad

$$\mathbb{B}_{(\Gamma,\gamma)} : \mathsf{Ty}_{\mathcal{E}}(\Gamma) \to \mathsf{Ty}_{\mathcal{E}}(\Gamma) \quad ,$$

given on a type $A \in \mathsf{Ty}_{\mathcal{E}}(\Gamma)$ by

$$\mathbb{B}_{(\Gamma,\gamma)} = (\Box A)[\gamma] \quad .$$

Lifting Grothendieck Universes to Coalgebras (cont'd)

The category of coalgebras $\mathcal{E}^{\square}$ can be enhanced into a model of dependent type theory.

At any $(\Gamma, \gamma) \in \mathcal{E}^{\square}$, we set

$$\mathsf{Ty}_{\mathcal{E}^{\square}}(\Gamma, \gamma) :\equiv \left(Ty_{\mathcal{E}}(\Gamma) \right)^{\left(\mathbb{B}_{(\Gamma,\gamma)}\right)} \quad .$$

We denote by $U \dashv F : \mathcal{E} \to \mathcal{E}^{\square}$ the forgetful-cofree adjunction of models.

Lifting Grothendieck Universes to Coalgebras (cont'd)

Theorem

Assume that both (the underlying category of) $\mathcal{E}$ and (the underlying functor of) $\Box : \mathcal{E} \to \mathcal{E}$ are Cartesian.

Then, if $\mathcal{E}$ admits a type classifier, so does $\mathcal{E}^{\Box}$.

Lifting Grothendieck Universes to Coalgebras (cont'd)

Proof.

We have isomorphisms

$$\begin{aligned} Cat(\mathcal{E}^{\Box})(\Delta, F(\mho_{\mathcal{E}})) &\cong Cat(\mathcal{E})(U\Delta, \mho_{\mathcal{E}}) \\ &\cong \mathrm{Ty}_{\mathcal{E}}(U\Delta) \quad , \end{aligned}$$

natural in $\Delta \in \mathcal{E}^{\Box}$. The comonad

$$\mathbb{B}_{\Delta} : \mathrm{Ty}_{\mathcal{E}}(U\Delta) \to \mathrm{Ty}_{\mathcal{E}}(U\Delta)$$

thus transports to one

$$Cat(\mathcal{E}^{\Box})(\Delta, F(\mho_{\mathcal{E}})) \to Cat(\mathcal{E}^{\Box})(\Delta, F(\mho_{\mathcal{E}})) \quad ,$$

again natural in $\Delta \in \mathcal{E}^{\Box}$.

Lifting Grothendieck Universes to Coalgebras (cont'd)

Proof (cont'd).

This internalizes to a comonad

$$\beta : F(\mho_{\mathcal{E}}) \to F(\mho_{\mathcal{E}}) \quad .$$

We set

$$\mho_{\mathcal{E}^\Box} :\equiv F(\mho_{\mathcal{E}})^{\beta} \quad .$$

Then, for any $\Delta \in \mathcal{E}^{\Box}$, we have

$$\begin{aligned} Cat(\mathcal{E}^{\Box})(\Delta, \mho_{\mathcal{E}^\Box}) &\equiv Cat(\mathcal{E}^{\Box})(\Delta, F(\mho_{\mathcal{E}})^{\beta}) \\ &\cong Cat(\mathcal{E}^{\Box})(\Delta, F(\mho_{\mathcal{E}}))^{Cat(\mathcal{E}^{\Box})(\Delta,\beta)} \\ &\cong (Ty_{\mathcal{E}}(U\Delta))^{(\mathbb{B}_{\Delta})} \\ &\equiv \mathrm{Ty}_{\mathcal{E}^\Box}(\Delta) \quad . \end{aligned}$$

□

Presheaves (Revisited)

The inclusion functor

$$i : |C| \hookrightarrow C$$

induces an adjunction of models

$$i^* \dashv i_* : Set^{|C|} \to Set^{C^{op}}$$

such that i^* is strictly comonadic and Cartesian. We thus (essentially) recover the Hofmann-Streicher universe via our theorem.

Explicitly, for each $I \in C$, we have a strictly comonadic adjunction

$$(i_* \mho_{Set^{|C|}})(I) \cong \mathcal{U}^{|C/I|} \leftrightarrows \mathcal{U}^{(C/I)^{op}} = \mho_{Set^{C^{op}}}(I) \quad .$$

Sheaves on a Topological Space

Let X be a $\mathcal{U}$-small topological space. The continuous inclusion function

$$i : |X| \hookrightarrow X$$

induces an adjunction of models

$$i^* \dashv i_* : \mathrm{Sh}(|X|) \to \mathrm{Sh}(X)$$

such that i^* is comonadic and Cartesian. Since $Set^{|X|} \simeq \mathrm{Sh}(|X|)$, we recover $\mathrm{Sh}(X)$ up to equivalence as the model of coalgebras for a Cartesian comonad on $Set^{|X|}$.

This model of coalgebras has a type classifier, 'lifted' from that of $Set^{|X|}$ by our theorem.

Sheaves Models with Enough Points

Definition

We will say that $\mathrm{Sh}(C, J)$ *has* **enough points** *if there exists a* $\mathcal{U}$*-small 'set of points'* A *and a Cartesian, comonadic morphism of models* $\mathrm{Sh}(C, J) \to Set^A$.

Obviously, when $\mathrm{Sh}(C, J)$ has enough points, we recover it up to equivalence as the model of coalgebras for a Cartesian comonad on some Set^A.

This model of coalgebras has a type classifier, 'lifted' from that of Set^A by our theorem.

Thanks for your attention!

References

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