Synthetic fibered $(\infty, 1)$-category theory

Jonathan Weinberger[1]

University of Birmingham

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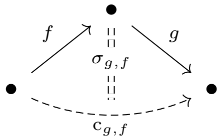
[1] My webpage: https://sites.google.com/view/jonathanweinberger

The concept of $(\infty, 1)$-category

- $(\infty, 1)$**-categories:** "categories weakly enriched in spaces"
- weak composition of 1-morphisms: uniquely up to contractibility



$$\mathrm{Comp}(g, f) = \{ \text{ composition data } \langle \mathrm{c}_{g,f}, \sigma_{g,f} \rangle \} \overset{!}{\simeq} \mathbf{1}$$

- Introduced by Boardman–Vogt as *quasi-categories* in 1973, later considerably developed by Joyal and Lurie
- Relevant in derived/spectral algebraic geometry, stable homotopy theory, higher algebra, topological field theories, . . .

Synthetic $(\infty, 1)$-categories in HoTT?

- In HoTT, the types are understood as homotopy types *aka* spaces *aka* ∞-groupoids $A \in \mathcal{S}$
- But $(\infty, 1)$-categories are more general.
- We have path types $(a =_A b)$, but what about directed hom types $(a \to_A b)$?
- Approach due to Riehl–Shulman and Joyal: Extend HoTT to reason about *simplicial* homotopy types *aka* simplicial spaces $X \in [\Delta^{\mathrm{op}}, \mathcal{S}]$.
- From those we can **internally** single out the $(\infty, 1)$-categories and ∞-groupoids, resp.
- By [Shu19], we can replace $\mathcal{S}$ by an arbitrary Grothendieck–Rezk–Lurie $(\infty, 1)$-topos $\mathcal{E}$. $\rightsquigarrow$ **synthetic internal $(\infty, 1)$-category theory**
- Our setting: Fibered $(\infty, 1)$-category theory in Riehl–Shulman's **simplicial HoTT**, oriented along Riehl–Verity's ∞**-cosmos theory**.

Previous and related work

- **On directed type theory and directed univalence:** Harper–Licata, Warren, Nuyts, Riehl–Shulman, Cavallo–Riehl–Sattler, Weaver–Licata, Buchholtz–W, Kudasov, Annenkov–Capriotti–Kraus–Sattler, Finster–Rice–Vicary, Cisinski–Nguyen, North, Altenkirch–Sestini . . .
- **On fibrations of $(\infty, 1)$-categories:** Joyal, Lurie, Ayala–Francis, Barwick–Dotto–Glasman–Nardin–Shah, Rasekh, Riehl–Verity . . .
- **On Segal spaces and Segal objects/internal $(\infty, 1)$-categories:** Rezk, Joyal–Tierney, Lurie, Kazhdan–Varshavsky, Boavido de Brito, Rasekh, Martini–Wolf . . .
- **Proof assistant for sHoTT:** Check out `rzk` developed by Kudasov—prototype interactive proof assistant with online live mode at: `https://github.com/fizruk/rzk`

HoTTEST talks

- Emily Riehl: *The synthetic theory of ∞-categories vs the synthetic theory of ∞-categories*, Mar 1, 2018
`https://www.youtube.com/watch?v=ge-9m1SsEmc`
- Denis-Charles Cisinski: *Univalence of the universal coCartesian fibration*, Apr 2, 2020
`https://www.youtube.com/watch?v=OnMUka9bLAw`
- Matthew Weaver: *A constructive model of directed univalence in bicubical sets*, Apr 16, 2020
`https://www.youtube.com/watch?v=kkfNjqSx4Nw`
- Ulrik Buchholtz: *(Co)cartesian families in simplicial type theory*, Apr 22, 2021
`https://www.youtube.com/watch?v=TOGx2F-MLi0`

sHoTT: Cubes, shapes, and topes

simplicial HoTT [RS17]: Multi-part contexts $\Xi \mid \Phi \mid \Gamma \vdash A$ with pre-type layers[2]

1. **Abstract cubes (*cube layer*):** Lawvere theory generated by directed interval $\mathbb{2}$

$$\frac{}{\mathbf{1}, \mathbb{2}\,\mathrm{cube}} \qquad \frac{}{\Xi \vdash \star : \mathbf{1}} \qquad \frac{}{\Xi \vdash 0, 1 : \mathbb{2}} \qquad \frac{I\,\mathrm{cube} \qquad J\,\mathrm{cube}}{I \times J\,\mathrm{cube}} \qquad \frac{(t : I) \in \Xi}{\Xi \vdash t : I} \qquad [\ldots]$$

2. **Subpolytopes (*tope layer*):** Intuitionistic theory of formulas φ in cube contexts Ξ

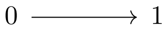
$$\frac{\varphi \in \Phi}{\Xi \mid \Phi \vdash \varphi} \qquad \frac{}{\Xi \vdash \bot, \top\,\mathrm{tope}} \qquad \frac{\Xi \vdash s : I \qquad \Xi \vdash t : I}{\Xi \vdash (s \equiv t)\,\mathrm{tope}} \qquad \frac{\Xi \vdash \varphi\,\mathrm{tope} \qquad \Xi \vdash \psi\,\mathrm{tope}}{\Xi \vdash (\varphi \wedge \psi), (\varphi \vee \psi)\,\mathrm{tope}}$$

$$\frac{}{x, y : \mathbb{2} \vdash (x \leq y)\,\mathrm{tope}} \qquad [\ldots]$$

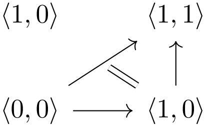
[2] cf. Cubical Type Theory

sHoTT: Examples of shapes

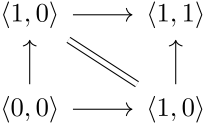
Δ^1



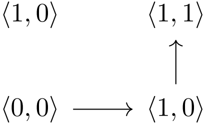
Δ^2



$\Delta^1 \times \Delta^1$



Λ^2_1



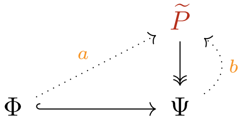
$$\Delta^1 :\equiv \{t : \mathbb{2} \mid \top\}, \quad \Delta^2 :\equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \leq t\},$$
$$\Delta^1 \times \Delta^1 \equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid \top\}, \quad \Lambda^2_1 :\equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid (s \equiv 0) \vee (t \equiv 1)\}$$

sHoTT: Extension types

Idea: "Π-types with strict side conditions". Originally due to Lumsdaine–Shulman.[3]

Input:

- shape inclusion $\Phi \hookrightarrow \Psi$
- family $P : \Psi \to \mathcal{U}$
- partial section $a : \Pi_{t:\Phi} P(t)$

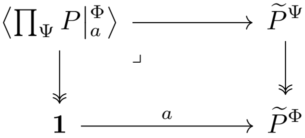


$\rightsquigarrow$

Extension type $\left\langle \prod_\Psi P\big|^{\Phi}_{a} \right\rangle$

with terms $b : \Pi_\Psi P$ such that $b|_\Phi \equiv a$.

Semantically:



[3] cf. also path types in Cubical Type Theory

Hom types I

Definition (Hom types, [RS17])

Let B be a type. Fix terms $a, b : B$. The type of *arrows in* B *from* a *to* b is the extension type

$$\hom_B(a,b) :\equiv (a \to_B b) :\equiv \left\langle \Delta^1 \to B \Big|^{\partial\Delta^1}_{[a,b]} \right\rangle.$$

Definition (Dependent hom types, [RS17])

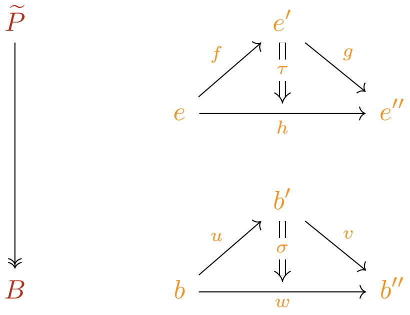
Let $P : B \to \mathcal{U}$ be family. Fix an arrow $u : \hom_B(a,b)$ in B and points $d : P\,a$, $e : P\,b$ in the fibers. The type of *dependent arrows in* P *over* u *from* d *to* e is the extension type

$$\mathrm{dhom}_{P,u}(d,e) :\equiv (d \to^P_u e) :\equiv \left\langle \prod_{t:\Delta^1} P(u(t)) \Big|^{\partial\Delta^1}_{[d,e]} \right\rangle.$$

Hom types II

We will also be considering types of 2-cells: For arrows u, v, w in B with f, g, h in P lying above, with appropriate co-/domains, let

$$\hom_B^2(u, v; w) :\equiv \left\langle \Delta^2 \to B \Big|^{\partial\Delta^2}_{[u,v,w]} \right\rangle, \quad \mathrm{dhom}_\sigma^{2,P}(f, g; h) :\equiv \left\langle \prod_{\langle t,s\rangle : \Delta^2} P(\sigma(t,s)) \Big|^{\partial\Delta^2}_{[f,g,h]} \right\rangle.$$



Segal, Rezk, and discrete(=groupoidal) types

Can now define synthetic ∞-categories[4] using shapes and extension types:

Definition (Synthetic ∞-categories, [RS17])

- **Synthetic pre-∞-category *aka* Segal type:** types A with *weak composition*, *i.e.*:

$$\iota : \Lambda_1^2 \hookrightarrow \Delta^2 \rightsquigarrow A^\iota : A^{\Delta^2} \xrightarrow{\simeq} A^{\Lambda_1^2} \qquad \text{(Joyal).}$$

- **Synthetic ∞-category *aka* Rezk type:** Segal types A satisfying *Rezk completeness/univalence*, *i.e.*

$$\mathrm{idtoiso}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \mathrm{iso}_A(x,y).$$

- **Synthetic ∞-groupoid *aka* discrete type:** types A such that *every arrow is invertible*, *i.e.*

$$\mathrm{idtoarr}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \hom_A(x,y).$$

[4] *Henceforth:* short for $(\infty, 1)$-categories

Cocartesian families: Motivation

- Any type family $P : B \to \mathcal{U}$ **transforms covariantly** in paths:

$$u : a =_B b \quad \rightsquigarrow \quad u_! : P\,a \to P\,b$$

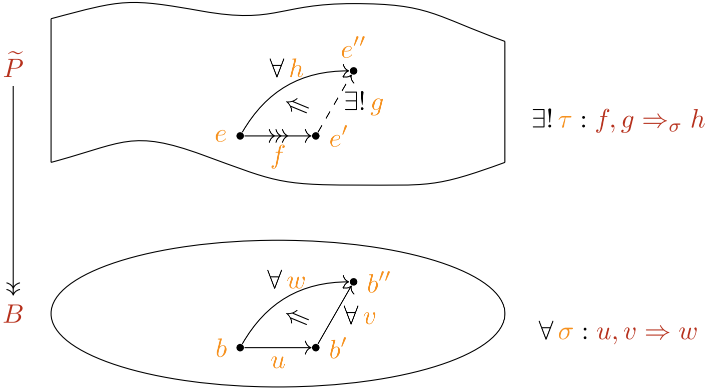
- What about the **directed** analogue? We'd like:

$$u : a \to_B b \quad \rightsquigarrow \quad u_! : P\,a \to P\,b$$

- Riehl–Shulman [RS17]: **groupoidal** case, where the fibers of P are discrete (**covariant families**). Discrete two-sided case.
- Buchholtz–W [BW21], W [W22]: generalization to **categorical** case, where the fibers of P are Rezk (**cocartesian families**).
- W [W22]: further extensions to include left exact, bivariant, fibered, and two-sided (*i.e.* mixed-variance) families.
- These are central notions of **fibrations** of synthetic $(\infty, 1)$-categories. They have important applications, and enjoy good properties such as **directed arrow induction** *aka* **type-theoretic Yoneda Lemmas** (originally due to [RS17], also in [RV22]).

Cocartesian arrows: Definition ([BW21], Ch. 3 in thesis [W22]) I

Intuitively: An arrow $f : e \to_u^P e'$ over $u : b \to_B b'$ is *cocartesian* if it satisfies the following universal property:



Thanks to Ulrik Buchholtz for the TikZ figures

Cocartesian arrows: Definition ([BW21], Ch. 3 in thesis [W22]) II

Definition (Cocartesian arrows (Buchholtz-W))

Let B be a type and $P : B \to \mathcal{U}$ be an inner family. Let $b, b' : B$, $u : \hom_B(b, b')$, and $e : P\,b$, $e' : P\,b'$. An arrow $f : \hom_{P\,u}(e, e')$ is a *(P-)cocartesian morphism* or *(P-)cocartesian arrow* iff

$$\mathrm{isCocartArr}_P\, f :\equiv \prod_{\sigma:\left\langle \Delta^2 \to B \middle|^{\Delta^1_0}_{u} \right\rangle} \prod_{h:\prod_{t:\Delta^1} P\,\sigma(t,t)} \mathrm{isContr}\left(\left\langle \prod_{\langle t,s\rangle:\Delta^2} P\sigma(t,s) \middle|^{\Lambda^2_0}_{[f,h]} \right\rangle \right).$$

Notice that being a cocartesian arrow is a proposition. Over a Segal base, this amounts to:

$$\mathrm{isCocartArr}_P\, f \simeq \prod_{b'':B} \prod_{v:\hom_B(b',b'')} \prod_{w:\hom_B(b,b'')} \prod_{\sigma:\hom^2_B(u,v;w)} \prod_{e'':P\,b''} \prod_{h:\mathrm{dhom}_{P\,w}(e,e'')} \mathrm{isContr}\left(\sum_{g:\mathrm{dhom}_{P\,v}(e',e'')} \mathrm{dhom}^2_{P\,\sigma}(f,g;h)\right)$$

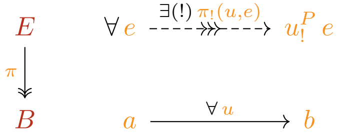
Cocartesian families: Definition ([BW21], Ch. 3 in thesis [W22])

Definition (Cocartesian family (Buchholtz–W))

Let B be a Rezk type and $P : B \to \mathcal{U}$ be a family such that $\widetilde{P}$ is a Rezk type. Then P is a *cocartesian family* if:

$$\mathrm{hasCocartLifts}\, P :\equiv \prod_{b,b':B} \prod_{u:b\to b'} \prod_{e:P\,b} \sum_{e':P\,b'} \sum_{f:e\to_u e'} \mathrm{isCocartArr}_P\, f$$

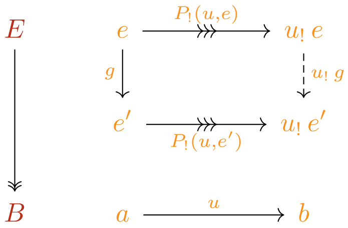
A map $\pi : E \twoheadrightarrow B$ is a *cocartesian fibration* iff $P :\equiv \mathrm{St}_B(\pi)$ is a cocartesian family.



$$\rightsquigarrow (-)_!^P : \prod_{a,b:B} (a \to_B b) \to P(a) \to P(b)$$

Cocartesian families: Functoriality ([BW21], Ch. 3 in thesis [W22])

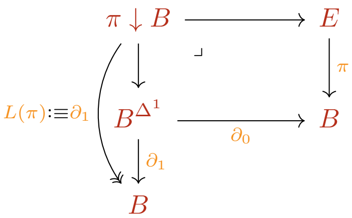
- Hence, any $u : a \to_B b$ induces a functor $u_! : P\,a \to P\,b$ acting on arrows as follows:



- Externally, this corresponds to a Cat-valued ∞-functor $B \to \mathrm{Cat}$, where Cat is the $(\infty,1)$-category of small $(\infty,1)$-categories.

Cocartesian families: Examples

1. For $g : C \to A \leftarrow B : f$, the comma projection $\partial_C : f \downarrow g \twoheadrightarrow C$.[5] (Hence, in particular the codomain projections $\partial_1 : A^{\Delta^1} \twoheadrightarrow A$.)
2. The *domain projection* $\partial_0 : A^{\Delta^1} \twoheadrightarrow A$, provided A has all pushouts.
3. For any map $\pi : E \to B$ between Rezk types, the *free cocartesian fibration*:



In particular, the desired UMP holds: $- \circ \iota : \mathrm{CocartFun}_B(L(\pi), \xi) \xrightarrow{\simeq} \mathrm{Fun}_B(\pi, \xi)$ for any cocartesian fibration $\xi : F \to B$.

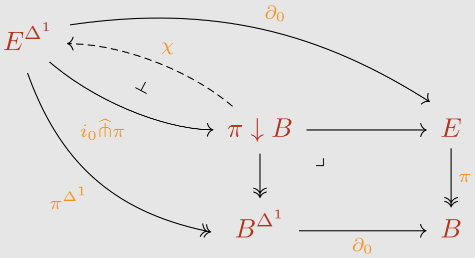
[5] $f \downarrow g \simeq \Sigma_{b:B, c:C}(f\, b \to_A g\, c)$

Cocartesian families: Characterization

Theorem (Chevalley criterion: Cocartesian families via lifting (W, cf. [RV22]))

Let B be a Rezk type. A given isoinner family $P : B \to \mathcal{U}$ is cocartesian if and only if the Leibniz cotensor map $i_0 \widehat{\pitchfork} \pi : E^{\Delta^1} \to \pi \downarrow B$ has a left adjoint right inverse:

$i_0 : \mathbf{1} \hookrightarrow \Delta^1 \rightsquigarrow$



The idea is that $\chi : \pi \downarrow B \to E^{\Delta^1}$ is the **lifting map** $\chi(u,e) = P_!(u,e)$. Chevalley criterion implies a lot of closure properties (cf. ∞-cosmoses)!

Yoneda Lemma for cocartesian families

Theorem (Dependent and absolute Yoneda Lemma (Buchholtz–W cf. [RV22]))

1. ***Dependent Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $Q : b \downarrow B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b is an equivalence:*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} Q \xrightarrow{\simeq} Q(\mathrm{id}_b)$$

2. ***Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $P : B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b as in*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} \partial_1^* P \xrightarrow{\simeq} P\, b$$

is an equivalence, where $\partial_1 : b \downarrow B \to B$.

For more, cf. Ulrik's HoTTEST 2021 talk

Cartesian and bicartesian families

- By (manual) dualization: obtain a theory of *cartesian* families $P : B \to \mathcal{U}$, with contravariant transport $u^* : P\,b \to P\,a$ and RARI condition.
- Combining both variances leads to *bicartesian* families, where $u_! \dashv u^* : P\,b \to P\,a$.
- *Examples:* Family fibration and Artin gluing, both if base has all pullbacks.
- *Application:* Fibered view of geometric morphisms (for 1-toposes: Bénabou, Moens, Jibladze, Streicher, Lietz, Frey . . .)
- Correspondence

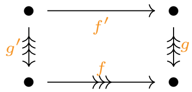
$$\{\text{g.m. } f : \mathcal{F} \to \mathcal{E}\} \simeq \{\text{topos fib. } p : \mathcal{X} \twoheadrightarrow \mathcal{E} \text{ with } \mathcal{F} \simeq p^{-1}(1_{\mathcal{E}})\}$$

via Artin gluing:

$$\mathcal{F} \underset{f^*\ \text{lex}}{\overset{f_*}{\underset{\top}{\rightleftarrows}}} \mathcal{E} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathcal{F} \downarrow f^* & \longrightarrow & \mathcal{F}^{\to} \\ {\scriptstyle \mathrm{gl}(f^*)}\downarrow & \lrcorner & \downarrow{\scriptstyle \partial_1} \\ \mathcal{E} & \xrightarrow{\quad f^* \quad} & \mathcal{F} \end{array}$$

Moens fibrations

- Q: More generally, which Grothendieck fibrations are of the form $\mathrm{gl}(F)$ f.s. lex functor F?
- A: *Lextensive* or *Moens fibrations*: lex bifibrations with stable and disjoint internal sums (*Moens' Thm.*)
- Recall: A bicomplete category $\mathcal{C}$ is *lextensive* if $\prod_{i \in I} \mathcal{C}/a_i \simeq \mathcal{C}/\coprod_{i \in I} a_i$ for all (small) families $(a_i)_{i \in I}$ of objects in $\mathcal{C}$.
- Fibrational version: a lex cart. fib. $\pi : E \twoheadrightarrow B$ is *lextensive* or a *Moens fibration* if:
  1. The fibration π is a bifibration and satisfies the *Beck–Chevalley condition*[6] (*internal sums*): A dependent square over a pb, as follows, is itself a pb iff f' is cocartesian:



  2. Cocartesian arrows in π are stable under pullback (*stability* of internal sums).
  3. Fiberwise diagonals of cocartesian arrows in π are cocartesian (*disjointness* of internal sums).

[6] Plays a role *e.g.* in: M. Hopkins, J. Lurie *Ambidexterity in $K(n)$-Local Stable Homotopy Theory*

Moens' Theorem for synthetic $(\infty, 1)$-categories I

We can adapt the proof from [Str21]:

Theorem (Moens' Theorem in simplicial HoTT (W, cf. [Str21]))

For a small lex Rezk type $B : \mathcal{U}$ the type

$$\mathrm{MoensFam}(B) :\equiv \sum_{P:B\to\mathcal{U}} \mathrm{isMoensFam}\, P$$

of $\mathcal{U}$-small Moens families is equivalent to the type

$$B \downarrow^{\mathrm{lex}} \mathrm{LexRezk} :\equiv \sum_{C:\mathrm{LexRezk}} (B \to^{\mathrm{lex}} C)$$

of lex functors from B into the type $\mathrm{LexRezk}$ of $\mathcal{U}$-small lex Rezk types.

Moens' Theorem for synthetic $(\infty, 1)$-categories II

Idea: Quasi-inverses given by "terminal transport" and gluing

$$\mathrm{MoensFam}(B) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} B \downarrow^{\mathrm{lex}} \mathrm{LexRezk} \qquad \simeq$$

i.e.

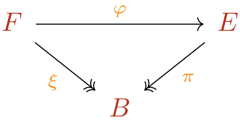
$$\Phi(P : B \to \mathcal{U}) \equiv \langle P\,z, \lambda b.(!_b)_!(\zeta_b)\rangle : B \to P\,z, \quad \Psi(F : B \to C) :\equiv \mathrm{St}_B\left(\mathrm{gl}(F) : C \downarrow F \twoheadrightarrow B\right) :$$

with $z : B$ terminal, and $\zeta : \Pi_B P$ picking the terminal element in each fiber:

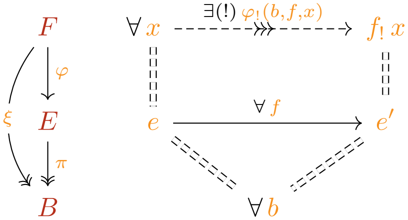
$$\begin{array}{ccc} E & & \zeta_b \longrightarrow\!\!\!\!\!\ggg\!\!\!\longrightarrow \Phi_P(b) \\ {\scriptstyle \pi_P}\downarrow & & \\ B & & b \overset{!_b}{\dashrightarrow} z \end{array}$$

Sliced cocartesian families (Ch. 5 in thesis [W22])

For $\xi : F \twoheadrightarrow B$, $\pi : E \twoheadrightarrow B$, a fibered functor



is a *sliced cocartesian family* over B if:



Externally, corresponds to cocartesian fibrations internal to Cat/B ("fibered fibration").

Two-sided cartesian families (Ch. 5 in thesis [W22])

A span

$$A \xleftarrow{\xi} E \xrightarrow{\pi} B \qquad \leftrightsquigarrow \qquad E \xrightarrow{\langle \xi, \pi \rangle} A \times B$$

is a *two-sided cartesian fibration* if

$$\begin{array}{ccccccc} E & \quad & v^* e & \overset{\exists(!)\, \pi^*(v,e)}{\dashleftarrow} & \forall\, e & \overset{\exists(!)\, \xi_!(u,e)}{\dashrightarrow} & u_!\, e \\ \downarrow{\scriptstyle \langle \xi, \pi \rangle} & & & & & & \\ A \times B & & a & = & a & \xrightarrow{\forall\, u} & a' \\ & & & & & & \\ & & b' & \xrightarrow{\forall\, v} & b & = & b \end{array}$$

and the lifts *commute*, *i.e.* canonically

$$u_!\, v^*\, e =_{P(a,b)} v^*\, u_!\, e.$$

Externally, corresponds to ∞-functors $B^{\mathrm{op}} \times A \to \mathrm{Cat}$ ("$(\infty, 1)$-categorical distributors", a kind of higher relation).

Properties of two-sided cartesian families

- **∞-Cosmological closure properties:** By considering two-sided cart. families $P : A \to B \to \mathcal{U}$ as certain "fibered" fibrations:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & A \times B \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

- **(Dependent) Yoneda Lemma for two-sided families:** Let $Q : a \downarrow A \times B \downarrow b \to \mathcal{U}$ be a two-sided family. For $a : A$, $b : B$, evaluation is an equiv.:

$$\mathrm{ev}_{\mathrm{id}_{\langle a,b \rangle}} : \left(\prod_{a \downarrow A \times B \downarrow b}^{\mathrm{2sCart}} Q \right) \xrightarrow{\simeq} Q(\mathrm{id}_a, \mathrm{id}_b)$$

Some WIP

1. Rezk universes and flat *aka* $(\infty, 1)$-Conduché fibrations (needs cohesion)
2. Opposites and twisted arrow types ((multi-)modal framework à la Licata–Riley–Shulman/Gratzer–Kavvos–Nuyts–Birkedal)
3. Higher algebra

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Thank you for your attention!