A Type Theory for Strictly Unital ∞-Categories

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arXiv:2007.08307
https://github.com/ericfinster/catt.io

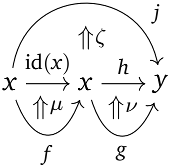
Homotopy Type Theory Electronic Seminar Talks
3 December 2020

Paths modulo units

Consider these cells in some higher structure, like a 2-groupoid:

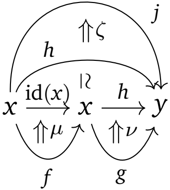
$x : \star$	$f : x \to x$	$h : x \to y$	$\mu : f \to \mathrm{id}(x)$	$\zeta : h \to j$
$y : \star$	$g : x \to y$	$j : x \to y$	$\nu : g \to h$	

We might try to compose them as follows:

(I) 

$(\mu \bullet_0 \nu) \bullet_1 \zeta$

Simple, but invalid

(II) 

$(\mu \bullet_0 \nu) \bullet_1 \mathsf{coh} \bullet_1 \zeta$

Verbose, but correct

But wait—aren't these units somehow trivial? Ideally, we would like to:

- have the type checker accept (I) ✓
- on request, "inflate" (I) to (II), inserting missing coherences WIP!
- on request, "deflate" (II) to (I), removing trivial structure ✓

Plan for today

- Recall Catt, a type theory for weak ∞-categories (Finster & Mimram, arXiv:1706.02866)
- Give a reduction relation on terms which "removes unit structure", and show it's confluent and terminating
- Define the new type theory $\mathsf{Catt}_{\mathsf{su}}$, by using our reduction relation to generate definitional equality for Catt
- Models of $\mathsf{Catt}_{\mathsf{su}}$ are *strictly unital ∞-categories*, and we explore their properties
- Investigate nontrivial examples, including Eckmann-Hilton and the Syllepsis
- Speculate on possible future application of these ideas to Martin-Löf identity types

Catt **overview**

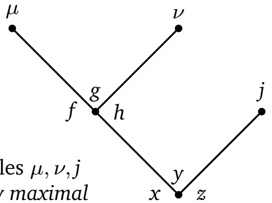
Contexts $\Gamma, \Delta, \ldots$ are lists of variables-with-types: $x : A, y : B, \ldots, z : C$	$\Gamma \vdash$	"Γ *is the generating data for a free* ∞*-category* $\widetilde{\Gamma}$"
Types $A, B, C, \ldots$ are trivial, or pairs of parallel terms: $\star \qquad u \to v$	$\Gamma \vdash A$	"*in* $\widetilde{\Gamma}$*, there is a hom-set* A"
Terms $t, u, v, \ldots$ are variables, coherences, or composites: $x \quad \mathsf{coh}(\Gamma : A)[\sigma] \quad \mathsf{comp}(\Gamma : A)[\sigma]$	$\Gamma \vdash t : A$	"*in* $\widetilde{\Gamma}$*, there is a morphism* t *in the hom-set* A"
Substitutions $\sigma : \Gamma \to \Delta$ are functions $\sigma : \mathsf{var}(\Gamma) \to \mathsf{tm}(\Delta)$	$\Delta \vdash \sigma : \Gamma$	"*there is a strict* ∞*-functor* $\sigma : \widetilde{\Gamma} \to \widetilde{\Delta}$"

No definitional equality—"Catt does not compute".

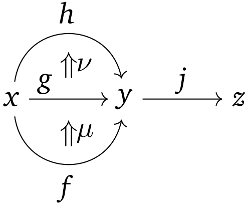
Catt **pasting contexts**

In Catt we can characterize the pasting contexts inductively.

We can illustrate this with *Batanin trees*.



Leaf variables μ, ν, j have *locally maximal* dimension



$$\Gamma = x : \star,\ y : \star,\ f : x \to y,\ g : x \to y,\ \mu : f \to g,\ h : x \to y,\ \nu : g \to h,\ z : \star,\ j : y \to z$$

We can also define the *boundaries* $\partial^{\pm}$ of a pasting context, in this case:

$$\partial^{+} = \{x : \star,\ y : \star,\ h : x \to y,\ z : \star,\ j : y \to z\} \qquad x \xrightarrow{h} y \xrightarrow{j} z$$

$$\partial^{-} = \{x : \star,\ y : \star,\ f : x \to y,\ z : \star,\ j : y \to z\} \qquad x \xrightarrow{f} y \xrightarrow{j} z$$

Catt **term construction**

"in a pasting context, parallel full terms can be filled"

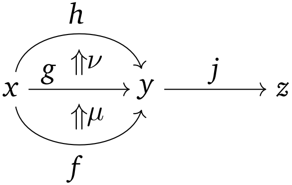
We can construct terms as follows, when Γ is a pasting context:

$$\frac{\partial^-(\Gamma) \vdash u : A \quad \partial^+(\Gamma) \vdash v : A}{\Gamma \vdash \mathsf{comp}(\Gamma, u, v) : u \to v}$$

Side condition: u, v are "full", using every variable of their contexts.

$\partial^+(\Gamma)$ $\qquad x \xrightarrow{h} y \xrightarrow{j} z$ $\qquad \delta^+(\Gamma) \vdash u : A$

Γ



$\boxed{\Gamma \vdash \mathsf{comp}(\Gamma, u, v) : u \to_A v}$

$\partial^-(\Gamma)$ $\qquad x \xrightarrow{f} y \xrightarrow{j} z$ $\qquad \delta^-(\Gamma) \vdash u : A$

This is a conceptually profound idea.

Catt **term construction**

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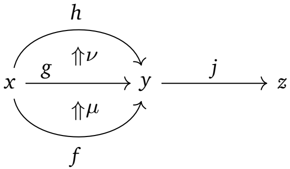
Side condition: u, v are "full", using every variable of their contexts.

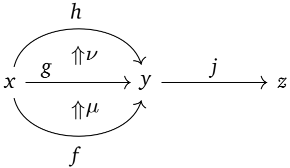
Γ



$\Gamma \vdash v : A$

$$\boxed{\Gamma \vdash \mathsf{coh}(\Gamma, u, v) : u \to v}$$

Γ



$\Gamma \vdash u : A$

Catt **term construction**

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Side condition: u, v are "full", using every variable of their contexts.

Here are some examples:

- $\mathsf{comp}(x \xrightarrow{f} y \xrightarrow{g} z, x, z) : x \to z$ gives the binary composite $f \bullet g$
- $\mathsf{comp}(x \xrightarrow{f} y, x, y) : x \to y$ gives the unary composite (f)
- $\mathsf{coh}(x, x, x) : x \to x$ gives the identity 1-cell $\mathrm{id}(x)$

To obtain richer terms, we can substitute:

- $\mathsf{comp}(x \xrightarrow{f} y \xrightarrow{g} z, x, z)[p, q]$ gives the binary composite $p \bullet q$
- $\mathsf{coh}(x \xrightarrow{f} y, \mathrm{id}(x) \bullet f, f)$ gives the unitor λ_f
- $\mathsf{coh}(x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w, (f \bullet g) \bullet h, f \bullet (g \bullet h))$ gives associator $\alpha_{f,g,h}$

Every Catt term is a *variable*, a *composite*, or a *coherence*:

$$x \qquad \mathsf{comp}(\Gamma, u, v)[\sigma] \qquad \mathsf{coh}(\Gamma, u, v)[\sigma]$$

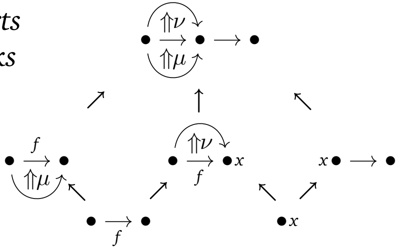
Globular sums

"pasting contexts are colimits of disks"

Definition. The category Catt^p has pasting contexts as objects, and substitutions as morphisms.

Theorem. *In* Catt^p*, pasting contexts are colimits of locally-maximal disks ("globular sums").*

Definition. An ∞*-category* is a presheaf $(\mathsf{Catt}^\mathsf{p})^{\mathsf{op}} \to \mathsf{Set}$ preserving globular sums.



Known to agree with the definition of *contractible* ∞*-category* (Grothendieck, Maltsiniotis, Batanin, Leinster, Brunerie) via recent work of Dmitri Ara, John Bourke and Thibaut Benjamin.

Lightweight approach:

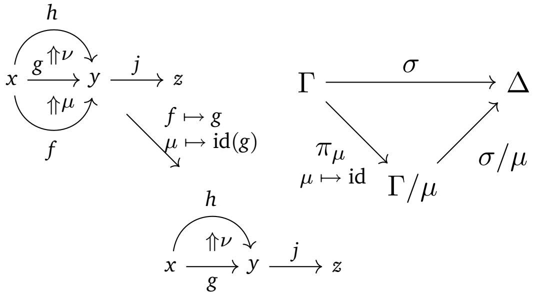
- no globular extension technology (Grothendieck/Maltsiniotis)
- no globular operad technology (Batanin/Leinster)

P Reduction

"prune identity arguments of a comp *or* coh*"*

Suppose $\mu \in \mathsf{var}(\Gamma)$ is locally maximal, with $\mu[\sigma]$ an identity.

Then σ factorizes via Γ/μ, with $\mu[\pi_\mu] = \mathrm{id}$:



The intuition is that μ has been collapsed, or "pruned".

We define the reduction as follows:

$$\mathsf{comp}(\Gamma, u, v)[\sigma] \rightsquigarrow_{\mathrm{P}} \mathsf{comp}(\Gamma/\mu, u[\pi_\mu], v[\pi_\mu])[\sigma/\mu]$$

$$\mathsf{coh}(\Gamma, u, v)[\sigma] \rightsquigarrow_{\mathrm{P}} \mathsf{coh}(\Gamma/\mu, u[\pi_\mu], v[\pi_\mu])[\sigma/\mu]$$

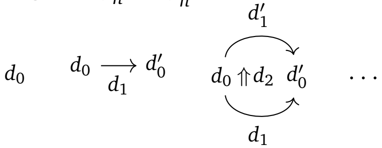
D Reduction

"simplify unary composites"

We define the *n-sphere type* S^n and the *n-disk context* D^n recursively:

$$\mathrm{D}^0 := \{d_0 : S^{-1}\}$$
$$\mathrm{D}^{n+1} := \{\mathrm{D}^n, d'_n : S^{n-1}, d_{n+1} : S^n\}$$

$$S^{-1} := \star$$
$$S^n := d_n \to d'_n$$



Then for any n-cell u with $n > 0$, we can build its *unary composite:*

$$\mathsf{comp}(D^n, d_{n-1}, d'_{n-1})[u] \rightsquigarrow_{\mathrm{D}} u$$

This reduces to u itself.

L Reduction

"eliminate loops"

Consider a term as follows:

$$\mathsf{coh}(\Gamma, u, u)[\sigma] : u[\sigma] \to u[\sigma]$$

This "coherence law" says "$u[\sigma] = u[\sigma]$".

But this is obvious, and has a canonical witness:

$$\mathrm{id}(u[\sigma]) : u[\sigma] \to u[\sigma]$$

So it seems reasonable to eliminate these terms:

$$\mathsf{coh}(\Gamma, u, u)[\sigma] \rightsquigarrow_{\mathrm{L}} \mathrm{id}(u[\sigma])$$

Examples

$$\mathsf{comp}(\Gamma, u, v)[\sigma] \rightsquigarrow_{\mathrm{P}} \mathsf{comp}(\Gamma/\mu, u[\pi_\mu], v[\pi_\mu])[\sigma/\mu]$$
$$\mathsf{comp}(D^n, d_{n-1}, d'_{n-1})[u] \rightsquigarrow_{\mathrm{D}} u$$
$$\mathsf{coh}(\Gamma, u, u) \rightsquigarrow_{\mathrm{L}} \mathrm{id}(u[\sigma])$$

To get normalizing reductions, we extend $\rightsquigarrow_{\mathrm{P}}$, $\rightsquigarrow_{\mathrm{D}}$ and $\rightsquigarrow_{\mathrm{L}}$ to subterms, and add a single additional rule: never reduce the head of an identity.

- *Identity composite.* $f \bullet \mathrm{id}(y) \equiv \mathsf{comp}(x \xrightarrow{f} y \xrightarrow{g} z, x, z)[f, \mathrm{id}(y)]$
  $\rightsquigarrow_{\mathrm{P}} \mathsf{comp}(x \xrightarrow{f} y, x, y)[f]$
  $\rightsquigarrow_{\mathrm{D}} f$ ✓

- *Left unitor.* $\mathsf{coh}(x \xrightarrow{f} y, \mathrm{id}(x) \bullet f, f)$
  $\rightsquigarrow_{\mathrm{P}} \mathsf{coh}(x \xrightarrow{f} y, (f), f)$
  $\rightsquigarrow_{\mathrm{D}} \mathsf{coh}(x \xrightarrow{f} y, f, f) \equiv \mathrm{id}(f)$ ✓

- *Associator with identity.*

$$\alpha_{f,\mathrm{id}(y),g} \equiv \mathsf{coh}(x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w, (f \bullet g) \bullet h, f \bullet (g \bullet h))[f, \mathrm{id}(y), g]$$
$$\rightsquigarrow_{\mathrm{P}} \mathsf{coh}(x \xrightarrow{f} y \xrightarrow{g} z, (f \bullet \mathrm{id}(y)) \bullet g, f \bullet (\mathrm{id}(y) \bullet g))[f, g]$$
$$\rightsquigarrow_{\mathrm{P}} \rightsquigarrow_{\mathrm{P}} \mathsf{coh}(x \xrightarrow{f} y \xrightarrow{g} z, (f) \bullet g, f \bullet (g))[f, g]$$
$$\rightsquigarrow_{\mathrm{D}} \rightsquigarrow_{\mathrm{D}} \mathsf{coh}(x \xrightarrow{f} y \xrightarrow{g} z, f \bullet g, f \bullet g)[f, g]$$
$$\rightsquigarrow_{\mathrm{L}} \mathrm{id}(f \bullet g) \quad ✓$$

Results

Theorem. Reduction is terminating and has unique normal forms.

Definition. $\mathsf{Catt}_{\mathsf{su}}$ is obtained by extending Catt with definitional equality, defining $t = t'$ just when t, t' have the same normal form.

Terms in $\mathsf{Catt}_{\mathsf{su}}$ "compute" to their strictly unital normal form.

There is an obvious full projection functor $\pi : \mathsf{Catt}^{\mathsf{p}} \to \mathsf{Catt}^{\mathsf{p}}_{\mathsf{su}}$.

Definition. A *strictly unital* ∞*-category* is an ∞-category $(\mathsf{Catt}^{\mathsf{p}})^{\mathsf{op}} \to \mathsf{Set}$, which factors through π.

Appears to identify *more* terms than the definition of Batanin, Cisinski and Weber (arXiv:1209.2776), which has analogues of $\rightsquigarrow_{\mathrm{P}}$ and $\rightsquigarrow_{\mathrm{D}}$, but not $\rightsquigarrow_{\mathrm{L}}$.

Conjecture (WIP). Every ∞-category is weakly equivalent to a strictly unital ∞-category.

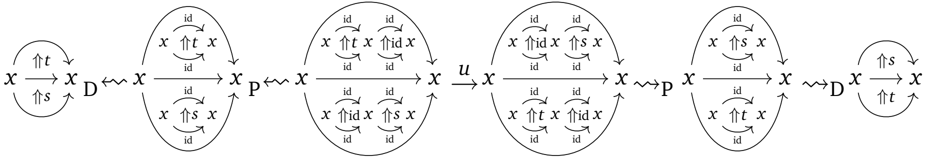
$\Gamma := \{x : \star,\ s : \mathrm{id}(x) \to \mathrm{id}(x),\ t : \mathrm{id}(x) \to \mathrm{id}(x)\}$

Eckmann-Hilton

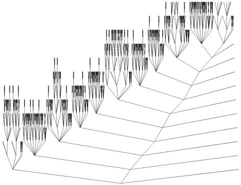
In Γ, the *Eckmann-Hilton 3-cell* has the following type:

$$\mathsf{EH}_{s,t} : s \bullet_1 t \to t \bullet_1 s$$

In $\mathsf{Catt}_{\mathsf{su}}$ we can construct it as an interchanger u:

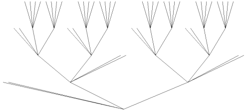


We can also formalize it in Catt, with the following syntax tree:



Catt syntax tree
1224 vertices

=



$\mathsf{Catt}_{\mathsf{su}}$ syntax tree
60 vertices

(20 times smaller)

$$\Delta := \{x : \star,\ s : \mathrm{id}(\mathrm{id}(x)) \to \mathrm{id}(\mathrm{id}(x)),\ t : \mathrm{id}(\mathrm{id}(x)) \to \mathrm{id}(\mathrm{id}(x))\}$$

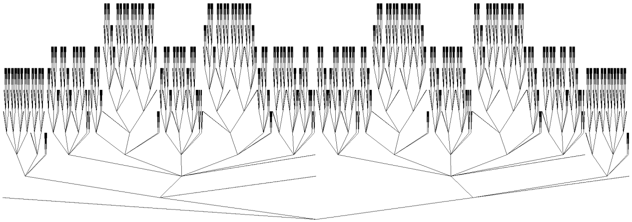
Syllepsis

In Δ, the *Syllepsis 5-cell* has the following type:

$$\mathsf{SY}_{s,t} : \mathsf{EH}_{s,t} \bullet_3 \mathsf{EH}_{t,s}^{-1} \to \mathrm{id}(s \bullet_2 t)$$

Geometrically, it says “the double braid is isotopic to the identity”.

We can construct it in $\mathsf{Catt}_{\mathsf{su}}$. Its syntax tree has 2,713 vertices:

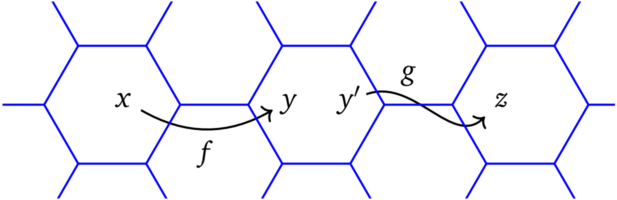


Cannot yet construct $\mathsf{SY}_{s,t}$ in Catt. (Would follow from WIP.)

Estimate Catt SY syntax tree size $\sim$ 100,000 vertices.

Outlook

Path types are *not* contractible . . .



. . . but they *can* be carved into contractible pieces.

Can we gain this advantage for Martin-Löf identity types, maybe via a more geometrical notion of composition?

Could this go some way to alleviate the burden of proof-relevance?

Could these ideas of semistrictness apply beyond path types?

Thanks for listening!