The Constructive Kan–Quillen Model Structure

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Theorem (Quillen)

The category of simplicial sets carries a proper cartesian model structure where

- *weak equivalences are the weak homotopy equivalences,*
- *fibrations are the Kan fibrations,*
- *cofibrations are the monomorphisms.* □

Can (a version of) this theorem be proven constructively?

A constructive version of the model structure would be useful in

- study of models of Homotopy Type Theory;
- understanding homotopy theory of simplicial sheaves.

The *category of simplices* Δ has

- totally ordered sets $[m] = \{0 < \ldots < m\}$ for $m \in \mathbb{N}$ as objects,
- order preserving maps between them as morphisms.

Morphisms of Δ are called *simplicial operators*.

- The injective ones are called *face operators*.
- An *elementary face operator* $\delta_i \colon [m-1] \to [m]$ omits $i \in [m]$.

A *simplicial set* is a presheaf over Δ, i.e., a functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$.

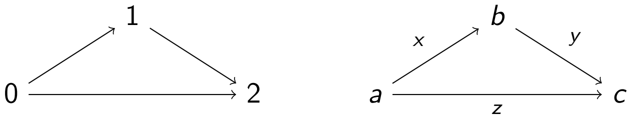
- Elements of X_m are called *m-dimensional simplices* of X.
- The action of simplicial operators describes how simplices of X are attached to each other.

- The elements of X_0 are called the *vertices* or *points* of X.
- The elements of X_1 are called the *edges* of X.
  If $x \in X_1$, $a = x\delta_1$ and $b = x\delta_0$ $(a, b \in X_0)$, then x is an edge from a to b.

$$0 \longrightarrow 1 \qquad\qquad a \xrightarrow{x} b$$

- We can enumerate faces of $u \in X_2$ as follows:

$$\begin{aligned} x &= u\delta_2 & a &= u\delta_2\delta_1 = u\delta_1\delta_1 \\ y &= u\delta_0 & b &= u\delta_2\delta_0 = u\delta_0\delta_1 \\ z &= u\delta_1 & c &= u\delta_0\delta_0 = u\delta_1\delta_0 \end{aligned}$$



The surjective simplicial operators are called *degeneracy operators*. The category Δ admits a presentation in terms of elementary face operators and elementary degeneracy operators.

Morphisms of simplicial sets are morphisms of presheaves, they are called *simplicial maps*.

Topology: simplicial sets are models of (triangulated) topological spaces.

Type theory: simplicial sets are interpretations of types:

- vertices correspond to elements of a type,
- edges correspond to elements of its identity type.

The *m-simplex* is the representable simplicial set $\Delta[m]$, i.e., $\Delta[m]_k = \{[k] \to [m]\}$.

The 1-simplex serves as an *interval object*.

A *homotopy* between simplicial maps $f, f' \colon X \to Y$ is

$$X \times \Delta[1] \longrightarrow Y \qquad\qquad X \longrightarrow Y^{\Delta[1]}$$

that restricts to f and f' over 0 and 1.

A map $f \colon X \to Y$ is a *homotopy equivalence* if there is $g \colon Y \to X$ gf is homotopic to id_X and fg is homotopic to id_Y (via zig-zags of homotopies).

The horn $\Lambda^1[2]$

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & \nearrow & & \searrow & \\
0 & & & & 2
\end{array}
$$

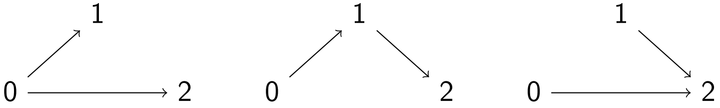
is homotopy equivalent to $\Delta[0]$, but not via a single homotopy.

The spine of $\mathbb{N}$

$$
0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots
$$

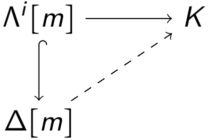
is **not** homotopy equivalent to $\Delta[0]$. (But it should be. . .)

The *horn* $\Lambda^i[m]$ is the simplicial subset of $\Delta[m]$ spanned by all faces of $\mathrm{id}_{[m]} \in \Delta[m]$ except for $\mathrm{id}_{[m]}$ itself and δ_i.



$\Lambda^0[2]$ $\Lambda^1[2]$ $\Lambda^2[2]$

A simplicial set K is a *Kan complex* if it has the right lifting property with respect to all horn inclusions.



A map $f\colon X \to Y$ is a *weak homotopy equivalence* if for all Kan complexes K, the induced map

$$K^Y \xrightarrow{f^*} K^X$$

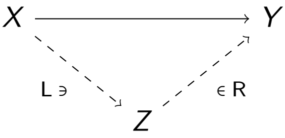
is a homotopy equivalence.

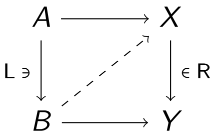
A map $X \to Y$ is a *Kan fibration* if it has the right lifting property with respect to all horn inclusions.

$$\begin{array}{ccc} \Lambda^i[m] & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta[m] & \longrightarrow & Y \end{array}$$

A *weak factorisation system* on a category $\mathcal{M}$ is a pair of classes of morphisms (L, R) such that

- every morphism factors as a morphism of L followed by a morphism of R,
- a morphism is in L if and only if it has the left lifting property with respect to all morphisms of R,
- a morphism is in R if and only if it has the left lifting property with respect to all morphisms of L.





A class of morphism W has the *2-out-of-3 property* when for all pairs of composable morphisms f and g if any two of f, g, gf are in W, then so is the third one.

A *model structure* on a category $\mathcal{M}$ is a triple of classes of morphisms: *weak equivalences*, *fibrations* and *cofibrations*

- (cofibrations, acyclic fibrations) is a weak factorisation system,
- (acyclic cofibrations, fibrations) is a weak factorisation system,
- weak equivalences satisfy 2-out-of-3.

[acyclic (co)fibration = (co)fibration and a weak equivalence]

Theorem (Quillen)

The category of simplicial sets carries a proper cartesian model structure where

- *weak equivalences are the weak homotopy equivalences,*
- *fibrations are the Kan fibrations,*
- *cofibrations are the monomorphisms.* □

In Constructive Zermelo–Fraenkel set theory (CZF) we have:

Theorem

The category of simplicial sets carries a proper cartesian model structure where

- *weak equivalences are the weak homotopy equivalences,*
- *fibrations are the Kan fibrations,*
- *cofibrations are the Reedy decidable inclusions.*

- S. Henry, *A constructive account of the Kan-Quillen model structure and of Kan's* Ex^∞ *functor* `arXiv:1905.06160`
- N. Gambino, C. Sattler, K. Szumiło, *The Constructive Kan–Quillen Model Structure: Two New Proofs* `arXiv:1907.05394`
- (for type theoretic applications) N. Gambino, S. Henry, *Towards a constructive simplicial model of Univalent Foundations* `arXiv:1905.06281`

Let $i: A \to B$ and $p: X \to Y$ be simplicial maps. $Prob(i, p)$ is the set of all squares of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y. \end{array}$$

We say that p has the *right lifting property* with respect to i if the map $\mathrm{sSet}(B, X) \to \mathrm{Prob}(i, p)$ has a section.

A *decidable inclusion* is a function $A \to X$ between sets such that $X \cong A \sqcup C$ for some set C. Logically: $\forall x \in X.\, x \in A \vee x \notin A$.

Decidable inclusions and split surjections form a weak factorisation system on Set.

A *Kan fibration* is a map with the right lifting property with respect to all horn inclusions $\Lambda^i[m] \to \Delta[m]$. A *trivial cofibration* is a map with the left lifting property with respect to all Kan fibrations.

The *boundary* $\partial\Delta[m]$ is the simplicial subset of $\Delta[m]$ spanned by all faces of $\mathrm{id}_{[m]} \in \Delta[m]$ except for $\mathrm{id}_{[m]}$.

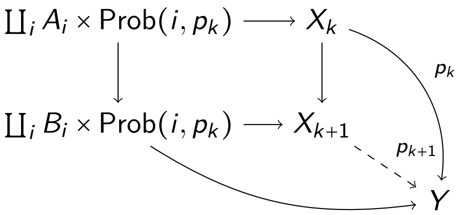
A *trivial fibration* is a map with the right lifting property with respect to all boundary inclusions $\partial\Delta[m] \to \Delta[m]$. A *cofibration* is a map with the left lifting property with respect to all trivial fibrations.

These form two weak factorisation systems by the constructive version of the small object argument.

Theorem

If I is a set of levelwise decidable inclusions between finite simplicial sets, then there is a weak factorisation system $(\mathsf{L}_I, \mathsf{R}_I)$ where a morphism is in R_I if and only if it has the right lifting property with respect to I.

Take $p_0 : X_0 \to Y$ and:



Set $X_\infty = \operatorname{colim} X_k$. Then $X_0 \to X_\infty \to Y$ is the required factorisation. ($X_\infty \to Y \in \mathsf{R}_I$ by levelwise decidability of I.)

A simplicial set A is cofibrant if and only if

- for all degeneracy operators $[m] \twoheadrightarrow [n]$, the map $A_n \to A_m$ is a decidable inclusion.

$u\sigma_0 \in A_2$ $\quad$ $u \in A_1$ $\quad$ $u\sigma_1 \in A_2$

- $L_m X \to X_m$ is a decidable inclusion for all m ($L_m X$ is the *latching object*: the set of all *degenerate* m-simplices of X.)

A map $A \to B$ is a cofibration if and only if

- for all degeneracy operators $[m] \twoheadrightarrow [n]$, the map $B_n \sqcup_{A_n} A_m \to B_m$ is a decidable inclusion.
- it is a *Reedy decidable inclusion*, i.e., for all m, the map $L_m B \sqcup_{L_m A} A_m \to B_m$ is a decidable inclusion.

Cofibrations and trivial cofibrations satisfy the pushout product property, i.e., if $A \to B$ and $C \to D$ are cofibrations, then so is

$$A \times D \sqcup_{A \times C} B \times C \longrightarrow B \times D$$

which is trivial if one of $A \to B$ or $C \to D$ is.

Proof: explicit combinatorics of horns and boundaries.

Some consequences:

- If A is cofibrant and $X \to Y$ is a (trivial) fibration, then so is $X^A \to Y^A$.
- If K is a Kan complex and $A \to B$ is a (trivial) cofibration, then $K^B \to K^A$ is a (trivial) fibration.
- If A is cofibrant and K is a Kan complex, then K^A is a Kan complex.

A *strong cofibrant replacement* of X is a cofibrant simplicial set $\widetilde{X}$ equipped with a trivial fibration $\widetilde{X} \to X$.

A map $f\colon X \to Y$ is a *weak homotopy equivalence* if

- (X and Y cofibrant Kan complexes) it is a homotopy equivalence;
- (X and Y Kan complexes) it has a strong cofibrant replacement that is a weak homotopy equivalence;
- (X and Y cofibrant) if $f^*\colon K^Y \to K^X$ is a weak homotopy equivalence for every Kan complex K;
- (X and Y arbitrary) it has a strong cofibrant replacement that is a weak homotopy equivalence.

Weak homotopy equivalences satisfy *2-out-of-6*, i.e., given

$$W \xrightarrow{\quad f \quad} X \xrightarrow{\quad g \quad} Y \xrightarrow{\quad h \quad} Z$$

if gf and hg are weak equivalences, then so are f, g, h and hgf.

Theorem

The category of Kan complexes is a fibration category, i.e.,

- *It has a terminal object and all objects are fibrant.*
- *Pullbacks along fibrations exist and (acyclic) fibrations are stable under pullback.*
- *Every morphism factors as a weak equivalence followed by a fibration.*
- *Weak equivalences satisfy the 2-out-of-6 property.*
- *It has products and (acyclic) fibrations are stable under products.*
- *It has limits of towers of fibrations and (acyclic) fibrations are stable under such limits.*

Lemma

Given $p\colon X \to Y$ and $q\colon Y \to Z$, if p and qp are trivial fibrations, then so is q. $\square$

Lemma

A Kan fibration $p\colon X \to Y$ between Kan complexes is acyclic if and only if it is trivial.

If X and Y are cofibrant: use the pushout product property to strictify a homotopy inverses of p to a *deformation section*, i.e., a map $s\colon Y \to X$ such that $ps = \mathrm{id}_Y$ and $sp \simeq \mathrm{id}_X$ over Y. Use that section to solve lifting problems against boundary inclusions.
For general X and Y, use the previous lemma:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\text{triv}} & X \\ \big\downarrow & & \big\downarrow \sim \\ \widetilde{Y} & \xrightarrow[\text{triv}]{} & Y \end{array}$$

Theorem

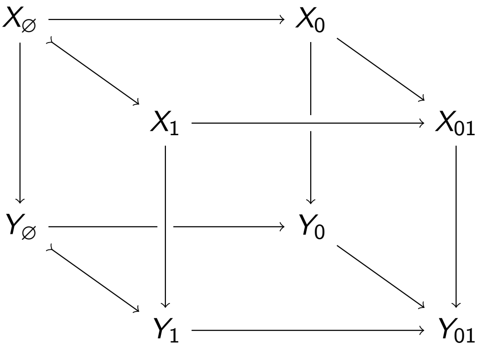
The category of cofibrant simplicial sets is a cofibration category, i.e.,

- *It has an initial object and all objects are cofibrant.*
- *Pushouts along cofibrations exist and (acyclic) cofibrations are stable under pushout.*
- *Every morphism factors as a cofibration followed by a weak equivalence.*
- *Weak equivalences satisfy the 2-out-of-6 property.*
- *It has coproducts and (acyclic) cofibrations are stable under coproducts.*
- *It has colimits of sequences of cofibrations and (acyclic) cofibrations are stable under such colimits.*

Proof: dualise by applying $K^{(-)}$ for all Kan complexes K.

Lemma (Gluing Lemma)

In a cofibration category



if top and bottom squares are pushouts along cofibrations and all $X_\varnothing \to Y_\varnothing$, $X_0 \to Y_0$ *and* $X_1 \to Y_1$ *are weak equivalences, then so is* $X_{01} \to Y_{01}$. □

A *bisimplicial set* is a presheaf over $\Delta \times \Delta$. It can be seen as a simplicial object in sSet in two ways.

A bisimplicial set X is *cofibrant* if it satisfies the equivalent conditions:

- it is Reedy cofibrant over Set, i.e., $L_{m,n}X \to X_{m,n}$ is a decidable inclusion for all m and n.
- it is Reedy cofibrant over sSet (in either direction), i.e., $L_m X \to X_m$ is a cofibration in sSet.

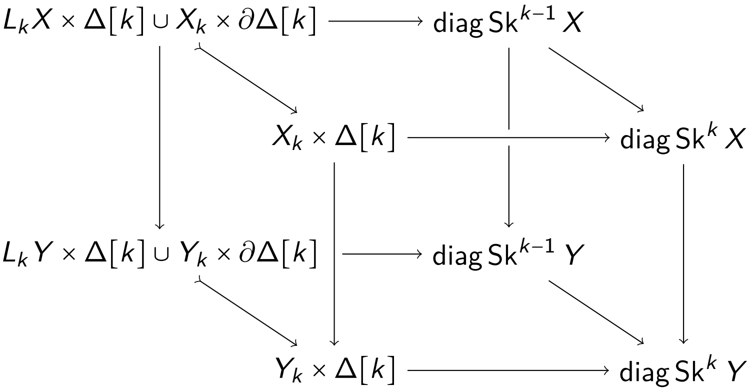
The *diagonal* of X is the simplicial set $[m] \mapsto X_{m,m}$.

The *k-skeleton* of X is the bisimplicial set

$$\mathrm{Sk}^k X = \mathrm{Lan}_{\Delta_{\le k} \to \Delta} X|\Delta_{\le k}.$$

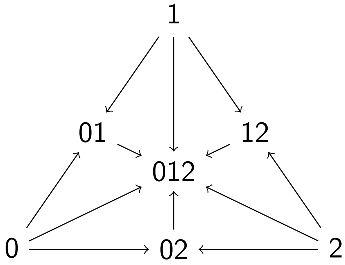
Proposition

If $X \to Y$ is a map between cofibrant bisimplicial sets such that $X_k \to Y_k$ is a weak homotopy equivalence for all k, then the induced map $\operatorname{diag} X \to \operatorname{diag} Y$ *is also a weak homotopy equivalence.*



Let $\mathrm{sd}[m]$ be the poset of non-empty subsets of $[m]$ ordered by inclusion. The *barycentric subdivision* of a simplicial set X is

$$\mathrm{Sd}\,X = \mathrm{colim}_{\Delta[m]\to X}\,\mathrm{N}\,\mathrm{sd}[m].$$



The order preserving map $\max\colon \mathrm{sd}[m] \to [m]$ induces a natural transformation $\mathrm{Sd}\,X \to X$.

$$\operatorname{Ex} X = \operatorname{sSet}(\operatorname{Sd}\Delta[-], X)$$
$$\operatorname{Ex}^\infty X = \operatorname{colim}(X \to \operatorname{Ex} X \to \operatorname{Ex}^2 X \to \dots)$$

Proposition

- Ex^∞ *preserves finite limits.*
- Ex^∞ *preserves Kan fibrations between cofibrant objects.*
- *If X is cofibrant, then $\operatorname{Ex}^\infty X$ is a Kan complex.*
- *If X is cofibrant, then $X \to \operatorname{Ex}^\infty X$ is a weak homotopy equivalence.*

The last statement is proven by argument of Latch–Thomason–Wilson.

$$
\begin{CD}
\mathrm{sSet}(\Delta[m]\times\Delta[0], X) @>>> \mathrm{sSet}(\Delta[m]\times\Delta[n], X)\\
@VVV @VVV\\
\mathrm{sSet}(\mathrm{Sd}\,\Delta[m]\times\Delta[0], X) @>>> \mathrm{sSet}(\mathrm{Sd}\,\Delta[m]\times\Delta[n], X)
\end{CD}
$$

$$
\begin{CD}
\bullet @>>> X^{\Delta[m]}\\
@VVV @VV{\simeq}V\\
\bullet @>>> X^{\mathrm{Sd}\,\Delta[m]}
\end{CD}
\qquad
\begin{CD}
X^{\Delta[0]} @>{\simeq}>> X^{\Delta[n]}\\
@VVV @VVV\\
\mathrm{Ex}(X^{\Delta[0]}) @>>{\simeq}> \mathrm{Ex}(X^{\Delta[n]})
\end{CD}
\qquad
\begin{CD}
X @>{\sim}>> \bullet\\
@VVV @VV{\sim}V\\
\mathrm{Ex}\,X @>>{\sim}> \bullet
\end{CD}
$$

Proposition

For a Kan fibration $p\colon X \to Y$, the following are equivalent:

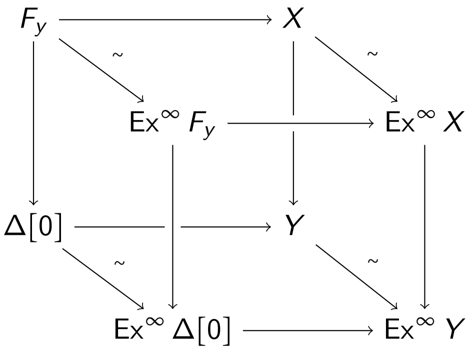
- *p is trivial,*
- *p is acyclic,*
- *p has contractible fibers.*

By the "cancellation lemma" we can assume that X and Y are cofibrant.

[trivial $\Rightarrow$ acyclic] Omitted (but straightforward).

[contractible fibers $\Rightarrow$ trivial] Take a lifting problem against $\partial\Delta[m] \to \Delta[m]$ and "contract" it to a fiber. Solve it in the fiber (which is a contractible Kan complex) and "uncontract" to the solution of the original problem.

[acyclic $\Rightarrow$ contractible fibers] (If X and Y are cofibrant) use Ex^∞:



Theorem (Quillen's Theorem A)

Let $f\colon I \to J$ be a functor between categories with decidable identities. If for every $y \in J$, $\mathrm{N}(f \downarrow y)$ is weakly contractible, then the map $\mathrm{N}\, I \to \mathrm{N}\, J$ is a weak homotopy equivalence.

One can construct a bisimplicial set Sf with maps

$$\mathrm{N}\, I \boxtimes \Delta[0] \longleftarrow Sf \longrightarrow \Delta[0] \boxtimes \mathrm{N}\, J$$

satisfying the Diagonal Lemma which yields

$$\begin{array}{ccccc} \mathrm{N}\, I & \xleftarrow{\sim} & \operatorname{diag} Sf & \xrightarrow{\sim} & \mathrm{N}\, J \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{N}\, J & \xleftarrow[\sim]{} & \operatorname{diag} S\,\mathrm{id}_J & \xrightarrow[\sim]{} & \mathrm{N}\, J. \end{array}$$

$$TX = \mathrm{N}(\Delta \downarrow X)$$

Proposition

- *T preserves colimits.*
- *T preserves cofibrations.*
- *TX is cofibrant for all X.*
- *$TX \to X$ is a weak homotopy equivalence for all X.*

Proof of the last statement:

- for $X = \Delta[m]$: $\Delta \downarrow X$ has a terminal object,
- for X cofibrant: by induction using Gluing Lemma etc.,
- for X arbitrary: T carries trivial fibrations to weak homotopy equivalences.

Lemma

If $p: X \to Y$ is a trivial fibration, then Tp is a weak homotopy equivalence.

Consider

$$\begin{array}{ccc} \Delta_\sharp \downarrow X & \xrightarrow{\text{Thm A}} & \Delta \downarrow X \\ \downarrow & & \downarrow \\ \Delta_\sharp \downarrow Y & \xrightarrow[\text{Thm A}]{} & \Delta \downarrow Y \end{array}$$

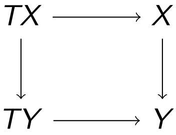
where $\Delta_\sharp$ is the category of face operators and construct a homotopy inverse to the left map by lifting all simplices of Y against p.

Proposition

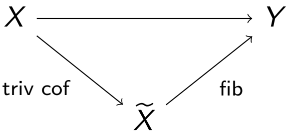
A cofibration $i\colon X \to Y$ is acyclic if and only if it is trivial.

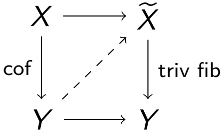
[trivial $\Rightarrow$ acyclic] If X and Y are cofibrant and K is a Kan complex, then $i^*\colon K^Y \to K^X$ is a trivial fibration.

For general X and Y, use T:



[acyclic $\Rightarrow$ trivial] Retract argument:

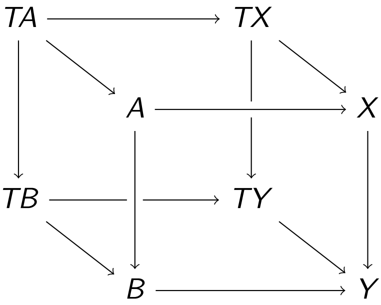




Proposition

The Kan–Quillen model structure is proper. (Weak equivalences are stable under pushouts along cofibrations and pullbacks along fibrations.)

[Left properness] Let $i\colon A \to B$ be a cofibration and $f\colon A \to X$ a weak homotopy equivalence.



[Right properness] Similarly, using Ex^{∞}.