The Constructive Kan–Quillen Model Structure

Karol Szumiło

University of Leeds

HoTTTEST 2020
Theorem (Quillen)

The category of simplicial sets carries a proper cartesian model structure where

- weak equivalences are the weak homotopy equivalences,
- fibrations are the Kan fibrations,
- cofibrations are the monomorphisms.

Can (a version of) this theorem be proven constructively?

A constructive version of the model structure would be useful in

- study of models of Homotopy Type Theory;
- understanding homotopy theory of simplicial sheaves.
The category of simplices $\Delta$ has
- totally ordered sets $[m] = \{0 < \ldots < m\}$ for $m \in \mathbb{N}$ as objects,
- order preserving maps between them as morphisms.

Morphisms of $\Delta$ are called simplicial operators.
- The injective ones are called face operators.
- An elementary face operator $\delta_i : [m - 1] \to [m]$ omits $i \in [m]$.

A simplicial set is a presheaf over $\Delta$, i.e., a functor $X : \Delta^{\text{op}} \to \text{Set}$.
- Elements of $X_m$ are called $m$-dimensional simplices of $X$.
- The action of simplicial operators describes how simplices of $X$ are attached to each other.
The elements of $X_0$ are called the **vertices** or **points** of $X$.
The elements of $X_1$ are called the **edges** of $X$.
If $x \in X_1$, $a = x\delta_1$ and $b = x\delta_0$ ($a, b \in X_0$), then $x$ is an edge from $a$ to $b$.

$$
\begin{array}{c}
0 \rightarrow 1 \\
\end{array}
\begin{array}{ccc}
0 & \xrightarrow{x} & 1 \\
\end{array}
\begin{array}{c}
a \rightarrow b \\
\end{array}
$$

We can enumerate faces of $u \in X_2$ as follows:

\begin{align*}
x &= u\delta_2 \\
y &= u\delta_0 \\
z &= u\delta_1 \\
a &= u\delta_2\delta_1 = u\delta_1\delta_1 \\
b &= u\delta_2\delta_0 = u\delta_0\delta_1 \\
c &= u\delta_0\delta_0 = u\delta_1\delta_0
\end{align*}
The surjective simplicial operators are called *degeneracy operators*. The category $\Delta$ admits a presentation in terms of elementary face operators and elementary degeneracy operators.

Morphisms of simplicial sets are morphisms of presheaves, they are called *simplicial maps*.

Topology: simplicial sets are models of (triangulated) topological spaces.

Type theory: simplicial sets are interpretations of types:
- vertices correspond to elements of a type,
- edges correspond to elements of its identity type.
The \textit{m-simplex} is the representable simplicial set $\Delta[m]$, i.e.,

$\Delta[m]_k = \{[k] \to [m]\}$.

The 1-simplex serves as an \textit{interval object}.

A \textit{homotopy} between simplicial maps $f, f': X \to Y$ is

$$
\begin{array}{c}
X \times \Delta[1] \longrightarrow Y \\
X \longrightarrow Y^\Delta[1]
\end{array}
$$

that restricts to $f$ and $f'$ over 0 and 1.

A map $f: X \to Y$ is a \textit{homotopy equivalence} if there is $g: Y \to X$ $gf$ is homotopic to $\text{id}_X$ and $fg$ is homotopic to $\text{id}_Y$ (via zig-zags of homotopies).
The horn $\Lambda^1[2]$

\[
\begin{array}{ccc}
0 & \rightarrow & 1 \\
& / & \\
& \downarrow & \\
& & 2
\end{array}
\]

is homotopy equivalent to $\Delta[0]$, but not via a single homotopy.

The spine of $\mathbb{N}$

\[
0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots
\]

is not homotopy equivalent to $\Delta[0]$. (But it should be...)
The *horn* \( \Lambda^i[m] \) is the simplicial subset of \( \Delta[m] \) spanned by all faces of \( \text{id}[m] \in \Delta[m] \) except for \( \text{id}[m] \) itself and \( \delta_i \).

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 2
\end{array}
\]

\( \Lambda^0[2] \)

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 2
\end{array}
\]

\( \Lambda^1[2] \)

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 2
\end{array}
\]

\( \Lambda^2[2] \)

A simplicial set \( K \) is a *Kan complex* if it has the right lifting property with respect to all horn inclusions.

\[
\begin{array}{ccc}
\Lambda^i[m] & \longrightarrow & K \\
\downarrow & & \downarrow \\
\Delta[m] & \longrightarrow
\end{array}
\]
A map $f: X \to Y$ is a **weak homotopy equivalence** if for all Kan complexes $K$, the induced map

$$K^Y \xrightarrow{f^*} K^X$$

is a homotopy equivalence.

A map $X \to Y$ is a **Kan fibration** if it has the right lifting property with respect to all horn inclusions.
A weak factorisation system on a category $\mathcal{M}$ is a pair of classes of morphisms $(L, R)$ such that

- every morphism factors as a morphism of $L$ followed by a morphism of $R$,
- a morphism is in $L$ if and only if it has the left lifting property with respect to all morphisms of $R$,
- a morphism is in $R$ if and only if it has the left lifting property with respect to all morphisms of $L$.

A class of morphism $W$ has the 2-out-of-3 property when for all pairs of composable morphisms $f$ and $g$ if any two of $f$, $g$, $gf$ are in $W$, then so is the third one.
A model structure on a category $\mathcal{M}$ is a triple of classes of morphisms: weak equivalences, fibrations and cofibrations

- (cofibrations, acyclic fibrations) is a weak factorisation system,
- (acyclic cofibrations, fibrations) is a weak factorisation system,
- weak equivalences satisfy 2-out-of-3.

[acyclic (co)fibration = (co)fibration and a weak equivalence]

Theorem (Quillen)

The category of simplicial sets carries a proper cartesian model structure where

- weak equivalences are the weak homotopy equivalences,
- fibrations are the Kan fibrations,
- cofibrations are the monomorphisms.
In Constructive Zermelo–Fraenkel set theory (CZF) we have:

**Theorem**

*The category of simplicial sets carries a proper cartesian model structure where*

- weak equivalences are the weak homotopy equivalences,
- fibrations are the Kan fibrations,
- cofibrations are the Reedy decidable inclusions.

Let $i: A \to B$ and $p: X \to Y$ be simplicial maps. $\text{Prob}(i, p)$ is the set of all squares of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow p \\
B & \longrightarrow & Y
\end{array}
$$

We say that $p$ has the right lifting property with respect to $i$ if the map $\text{sSet}(B, X) \to \text{Prob}(i, p)$ has a section.

A decidable inclusion is a function $A \to X$ between sets such that $X \cong A \sqcup C$ for some set $C$. Logically: $\forall x \in X. x \in A \lor x \notin A$.

Decidable inclusions and split surjections form a weak factorisation system on Set.
A Kan fibration is a map with the right lifting property with respect to all horn inclusions $\Lambda^i[m] \to \Delta[m]$. A trivial cofibration is a map with the left lifting property with respect to all Kan fibrations.

The boundary $\partial \Delta[m]$ is the simplicial subset of $\Delta[m]$ spanned by all faces of $\text{id}_[m] \in \Delta[m]$ except for $\text{id}_[m]$.

A trivial fibration is a map with the right lifting property with respect to all boundary inclusions $\partial \Delta[m] \to \Delta[m]$. A cofibration is a map with the left lifting property with respect to all trivial fibrations.

These form two weak factorisation systems by the constructive version of the small object argument.
Theorem

If $I$ is a set of levelwise decidable inclusions between finite simplicial sets, then there is a weak factorisation system $(L_I, R_I)$ where a morphism is in $R_I$ if and only if it has the right lifting property with respect to $I$.

Take $p_0 : X_0 \to Y$ and:

$$
\begin{align*}
\coprod_i A_i \times \text{Prob}(i, p_k) & \to X_k \\
\coprod_i B_i \times \text{Prob}(i, p_k) & \to X_{k+1}
\end{align*}
$$

Set $X_\infty = \text{colim} X_k$. Then $X_0 \to X_\infty \to Y$ is the required factorisation. ($X_\infty \to Y \in R_I$ by levelwise decidability of $I$.)
A simplicial set $A$ is cofibrant if and only if

- for all degeneracy operators $[m] \rightarrow [n]$, the map $A_n \rightarrow A_m$ is a decidable inclusion.

\[
\begin{array}{ccc}
  a & \rightarrow & b \\
  a & \rightarrow & b \\
  a & \rightarrow & b \\
  u\sigma_0 \in A_2 & u \in A_1 & u\sigma_1 \in A_2
\end{array}
\]

- $L_mX \rightarrow X_m$ is a decidable inclusion for all $m$ ($L_mX$ is the latching object: the set of all degenerate $m$-simplices of $X$.)

A map $A \rightarrow B$ is a cofibration if and only if

- for all degeneracy operators $[m] \rightarrow [n]$, the map $B_n \sqcup_{A_n} A_m \rightarrow B_m$ is a decidable inclusion.

- it is a Reedy decidable inclusion, i.e., for all $m$, the map $L_mB \sqcup_{L_mA} A_m \rightarrow B_m$ is a decidable inclusion.
Cofibrations and trivial cofibrations satisfy the pushout product property, i.e., if $A \to B$ and $C \to D$ are cofibrations, then so is

$$A \times D \sqcup_{A \times C} B \times C \to B \times D$$

which is trivial if one of $A \to B$ or $C \to D$ is.

Proof: explicit combinatorics of horns and boundaries.

Some consequences:

- If $A$ is cofibrant and $X \to Y$ is a (trivial) fibration, then so is $X^A \to Y^A$.
- If $K$ is a Kan complex and $A \to B$ is a (trivial) cofibration, then $K^B \to K^A$ is a (trivial) fibration.
- If $A$ is cofibrant and $K$ is a Kan complex, then $K^A$ is a Kan complex.
A strong cofibrant replacement of $X$ is a cofibrant simplicial set $\tilde{X}$ equipped with a trivial fibration $\tilde{X} \to X$.

A map $f : X \to Y$ is a weak homotopy equivalence if

- $(X$ and $Y$ cofibrant Kan complexes) it is a homotopy equivalence;
- $(X$ and $Y$ Kan complexes) it has a strong cofibrant replacement that is a weak homotopy equivalence;
- $(X$ and $Y$ cofibrant) if $f^* : K^Y \to K^X$ is a weak homotopy equivalence for every Kan complex $K$;
- $(X$ and $Y$ arbitrary) it has a strong cofibrant replacement that is a weak homotopy equivalence.

Weak homotopy equivalences satisfy 2-out-of-6, i.e., given

$$
\begin{array}{c}
W \\ ^f \downarrow \\
\rightarrow \\
X \\ ^g \downarrow \\
\rightarrow \\
Y \\ ^h \downarrow \\
\rightarrow \\
Z
\end{array}
$$

if $gf$ and $hg$ are weak equivalences, then so are $f$, $g$, $h$ and $hgf$. 
Theorem

The category of Kan complexes is a fibration category, i.e.,

- It has a terminal object and all objects are fibrant.
- Pullbacks along fibrations exist and (acyclic) fibrations are stable under pullback.
- Every morphism factors as a weak equivalence followed by a fibration.
- Weak equivalences satisfy the 2-out-of-6 property.
- It has products and (acyclic) fibrations are stable under products.
- It has limits of towers of fibrations and (acyclic) fibrations are stable under such limits.
Lemma

Given $p: X \to Y$ and $q: Y \to Z$, if $p$ and $qp$ are trivial fibrations, then so is $q$.

Lemma

A Kan fibration $p: X \to Y$ between Kan complexes is acyclic if and only if it is trivial.

If $X$ and $Y$ are cofibrant: use the pushout product property to strictify a homotopy inverses of $p$ to a deformation section, i.e., a map $s: Y \to X$ such that $ps = \text{id}_Y$ and $sp \simeq \text{id}_X$ over $Y$. Use that section to solve lifting problems against boundary inclusions. For general $X$ and $Y$, use the previous lemma:
Theorem
The category of cofibrant simplicial sets is a cofibration category, i.e.,

- It has an initial object and all objects are cofibrant.
- Pushouts along cofibrations exist and (acyclic) cofibrations are stable under pushout.
- Every morphism factors as a cofibration followed by a weak equivalence.
- Weak equivalences satisfy the 2-out-of-6 property.
- It has coproducts and (acyclic) cofibrations are stable under coproducts.
- It has colimits of sequences of cofibrations and (acyclic) cofibrations are stable under such colimits.

Proof: dualise by applying $K^{(-)}$ for all Kan complexes $K$. 
Lemma (Gluing Lemma)

In a cofibration category

If top and bottom squares are pushouts along cofibrations and all $X_\emptyset \to Y_\emptyset$, $X_0 \to Y_0$ and $X_1 \to Y_1$ are weak equivalences, then so is $X_{01} \to Y_{01}$.
A *bisimplicial set* is a presheaf over $\Delta \times \Delta$. It can be seen as a simplicial object in $\text{sSet}$ in two ways.

A bisimplicial set $X$ is *cofibrant* if it satisfies the equivalent conditions:

- it is Reedy cofibrant over $\text{Set}$, i.e., $L_{m,n}X \to X_{m,n}$ is a decidable inclusion for all $m$ and $n$.
- it is Reedy cofibrant over $\text{sSet}$ (in either direction), i.e., $L_mX \to X_m$ is a cofibration in $\text{sSet}$.

The *diagonal* of $X$ is the simplicial set $[m] \mapsto X_{m,m}$.

The *$k$-skeleton* of $X$ is the bisimplicial set

$$\text{Sk}^k X = \text{Lan}_{\Delta_{\leq k} \to \Delta} X|_{\Delta_{\leq k}}.$$
Proposition

If $X \to Y$ is a map between cofibrant bisimplicial sets such that $X_k \to Y_k$ is a weak homotopy equivalence for all $k$, then the induced map $\text{diag } X \to \text{diag } Y$ is also a weak homotopy equivalence.

\[
\begin{align*}
L_k X \times \Delta[k] \cup X_k \times \partial \Delta[k] & \longrightarrow \text{diag } \text{Sk}^{k-1} X \\
X_k \times \Delta[k] & \longrightarrow \text{diag } \text{Sk}^k X \\
L_k Y \times \Delta[k] \cup Y_k \times \partial \Delta[k] & \longrightarrow \text{diag } \text{Sk}^{k-1} Y \\
Y_k \times \Delta[k] & \longrightarrow \text{diag } \text{Sk}^k Y
\end{align*}
\]
Let $sd[m]$ be the poset of non-empty subsets of $[m]$ ordered by inclusion. The barycentric subdivision of a simplicial set $X$ is

$$Sd X = \text{colim}_{\Delta[m]\to X} N sd[m].$$

The order preserving map $\text{max}: sd[m] \to [m]$ induces a natural transformation $Sd X \to X$. 
\[ \text{Ex} \; X = \text{sSet}(\text{Sd} \; \Delta[\cdot], \; X) \]
\[ \text{Ex}^\infty \; X = \text{colim}(X \to \text{Ex} \; X \to \text{Ex}^2 \; X \to \ldots) \]

**Proposition**

- \( \text{Ex}^\infty \) preserves finite limits.
- \( \text{Ex}^\infty \) preserves Kan fibrations between cofibrant objects.
- If \( X \) is cofibrant, then \( \text{Ex}^\infty \; X \) is a Kan complex.
- If \( X \) is cofibrant, then \( X \to \text{Ex}^\infty \; X \) is a weak homotopy equivalence.

The last statement is proven by argument of Latch–Thomason–Wilson.
Proposition

For a Kan fibration \( p: X \to Y \), the following are equivalent:

- \( p \) is trivial,
- \( p \) is acyclic,
- \( p \) has contractible fibers.

By the “cancellation lemma” we can assume that \( X \) and \( Y \) are cofibrant.

[trivial \( \Rightarrow \) acyclic] Omitted (but straightforward).

[contractible fibers \( \Rightarrow \) trivial] Take a lifting problem against \( \partial \Delta[m] \to \Delta[m] \) and “contract” it to a fiber. Solve it in the fiber (which is a contractible Kan complex) and “uncontract” to the solution of the original problem.
[acyclic $\Rightarrow$ contractible fibers] (If $X$ and $Y$ are cofibrant) use $\text{Ex}^\infty$:

```
\begin{tikzcd}
F_y \arrow{r} & X \\
\text{Ex}^\infty F_y \arrow{u} \arrow{r} & \text{Ex}^\infty X \\
\Delta[0] \arrow{u} \arrow{r} \arrow{d} & Y \\
\text{Ex}^\infty \Delta[0] \arrow{r} & \text{Ex}^\infty Y
\end{tikzcd}
```
Theorem (Quillen’s Theorem A)

Let $f: I \to J$ be a functor between categories with decidable identities. If for every $y \in J$, $N(f \downarrow y)$ is weakly contractible, then the map $N I \to N J$ is a weak homotopy equivalence.

One can construct a bisimplicial set $Sf$ with maps

$$N I \times \Delta[0] \xleftarrow{} Sf \xrightarrow{} \Delta[0] \times N J$$

satisfying the Diagonal Lemma which yields

$$N I \xleftarrow{\sim} \text{diag } Sf \xrightarrow{\sim} N J$$

$$N J \xleftarrow{\sim} \text{diag } S \text{id}_J \xrightarrow{\sim} N J.$$
\[ TX = N(\Delta \downarrow X) \]

**Proposition**

- \( T \) preserves colimits.
- \( T \) preserves cofibrations.
- \( TX \) is cofibrant for all \( X \).
- \( TX \to X \) is a weak homotopy equivalence for all \( X \).

**Proof of the last statement:**

- for \( X = \Delta[m] \): \( \Delta \downarrow X \) has a terminal object,
- for \( X \) cofibrant: by induction using Gluing Lemma etc.,
- for \( X \) arbitrary: \( T \) carries trivial fibrations to weak homotopy equivalences.
Lemma

If $p: X \to Y$ is a trivial fibration, then $Tp$ is a weak homotopy equivalence.

Consider

\[
\begin{array}{ccc}
\Delta_\# \downarrow X & \xrightarrow{\text{Thm A}} & \Delta \downarrow X \\
\downarrow & & \downarrow \\
\Delta_\# \downarrow Y & \xrightarrow{\text{Thm A}} & \Delta \downarrow Y
\end{array}
\]

where $\Delta_\#$ is the category of face operators and construct a homotopy inverse to the left map by lifting all simplices of $Y$ against $p$. 
Proposition

A cofibration \( i: X \to Y \) is acyclic if and only if it is trivial.

[trivial \( \Rightarrow \) acyclic] If \( X \) and \( Y \) are cofibrant and \( K \) is a Kan complex, then \( i^*: K^Y \to K^X \) is a trivial fibration.

For general \( X \) and \( Y \), use \( T \):

\[
\begin{array}{ccc}
TX & \to & X \\
\downarrow & & \downarrow \\
TY & \to & Y
\end{array}
\]

[acyclic \( \Rightarrow \) trivial] Retract argument:
Proposition

The Kan–Quillen model structure is proper. (Weak equivalences are stable under pushouts along cofibrations and pullbacks along fibrations.)

[Left properness] Let \( i: A \to B \) be a cofibration and \( f: A \to X \) a weak homotopy equivalence.

[Right properness] Similarly, using \( \text{Ex}^\infty \).