Non-wellfounded sets in HoTT

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Motivation

Models of constructive set theory in HoTT

Let U be a universe of small types.

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  - Equality interpreted as the identity type

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  - Equality interpreted as the identity type
- Joint work with Håkon Gylterud.

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Want: The terminal coalgebra for the (U-restricted) powerset functor.

Idea: Dualise the V^0 construction of Gylterud.

Polynomial functors

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- The powerset functor is not polynomial.
  - Terminal coalgebra exists classically (relies on AC), but not yet constructively.

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- We **do** get terminality with respect to embeddings.

This work

- We **don't** get a terminal coalgebra for the powerset functor.
- We **do** get a fixed point for the powerset functor.
- We **do** get terminality with respect to embeddings.
- This becomes a model of constructive set theory with SAFA.

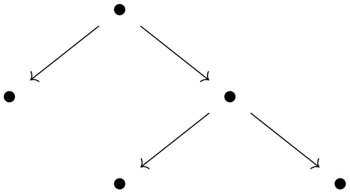
The type V^0

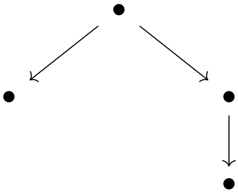
The type $W_{A:U} A$

The type $W_{A:U} A$ is the inductive type with a single constructor:

$$\text{sup} : \prod_{A:U} (A \to W_{A:U} A) \to W_{A:U} A$$

Example elements in $W_{A:U}\, A$





The functor T_U

Definition (T_U)

On types:

$$T_U : \text{Type} \to \text{Type}$$
$$T_U\, X := \sum_{A:U} A \to X$$

On maps:

$$T_U : (X \to Y) \to (T_U\; X \to T_U\; Y)$$
$$T_U\, g\, (A, f) := (A, g \circ f)$$

T_U is polynomial.

The initial T_U-algebra

Theorem

$(W_{A:U}\, A, \mathrm{sup})$ *is the initial* T_U*-algebra.*

The type V^0

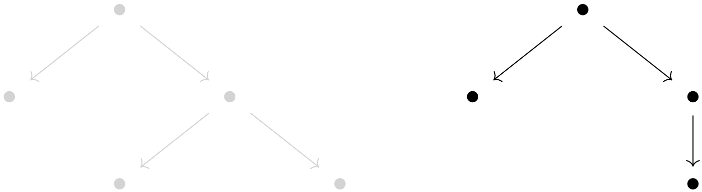
Definition (V^0)

Define the predicate:

$$\begin{aligned} \text{is-itset} &: \mathsf{W}_{A:U}\, A \to \text{Type} \\ \text{is-itset}\,(\sup\ A\, f) &:= (\text{is-emb}\ f) \times \prod_{a:A} \text{is-itset}\,(f\ a) \end{aligned}$$

Define the type $V^{\mathbf{0}} := \sum_{x:\mathsf{W}_{A:U}\, A} \text{is-itset}\ x$.

Example elements in V^0



The *U*-restricted powerset functor

Definition (P_U)

On types:

$$P_U : \mathsf{Type} \to \mathsf{Type}$$
$$P_U\, X := \sum_{A:U} A \hookrightarrow X$$

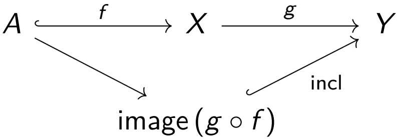
On maps:

$$P_U : (X \to Y) \to (P_U\, X \to P_U\, Y)$$
$$P_U\, g\,(A, f) := (\mathsf{image}\,(g \circ f), \mathsf{incl})$$

(This functor is **not** polynomial.)

The *U*-restricted powerset functor

In pictures:



The initial P_U-algebra

Theorem

V^0 *is the initial* P_U*-algebra.*

The type V^0_∞

Dualisation of V^0

Idea: Start from the terminal T_U-coalgebra and pick out the trees where the branchings are embeddings arbitrarily far down.

The type $M_{A:U}A$

- $(M_{A:U}A, \mathrm{desup}_\infty)$ is the terminal T_U-coalgebra.

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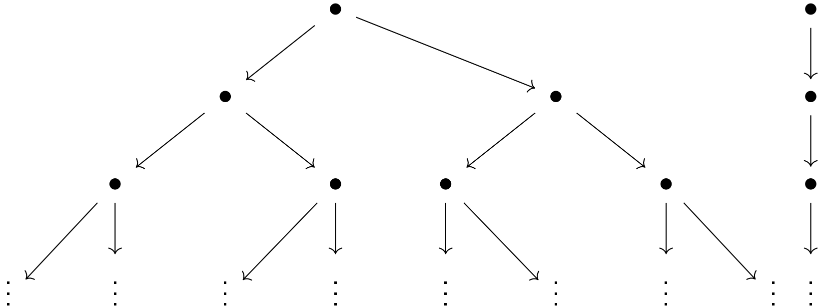
- For every T_U-coalgebra (X, m) let $\mathrm{corec}_T\,(X, m)$ denote the corresponding unique T_U-coalgebra homomorphism.

The type $M_{A:U}A$

- $(M_{A:U}A, \mathrm{desup}_\infty)$ is the terminal T_U-coalgebra.
- For every T_U-coalgebra (X, m) let $\mathrm{corec}_T\,(X, m)$ denote the corresponding unique T_U-coalgebra homomorphism.
- $M_{A:U}A$ can be constructed from inductive types [4].

Example elements of $M_{A:U}A$



Notation for T_U-coalgebras

Notation

Let X : Type and $m : X \to (\sum_{A:U} A \to X)$ be a T_U-coalgebra. For $x : X$, denote

$$\overline{x} : U \qquad \widetilde{x} : \overline{x} \to X$$

$$\overline{x} := \pi_0\,(m\,x) \qquad \widetilde{x} := \pi_1\,(m\,x)$$

The type V^0_∞

Definition (is-coitset$_n$)

Define the predicate:

$$\begin{aligned}
&\text{is-coitset} : \mathbb{N} \to M_{A:U} A \to \text{Type} \\
&\text{is-coitset}_0\, x := \text{is-emb}\ \widetilde{x} \\
&\text{is-coitset}_{(\mathsf{s}\ n)}\, x := \prod_{a:\overline{x}} \text{is-coitset}_n\, (\widetilde{x}\ a)
\end{aligned}$$

The type V^0_∞

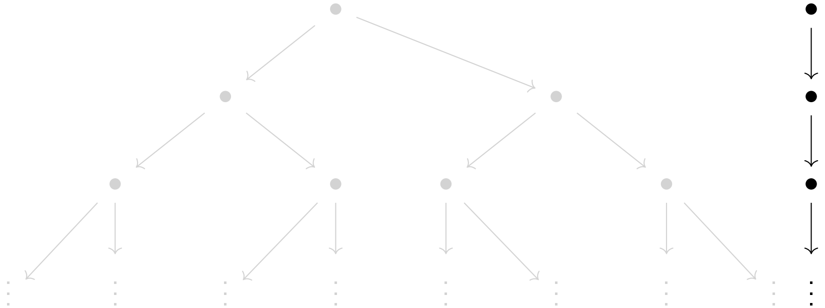
Definition (V^0_∞)

Define the predicate:

$$\text{is-coitset} : M_{A:U} A \to \text{Type}$$
$$\text{is-coitset}\, x := \prod_{n:\mathbb{N}} \text{is-coitset}_n\, x$$

Define the type $\mathsf{V}^{\mathbf{0}}_\infty := \sum_{x:M_{A:U}A} \text{is-coitset}\, x$

Example elements of V^0_∞



Results about V^0_∞

V^0_∞ is a fixed point for P_U

Theorem

There is an equivalence $\mathsf{V}^0_\infty \simeq \mathsf{P}_U\, \mathsf{V}^0_\infty$.

V^0_∞ is a fixed point for P_U

Outline of proof.

- We have $\mathsf{desup}_\infty : M_{A:U}A \simeq T_U(M_{A:U}A)$, with inverse sup_∞.

□

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Outline of proof.

- We have $\mathsf{desup}_\infty : M_{A:U} A \simeq T_U(M_{A:U} A)$, with inverse sup_∞.
- Define $\mathsf{desup}_0 : V^0_\infty \to T_U(M_{A:U} A)$ as

$$\mathsf{desup}_0 := \mathsf{desup}_\infty \circ \pi_0$$

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- Define $\mathsf{sup}_0 : P_U V^0_\infty \to M_{A:U} A$ as

$$\mathsf{sup}_0 (A, f) := \mathsf{sup}_\infty (A, \pi_0 \circ f)$$

□

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- Define $\mathrm{sup}_0 : P_U V^0_\infty \to M_{A:U}A$ as

$$\mathrm{sup}_0\,(A, f) := \mathrm{sup}_\infty\,(A, \pi_0 \circ f)$$

- Show that sup_0 in fact lands in V^0_∞.
- That desup_0 and sup_0 are inverses follows from the fact that desup_∞ and sup_∞ are inverses.

□

V^0_∞ is a model of constructive set theory

Since V^0_∞ is a fixed point for P_U, it is a model of constructive set theory. This goes back to Rieger 1957 [5].

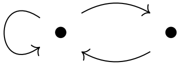
Terminality of V^0_∞?

Theorem

The P_U*-coalgebra* $(V^0_\infty, \mathrm{desup}_0)$ *is* ***not*** *terminal.*

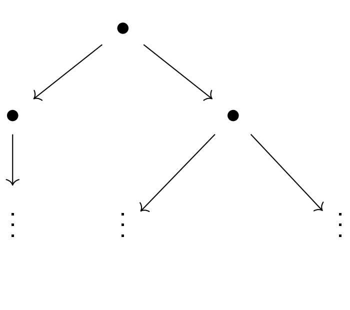
Terminality of V^0_∞?

Consider the following graph:



Terminality of V^0_∞?

The nodes are mapped by $\mathsf{corec}_\mathsf{T}$ to the corresponding unfolding trees:



Terminality of V^0_∞?

But there is also a P_U-coalgebra homomorphism which maps both nodes to the tree:

Terminality of V^0_∞?

So V^0_∞ is **not** the terminal P_U-coalgebra.

Terminality of V^0_∞ w.r.t. embeddings

Theorem

Let (X, m) *be a* P_U*-coalgebra such that* $\mathrm{corec}_T\,(X, m)$ *is an embedding. Then the following type is contractible:*

$$\sum_{f : P_U\text{-}Coalg\,\left(X, V^0_\infty\right)} \text{is-emb}\, f$$

Terminality of V^0_∞ w.r.t. embeddings

Center of contraction

- We show that the map $\mathsf{corec}_\mathsf{T}\,(X,m) : X \hookrightarrow \mathsf{M}_{A:U}A$ lands in V^0_∞. We need to show that for all $x : X$ and $n : \mathbb{N}$, $\text{is-coitset}_n\,(\mathsf{corec}_\mathsf{T}\ x)$.

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  - **Base case:** $(\mathsf{corec}_{\mathsf{T}}\ x) = \mathsf{corec}_{\mathsf{T}} \circ \widetilde{x}$, and the rhs is a composition of two embeddings.

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  - **Base case:** $\widetilde{(\mathrm{corec}_{\mathsf{T}}\ x)} = \mathrm{corec}_{\mathsf{T}} \circ \widetilde{x}$, and the rhs is a composition of two embeddings.
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  - **Induction step:** Let $a : \overline{x}$, by induction, $\mathsf{corec}_\mathsf{T}\,(\widetilde{x}\ a)$ is n-coiterative.
- So $\mathsf{corec}_\mathsf{T}\,(X, m) : X \hookrightarrow V^0_\infty$. And it is a P_U-coalgebra homomorphism because it is an embedding.

Terminality of V^0_∞ w.r.t. embeddings

Equality

- Let $f : X \hookrightarrow V^0_\infty$ be a P_U-coalgebra homomorphism. To show that $f = \mathsf{corec}_T$ it is enough to show that $\pi_0 \circ f = \pi_0 \circ \mathsf{corec}_T$.

Terminality of V^0_∞ w.r.t. embeddings

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- But $\pi_0 \circ f : X \to M_{A:U} A$ is a T_U-coalgebra since f is an embedding. By the terminality of $M_{A:U} A$ it follows that $\pi_0 \circ f = \pi_0 \circ \mathrm{corec}_T$.

Scott's anti-foundation axiom

Definition (Scott extensionality)

A graph G is **Scott extensional** if for any $a, b \in G$,

$$G_a \cong^t G_b \Rightarrow a = b$$

where $G_a \cong^t G_b$ means that the unfolding trees are isomorphic.

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Definition (Scott's anti-foundation axiom)

V is Scott extensional and every Scott extensional graph has a decoration [6].

Scott's anti-foundation axiom in our setting

In our setting:

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Scott's anti-foundation axiom in our setting

In our setting:

- V^0_∞ is Scott extensional, because equality is tree isomorphism.
- Given a P_U-coalgebra (X, m), to say that $\mathsf{corec}_\mathsf{T}\,(X, m)$ is an embedding is to say that equality between nodes is isomorphism of the corresponding unfolding trees.
- A decoration is precisely a P_U-coalgebra homomorphism.

Formalisation

The formalisation can be found at:
https://git.app.uib.no/hott/hott-set-theory/-/tree/2e98dd35

Future work

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  - Can we think of this as some type of non-wellfounded multisets?

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- Formulate SAFA in the model and formalise that it holds.
- Formalise counterexample to terminality.
- Investigate V^n_∞.
  - Can we think of this as some type of non-wellfounded multisets?
  - Do either a generalisation of AFA or SAFA hold?

References

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