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The synthetic theory of ∞-categories vs the synthetic theory of ∞-categories

joint with Dominic Verity and Michael Shulman

Homotopy Type Theory Electronic Seminar Talks

Abstract

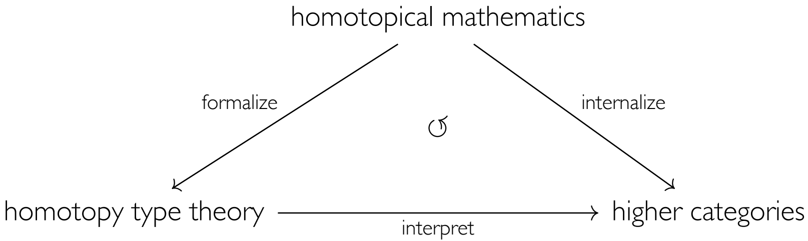
Homotopy type theory provides a "synthetic" framework that is suitable for developing the theory of mathematical objects with natively homotopical content. A famous example is given by $(\infty, 1)$-categories — aka ∞-categories — which are categories given by a collection of objects, a homotopy type of arrows between each pair, and a weak composition law.

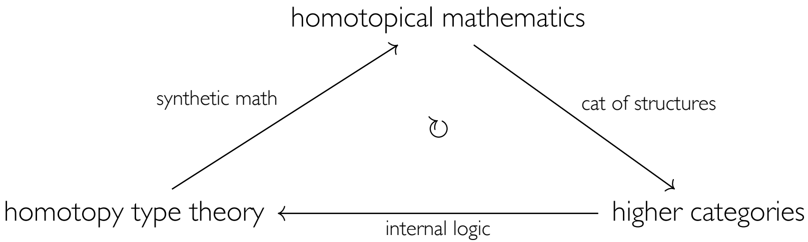
This talk will compare two "synthetic" developments of the theory of ∞-categories

- the first (with Verity) using 2-category theory and
- the second (with Shulman) using a simplicial augmentation of homotopy type theory due to Shulman

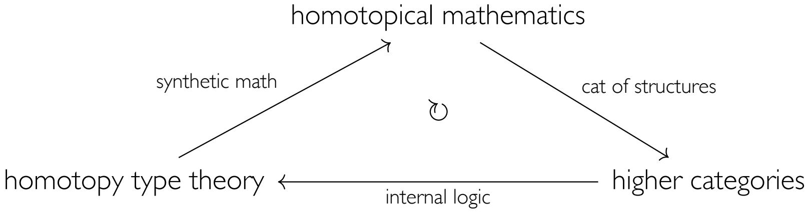
by considering in parallel their treatment of the theory of adjunctions between ∞-categories. The hope is to spark a discussion about the merits and drawbacks of various approaches to synthetic mathematics.

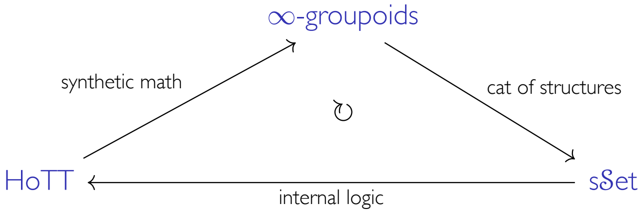
Homotopical trinitarianism?



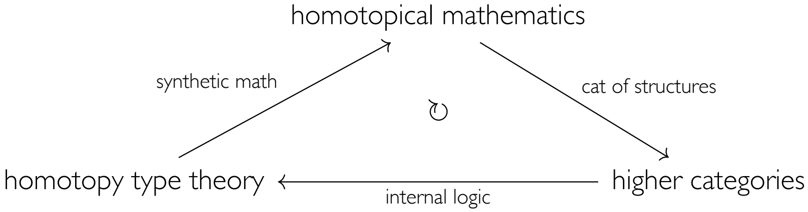


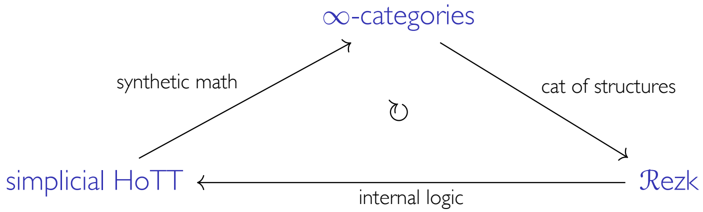
Synthetic homotopy theory



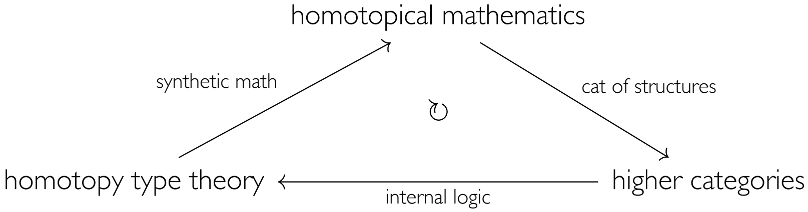


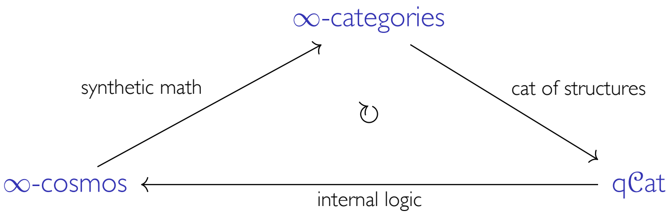
Synthetic ∞-category theory





Synthetic ∞-category theory





Plan

1. The synthetic theory of ∞-categories

2. The synthetic theory of ∞-categories

Plan

1. The semantic theory of ∞-categories
2. The synthetic theory of ∞-categories in an ∞-cosmos
3. The synthetic theory of ∞-categories in homotopy type theory
4. Discussion

I

The semantic theory of ∞-categories

The idea of an ∞-category

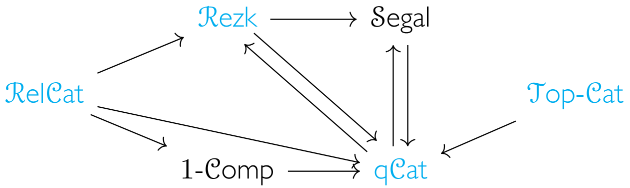
∞-categories are the nickname that Jacob Lurie gave to $(\infty, 1)$-categories: categories weakly enriched over homotopy types.

The schematic idea is that an ∞-category should have

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative (and unital) up to invertible 3-arrows
- with these witnesses coherent up to invertible arrows all the way up

The problem is that this definition is not very precise.

Models of ∞-categories



- topological categories and relative categories are strict objects but the correct maps between them are tricky to understand
- quasi-categories (originally weak Kan complexes) are the basis for the R-Verity synthetic theory of ∞-categories
- Rezk spaces (originally complete Segal spaces) are the basis for the R-Shulman synthetic theory of ∞-categories

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The synthetic theory of ∞-categories in an ∞-cosmos

∞-cosmoi

An ∞-cosmos is an axiomatization of the properties of q$\mathcal{C}$at.

The category of quasi-categories has:

- objects the quasi-categories A, B
- functors between quasi-categories $f\colon A \to B$, which define the points of a quasi-category $\mathsf{Fun}(A, B) = B^A$
- a class of isofibrations $E \twoheadrightarrow B$ with familiar closure properties
- so that (flexible weighted) limits of diagrams of quasi-categories and isofibrations are quasi-categories

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Theorem (R-Verity). q$\mathcal{C}$at, $\mathcal{R}$ezk, $\mathcal{S}$egal, and 1-$\mathcal{C}$omp define ∞-cosmoi.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
- 1-cells are the ∞-functors $f\colon A \to B$ in the ∞-cosmos
- 2-cells we call ∞-natural transformations $A \underset{g}{\overset{f}{\Downarrow\gamma}} B$ which are defined to be homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$

Prop (R-Verity). Equivalences in the homotopy 2-category

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \qquad A \underset{gf}{\overset{1_A}{\Downarrow\cong}} A \qquad B \underset{fg}{\overset{1_B}{\Downarrow\cong}} B$$

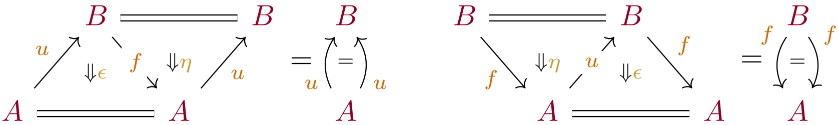
coincide with equivalences in the ∞-cosmos.

Adjunctions between ∞-categories

An adjunction consists of:

- ∞-categories A and B
- ∞-functors $u\colon A \to B$, $f\colon B \to A$
- ∞-natural transformations $\eta\colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow \mathrm{id}_A$

satisfying the triangle equalities



Write $f \dashv u$ to indicate that f is the left adjoint and u is the right adjoint.

The 2-category theory of adjunctions

Prop. Adjunctions compose:

$$C \underset{u'}{\overset{f'}{\rightleftarrows}} B \underset{u}{\overset{f}{\rightleftarrows}} A \quad \rightsquigarrow \quad C \underset{u'u}{\overset{ff'}{\rightleftarrows}} A$$

Prop. Adjoints to a given functor $u \colon A \to B$ are unique up to canonical isomorphism: if $f \dashv u$ and $f' \dashv u$ then $f \cong f'$.

Prop. Any equivalence can be promoted to an adjoint equivalence: if $u \colon A \xrightarrow{\sim} B$ then u is left and right adjoint to its equivalence inverse.

The universal property of adjunctions

Any ∞-category A has an ∞-category of arrows $\hom_A \twoheadrightarrow A \times A$ equipped with a generic arrow

$$\hom_A \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} A \qquad \Downarrow\kappa$$

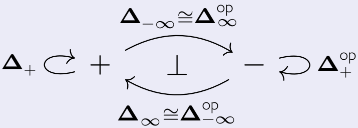
Prop. $A \underset{u}{\overset{f}{\rightleftarrows}} B$ with $f \dashv u$ ($\perp$) if and only if $\hom_A(f, A) \simeq_{A\times B} \hom_B(B, u)$.

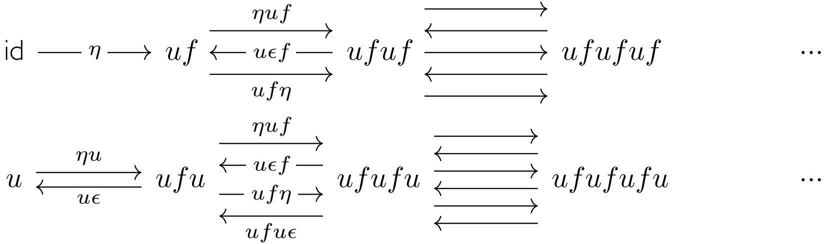
Prop. If $f \dashv u$ with unit η and counit ϵ then

- ηb is initial in $\hom_B(b, u)$ and
- ϵa is terminal in $\hom_A(f, a)$.

The free adjunction

Theorem (Schanuel-Street). Adjunctions in a 2-category $\mathcal{K}$ correspond to 2-functors $\mathcal{A}\mathsf{dj} \to \mathcal{K}$, where $\mathcal{A}\mathsf{dj}$, the free adjunction, is a 2-category:





Homotopy coherent adjunctions

A homotopy coherent adjunction in an ∞-cosmos $\mathcal{K}$ is a simplicial functor $\mathcal{A}\mathrm{dj} \to \mathcal{K}$. Explicitly, it picks out:

- a pair of objects $A, B \in \mathcal{K}$.
- homotopy coherent diagrams

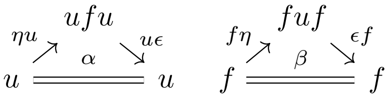
$$\begin{array}{cc} \mathbf{\Delta}_+ \to \mathrm{Fun}(B, B) & \mathbf{\Delta}_+^{\mathrm{op}} \to \mathrm{Fun}(A, A) \\ \mathbf{\Delta}_\infty \to \mathrm{Fun}(A, B) & \mathbf{\Delta}_\infty^{\mathrm{op}} \to \mathrm{Fun}(B, A) \end{array}$$

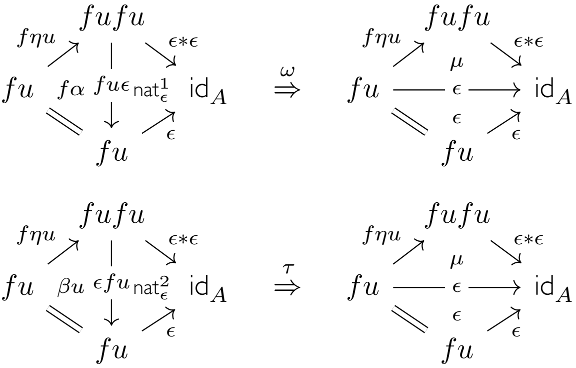
that are functorial with respect to the composition action of $\mathcal{A}\mathrm{dj}$.

Coherent adjunction data

A homotopy coherent adjunction is a simplicial functor $\mathcal{A}\mathrm{dj} \to \mathcal{K}$.

triangle equality witnesses





Existence of homotopy coherent adjunctions

Theorem (R-Verity). Any adjunction in the homotopy 2-category of an ∞-cosmos extends to a homotopy coherent adjunction.

Proof: Given adjunction data

- $u\colon A \to B$ and $f\colon B \to A$
- $\eta\colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow \mathrm{id}_A$
- α witnessing $u\epsilon \circ \eta u = \mathrm{id}_u$ and β witnessing $\epsilon f \circ f\eta = \mathrm{id}_f$

forget to either

- (f, u, η) or
- $(f, u, \eta, \epsilon, \alpha)$

and use the universal property of the unit η to extend all the way up.

Theorem (R-Verity). Moreover, the spaces of extensions from the data (f, u, η) or $(f, u, \eta, \epsilon, \alpha)$ are contractible Kan complexes.

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The synthetic theory of ∞-categories in homotopy type theory

The intended model

$\mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$	$\supset$	$\mathcal{R}\mathrm{eedy}$	$\supset$	$\mathcal{S}\mathrm{egal}$	$\supset$	$\mathcal{R}\mathrm{ezk}$
$\parallel$		$\parallel$		$\parallel$		$\parallel$
bisimplicial sets		types		types with composition		types with composition & univalence

Theorem (Shulman). Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

Theorem (Rezk). $(\infty, 1)$-categories are modeled by Rezk spaces aka complete Segal spaces.

The bisimplicial sets model of homotopy type theory has:

- an interval type I, parametrizing paths inside a general type
- a directed interval type $\mathbb{2}$, parametrizing arrows inside a general type

Paths and arrows

- The identity type for A depends on two terms in A:

$$x, y : A \vdash x =_A y$$

  and a term $p : x =_A y$ defines a path in A from x to y.
- The hom type for A depends on two terms in A:

$$x, y : A \vdash \mathbf{hom}_A(x, y)$$

  and a term $f : \mathbf{hom}_A(x, y)$ defines an arrow in A from x to y.

Hom types are defined as instances of extension types axiomatized in a three-layered type theory with (simplicial) shapes due to Shulman

$$\mathbf{hom}_A(x, y) := \left\langle \begin{array}{ccc} 1+1 & \xrightarrow{[x,y]} & A \\ \downarrow & \nearrow & \\ 2 & & \end{array} \right\rangle$$

Semantically, hom types $\sum_{x,y:A} \mathbf{hom}_A(x, y)$ recover the ∞-category of arrows $\mathbf{hom}_A \twoheadrightarrow A \times A$ in the ∞-cosmos $\mathcal{R}ezk$.

Segal, Rezk, and discrete types

- A type A is Segal if every composable pair of arrows has a unique composite: if for every $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$

$$\left\langle \begin{array}{ccc} \Lambda^2_1 & \xrightarrow{[f,g]} & A \\ \downarrow & \nearrow & \\ \Delta^2 & & \end{array} \right\rangle \qquad \text{is contractible.}$$

- A Segal type A is Rezk if every isomorphism is an identity: if

$$\text{id-to-iso} : \prod_{x,y:A} (x =_A y) \to (x \cong_A y) \qquad \text{is an equivalence.}$$

- A type A is discrete if every arrow is an identity: if

$$\text{id-to-arr} : \prod_{x,y:A} (x =_A y) \to \hom_A(x, y) \qquad \text{is an equivalence.}$$

Prop. A type is discrete if and only if it is Rezk and all of its arrows are isomorphisms — the discrete types are the ∞-groupoids.

The 2-category of Segal types

Prop (R-Shulman).

- Any function $f\colon A \to B$ between Segal types preserves identities and composition. Morever, the type $A \to B$ of functors is again a Segal type.
- Given functors $f, g\colon A \to B$ between Segal types there is an equivalence

$$\hom_{A \to B}(f, g) \xrightarrow{\sim} \prod_{a:A} \hom_B(fa, ga)$$

- Terms $\gamma : \hom_{A \to B}(f, g)$, called natural transformations, are natural and can be composed vertically and horizontally up to typal equality.

Incoherent adjunction data

A quasi-diagrammatic adjunction between types A and B consists of

- functors $u\colon A \to B$ and $f\colon B \to A$
- natural transformations $\eta\colon \underset{B\to B}{\mathbf{hom}}(\mathrm{id}_B, uf)$, $\epsilon\colon \underset{A\to A}{\mathbf{hom}}(fu, \mathrm{id}_A)$
- witnesses $\alpha : u\epsilon \circ \eta u = \mathrm{id}_u$ and $\beta : \epsilon f \circ f\eta = \mathrm{id}_f$

A (quasi*-)transposing adjunction between types A and B consists of functors $u\colon A \to B$ and $f\colon B \to A$ and a family of equivalences

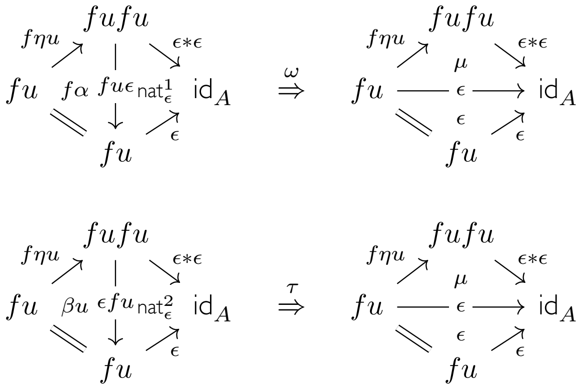
$$\prod_{a:A,b:B} \mathbf{hom}_A(fb, a) \simeq \mathbf{hom}_B(b, ua)$$

(*together with their quasi-inverses and the witnessing homotopies).

Theorem(R-Shulman). Given functors $u\colon A \to B$ and $f\colon B \to A$ between Segal types the type of quasi-transposing adjunctions $f \dashv u$ is equivalent to the type of quasi-diagrammatic adjunctions $f \dashv u$.

Coherent adjunction data

A half-adjoint diagrammatic adjunction consists of:



Theorem (R-Shulman). Given functors $u \colon A \to B$ and $f \colon B \to A$ between Segal types the type of transposing adjunctions $f \dashv u$ is equivalent to the type of half-adjoint diagrammatic adjunctions $f \dashv u$.

Uniqueness of coherent adjunction data

If $\eta\colon \hom_{B\to B}(\mathrm{id}_B, uf)$ is a unit, then that adjunction is uniquely determined:

Theorem (R-Shulman). Given Segal types A and B, functors $u\colon A \to B$ and $f\colon B \to A$, and a natural transformation $\eta\colon \hom_{B\to B}(\mathrm{id}_B, uf)$ the following are equivalent propositions:

- The type of $(\epsilon, \alpha, \beta, \mu, \omega, \tau)$ extending (f, u, η) to a half-adjoint diagrammatic adjunction.
- The propositional truncation of the type of $(\epsilon, \alpha, \beta)$ extending (f, u, η) to a quasi-diagrammatic adjunction.

Theorem (R-Shulman). Given the data $(f, u, \eta, \epsilon, \alpha)$ as in a quasi-diagrammtic adjunction, the following are equivalent propositions:

- The type of $(\beta, \mu, \omega, \tau)$ extending this data to a half-adjoint diagrammatic adjunction.
- The propositional truncation of the type of β extending this data to a quasi-diagrammatic adjunction.

Where does Rezk-completeness come in?

For Rezk types — the synthetic ∞-categories — adjoints are literally unique, not just "unique up to isomorphism":

Theorem (R-Shulman). Given a Segal type B, a Rezk type A, and a functor $u\colon A \to B$, the following types are equivalent propositions:

- The type of transposing left adjoints of u.
- The type of half-adjoint diagrammatic left adjoints of u.
- The propositional truncation of the type of quasi-diagrammatic left adjoints of u.

4

Discussion

Closing thoughts

- In an ∞-cosmos, we prove that there exists a quasi-diagrammatic adjunction if and only if there exists a quasi-transposing adjunction. In simplicial HoTT, we prove the types of such are equivalent, which conveys more information (though I'm not exactly sure what).
- The ∞-cosmos $\mathcal{R}$ezk does not see Segal or ordinary types — because we've axiomatized the fibrant objects, rather than the full model category.
- It seems to be much easier to produce an ∞-cosmos than to define a model of simplicial HoTT.
- But overall the experiences of working with either approach to the synthetic theory of ∞-categories are strikingly similar — and I'm not sure I entirely understand why that is.

References

For more on homotopical trinitarianism, see:

Michael Shulman

- Homotopical trinitarianism: a perspective on homotopy type theory, home.sandiego.edu/∼shulman/papers/trinity.pdf

For more on the synthetic theory of ∞-categories, see:

Emily Riehl and Dominic Verity

- ∞-category theory from scratch, arXiv:1608.05314
- ∞-Categories for the Working Mathematician, www.math.jhu.edu/∼eriehl/ICWM.pdf

Emily Riehl and Michael Shulman

- A type theory for synthetic ∞-categories, Higher Structures 1(1):116–193, 2017; arXiv:1705.07442

Thank you!