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The synthetic theory of ∞ -categories vs the synthetic theory of ∞ -categories

joint with Dominic Verity and Michael Shulman

Homotopy Type Theory Electronic Seminar Talks



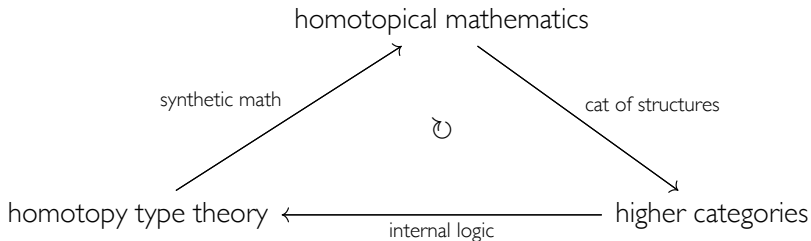
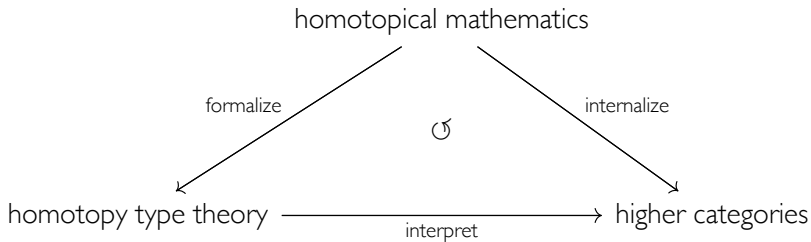
Homotopy type theory provides a “synthetic” framework that is suitable for developing the theory of mathematical objects with natively homotopical content. A famous example is given by $(\infty, 1)$ -categories — aka ∞ -categories — which are categories given by a collection of objects, a homotopy type of arrows between each pair, and a weak composition law.

This talk will compare two “synthetic” developments of the theory of ∞ -categories

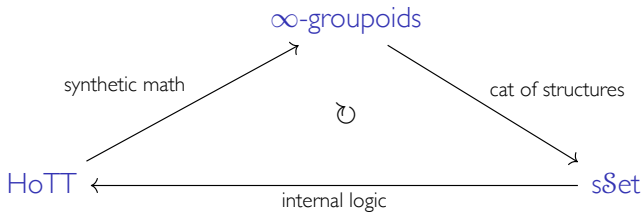
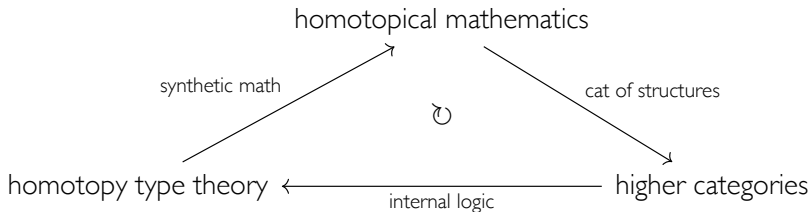
- the first (with Verity) using 2-category theory and
- the second (with Shulman) using a simplicial augmentation of homotopy type theory due to Shulman

by considering in parallel their treatment of the theory of adjunctions between ∞ -categories. The hope is to spark a discussion about the merits and drawbacks of various approaches to synthetic mathematics.

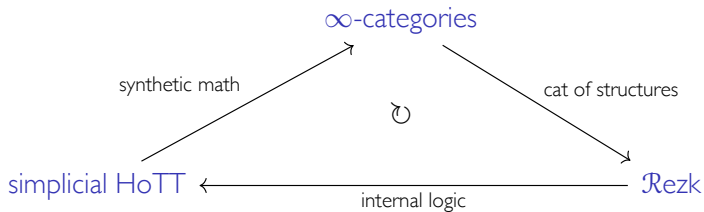
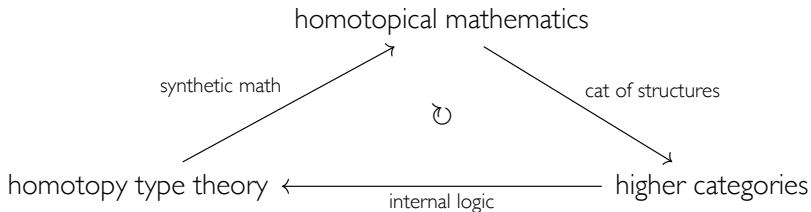
Homotopical trinitarianism?



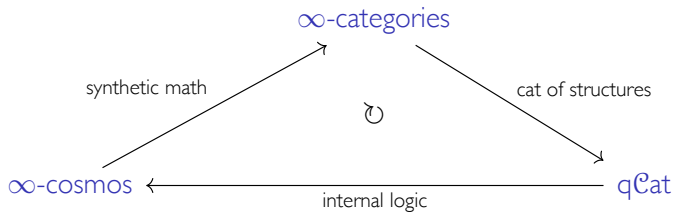
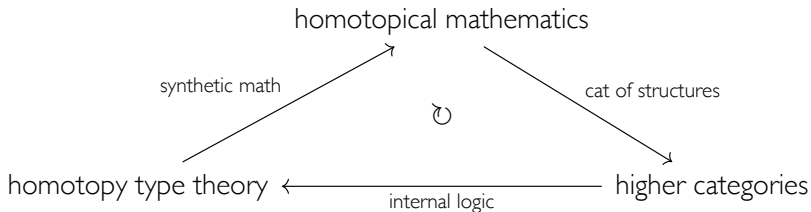
Synthetic homotopy theory



Synthetic ∞ -category theory



Synthetic ∞ -category theory



Plan



1. The synthetic theory of ∞ -categories
2. The synthetic theory of ∞ -categories



1. The semantic theory of ∞ -categories
2. The synthetic theory of ∞ -categories in an ∞ -cosmos
3. The synthetic theory of ∞ -categories in homotopy type theory
4. Discussion



The semantic theory of ∞ -categories

The idea of an ∞ -category



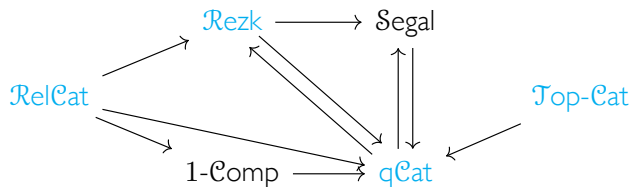
∞ -categories are the nickname that Jacob Lurie gave to $(\infty, 1)$ -categories: categories **weakly enriched** over homotopy types.

The schematic idea is that an ∞ -category should have

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative (and unital) up to invertible 3-arrows
- with these witnesses coherent up to invertible arrows all the way up

The problem is that this definition is not very precise.

Models of ∞ -categories



- **topological categories** and **relative categories** are strict objects but the correct maps between them are tricky to understand
- **quasi-categories** (originally **weak Kan complexes**) are the basis for the R-Verity synthetic theory of ∞ -categories
- **Rezk spaces** (originally **complete Segal spaces**) are the basis for the R-Shulman synthetic theory of ∞ -categories



2

The synthetic theory of ∞ -categories
in an ∞ -cosmos



An ∞ -cosmos is an axiomatization of the properties of \mathbf{qCat} .

The category of quasi-categories has:

- objects the quasi-categories A, B
- functors between quasi-categories $f: A \rightarrow B$, which define the points of a quasi-category $\mathbf{Fun}(A, B) = B^A$
- a class of isofibrations $E \twoheadrightarrow B$ with familiar closure properties
- so that (flexible weighted) limits of diagrams of quasi-categories and isofibrations are quasi-categories



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- so that (flexible weighted) limits of diagrams of ∞ -categories and isofibrations are ∞ -categories

Theorem (R-Verity). \mathbf{qCat} , Rezk, Segal, and 1-Comp define ∞ -cosmoi.

The homotopy 2-category



The **homotopy 2-category** of an ∞ -cosmos is a strict 2-category whose:

- objects are the ∞ -categories A, B in the ∞ -cosmos
- 1-cells are the ∞ -functors $f: A \rightarrow B$ in the ∞ -cosmos
- 2-cells we call ∞ -natural transformations $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \gamma \\ \xrightarrow{g} \end{array} B$ which are defined to be homotopy classes of 1-simplices in $\text{Fun}(A, B)$

Prop (R-Verity). Equivalences in the homotopy 2-category

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \gamma \\ \xrightarrow{g} \end{array} & B \\ A & \begin{array}{c} \xrightarrow{1_A} \\ \Downarrow \cong \\ \xrightarrow{gf} \end{array} & A \\ B & \begin{array}{c} \xrightarrow{1_B} \\ \Downarrow \cong \\ \xrightarrow{fg} \end{array} & B \end{array}$$

coincide with **equivalences** in the ∞ -cosmos.

Adjunctions between ∞ -categories



An adjunction consists of:

- ∞ -categories A and B
- ∞ -functors $u: A \rightarrow B, f: B \rightarrow A$
- ∞ -natural transformations $\eta: \text{id}_B \Rightarrow uf$ and $\epsilon: fu \Rightarrow \text{id}_A$

satisfying the triangle equalities

$$\begin{array}{ccc}
 \begin{array}{c}
 B \xlongequal{\quad} B \\
 \begin{array}{ccc}
 u \nearrow & \searrow f & \searrow u \\
 \downarrow \epsilon & & \downarrow \eta \\
 A \xlongequal{\quad} A & & A \xlongequal{\quad} A
 \end{array}
 \end{array}
 & = &
 \begin{array}{c}
 B \\
 \left(\begin{array}{c} \nearrow \\ = \\ \searrow \end{array} \right)_u \\
 A
 \end{array}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c}
 B \xlongequal{\quad} B \\
 \begin{array}{ccc}
 \searrow f & \searrow u & \searrow f \\
 \downarrow \eta & & \downarrow \epsilon \\
 A \xlongequal{\quad} A & & A \xlongequal{\quad} A
 \end{array}
 \end{array}
 & = &
 \begin{array}{c}
 B \\
 \left(\begin{array}{c} \searrow \\ = \\ \searrow \end{array} \right)_f \\
 A
 \end{array}
 \end{array}
 \end{array}$$

Write $f \dashv u$ to indicate that f is the left adjoint and u is the right adjoint.

The 2-category theory of adjunctions



Prop. Adjunctions compose:

$$C \begin{array}{c} \xrightarrow{f'} \\ \perp \\ \xleftarrow{u'} \end{array} B \begin{array}{c} \xrightarrow{f} \\ \perp \\ \xleftarrow{u} \end{array} A \quad \rightsquigarrow \quad C \begin{array}{c} \xrightarrow{ff'} \\ \perp \\ \xleftarrow{u'u} \end{array} A$$

Prop. Adjoints to a given functor $u: A \rightarrow B$ are unique up to canonical isomorphism: if $f \dashv u$ and $f' \dashv u$ then $f \cong f'$.

Prop. Any equivalence can be promoted to an adjoint equivalence: if $u: A \xrightarrow{\sim} B$ then u is left and right adjoint to its equivalence inverse.

The universal property of adjunctions



Any ∞ -category A has an ∞ -category of arrows $\mathbf{hom}_A \rightarrow A \times A$ equipped with a generic arrow

$$\begin{array}{ccc} & \xrightarrow{\text{dom}} & \\ \mathbf{hom}_A & \Downarrow \kappa & A \\ & \xrightarrow{\text{cod}} & \end{array}$$

Prop. $A \begin{array}{c} \xleftarrow{f} \\ \perp \\ \xrightarrow{u} \end{array} B$ if and only if $\mathbf{hom}_A(f, A) \simeq_{A \times B} \mathbf{hom}_B(B, u)$.

Prop. If $f \dashv u$ with unit η and counit ϵ then

- ηb is initial in $\mathbf{hom}_B(b, u)$ and
- ϵa is terminal in $\mathbf{hom}_A(f, a)$.

The free adjunction



Theorem (Schanuel-Street). Adjunctions in a 2-category \mathcal{K} correspond to 2-functors $\mathcal{A}dj \rightarrow \mathcal{K}$, where $\mathcal{A}dj$, the **free adjunction**, is a 2-category:

$$\begin{array}{c}
 \Delta_{-\infty} \cong \Delta_{\infty}^{\text{op}} \\
 \Delta_{+} \curvearrowright + \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} - \curvearrowleft \Delta_{+}^{\text{op}} \\
 \Delta_{\infty} \cong \Delta_{-\infty}^{\text{op}}
 \end{array}$$

$$\begin{array}{ccccccc}
 \text{id} & \xrightarrow{\eta} & uf & \begin{array}{c} \xrightarrow{\eta uf} \\ \xleftarrow{uf\eta} \end{array} & ufuf & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} & ufufuf & \dots \\
 u & \begin{array}{c} \xrightarrow{\eta u} \\ \xleftarrow{u\epsilon} \end{array} & ufu & \begin{array}{c} \xrightarrow{\eta uf} \\ \xleftarrow{uf\eta} \\ \xrightarrow{\eta uf} \\ \xleftarrow{uf\eta} \\ \xrightarrow{ufu\epsilon} \end{array} & ufufu & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & ufufufu & \dots
 \end{array}$$

Homotopy coherent adjunctions



A **homotopy coherent adjunction** in an ∞ -cosmos \mathcal{K} is a simplicial functor $\mathcal{A}dj \rightarrow \mathcal{K}$. Explicitly, it picks out:

- a pair of objects $A, B \in \mathcal{K}$.
- homotopy coherent diagrams

$$\begin{array}{ll} \Delta_+ \rightarrow \text{Fun}(B, B) & \Delta_+^{\text{op}} \rightarrow \text{Fun}(A, A) \\ \Delta_\infty \rightarrow \text{Fun}(A, B) & \Delta_\infty^{\text{op}} \rightarrow \text{Fun}(B, A) \end{array}$$

that are functorial with respect to the composition action of $\mathcal{A}dj$.

Coherent adjunction data



A homotopy coherent adjunction is a simplicial functor $\mathcal{A}dj \rightarrow \mathcal{K}$.

triangle equality witnesses

$$\begin{array}{ccc}
 & ufu & \\
 \eta u \nearrow & \alpha & \searrow u\epsilon \\
 u & \underline{\underline{=}} & u
 \end{array}
 \quad
 \begin{array}{ccc}
 & fuf & \\
 f\eta \nearrow & \beta & \searrow \epsilon f \\
 f & \underline{\underline{=}} & f
 \end{array}$$

$$\begin{array}{ccc}
 & fufu & \\
 f\eta u \nearrow & | & \searrow \epsilon * \epsilon \\
 fu & f\alpha \quad fu\epsilon \text{nat}_\epsilon^1 & \searrow \epsilon \\
 \Downarrow & \downarrow & \nearrow \epsilon \\
 & fu &
 \end{array}$$

 \Rightarrow

$$\begin{array}{ccc}
 & fufu & \\
 f\eta u \nearrow & \mu & \searrow \epsilon * \epsilon \\
 fu & \epsilon & \longrightarrow \text{id}_A \\
 \Downarrow & \epsilon & \nearrow \epsilon \\
 & fu &
 \end{array}$$

$$\begin{array}{ccc}
 & fufu & \\
 f\eta u \nearrow & | & \searrow \epsilon * \epsilon \\
 fu & \beta u \quad \epsilon fu \text{nat}_\epsilon^2 & \searrow \epsilon \\
 \Downarrow & \downarrow & \nearrow \epsilon \\
 & fu &
 \end{array}$$

 \Rightarrow

$$\begin{array}{ccc}
 & fufu & \\
 f\eta u \nearrow & \mu & \searrow \epsilon * \epsilon \\
 fu & \epsilon & \longrightarrow \text{id}_A \\
 \Downarrow & \epsilon & \nearrow \epsilon \\
 & fu &
 \end{array}$$

Existence of homotopy coherent adjunctions



Theorem (R-Verity). Any adjunction in the homotopy 2-category of an ∞ -cosmos extends to a homotopy coherent adjunction.

Proof: Given adjunction data

- $u: A \rightarrow B$ and $f: B \rightarrow A$
- $\eta: \text{id}_B \Rightarrow uf$ and $\epsilon: fu \Rightarrow \text{id}_A$
- α witnessing $u\epsilon \circ \eta u = \text{id}_u$ and β witnessing $\epsilon f \circ f\eta = \text{id}_f$

forget to either

- (f, u, η) or
- $(f, u, \eta, \epsilon, \alpha)$

and use the universal property of the unit η to extend all the way up.

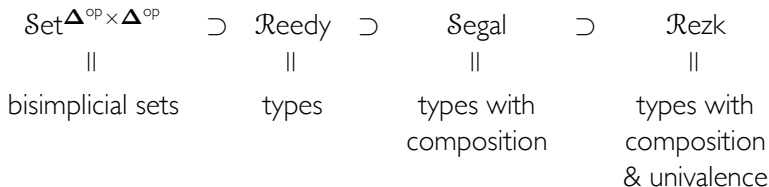
Theorem (R-Verity). Moreover, the spaces of extensions from the data (f, u, η) or $(f, u, \eta, \epsilon, \alpha)$ are contractible Kan complexes.



3

The synthetic theory of ∞ -categories
in homotopy type theory

The intended model



Theorem (Shulman). Homotopy type theory is modeled by the category of **Reedy fibrant** bisimplicial sets.

Theorem (Rezk). $(\infty, 1)$ -categories are modeled by **Rezk spaces** aka complete Segal spaces.

The bisimplicial sets model of homotopy type theory has:

- an interval type I , parametrizing **paths** inside a general type
- a directed interval type $\mathbb{2}$, parametrizing **arrows** inside a general type

Paths and arrows



- The **identity type** for A depends on two terms in A :

$$x, y : A \vdash x =_A y$$

and a term $p : x =_A y$ defines a **path** in A from x to y .

- The **hom type** for A depends on two terms in A :

$$x, y : A \vdash \mathbf{hom}_A(x, y)$$

and a term $f : \mathbf{hom}_A(x, y)$ defines an **arrow** in A from x to y .

Hom types are defined as instances of **extension types** axiomatized in a three-layered type theory with (simplicial) shapes due to Shulman

$$\mathbf{hom}_A(x, y) := \left\langle \begin{array}{ccc} 1 + 1 & \xrightarrow{[x, y]} & A \\ \downarrow & \nearrow & \\ 2 & & \end{array} \right\rangle$$

Semantically, **hom types** $\sum_{x, y: A} \mathbf{hom}_A(x, y)$ recover the ∞ -category of arrows $\mathbf{hom}_A \rightarrow A \times A$ in the ∞ -cosmos \mathbf{Rezk} .

Segal, Rezk, and discrete types



- A type A is **Segal** if every composable pair of arrows has a unique composite: if for every $f : \mathbf{hom}_A(x, y)$ and $g : \mathbf{hom}_A(y, z)$

$$\left\langle \begin{array}{ccc} \Lambda_1^2 & \xrightarrow{[f,g]} & A \\ \Downarrow & \dashrightarrow & \uparrow \\ \Delta^2 & & \end{array} \right\rangle \quad \text{is contractible.}$$

- A Segal type A is **Rezk** if every isomorphism is an identity: if

$$\text{id-to-iso} : \prod_{x,y:A} (x =_A y) \rightarrow (x \cong_A y) \quad \text{is an equivalence.}$$

- A type A is **discrete** if every arrow is an identity: if

$$\text{id-to-arr} : \prod_{x,y:A} (x =_A y) \rightarrow \mathbf{hom}_A(x, y) \quad \text{is an equivalence.}$$

Prop. A type is discrete if and only if it is Rezk and all of its arrows are isomorphisms — the discrete types are the ∞ -groupoids.

The 2-category of Segal types



Prop (R-Shulman).

- Any function $f: A \rightarrow B$ between Segal types preserves identities and composition. Moreover, the type $A \rightarrow B$ of functors is again a Segal type.
- Given functors $f, g: A \rightarrow B$ between Segal types there is an equivalence

$$\mathbf{hom}_{A \rightarrow B}(f, g) \xrightarrow{\sim} \prod_{a:A} \mathbf{hom}_B(fa, ga)$$

- Terms $\gamma: \mathbf{hom}_{A \rightarrow B}(f, g)$, called natural transformations, are natural and can be composed vertically and horizontally up to typal equality.

Incoherent adjunction data



A quasi-diagrammatic adjunction between types A and B consists of

- functors $u: A \rightarrow B$ and $f: B \rightarrow A$
- natural transformations $\eta: \mathbf{hom}_{B \rightarrow B}(\text{id}_B, uf)$, $\epsilon: \mathbf{hom}_{A \rightarrow A}(fu, \text{id}_A)$
- witnesses $\alpha: u\epsilon \circ \eta u = \text{id}_u$ and $\beta: \epsilon f \circ f\eta = \text{id}_f$

A (quasi*-)transposing adjunction between types A and B consists of functors $u: A \rightarrow B$ and $f: B \rightarrow A$ and a family of equivalences

$$\prod_{a:A, b:B} \mathbf{hom}_A(fb, a) \simeq \mathbf{hom}_B(b, ua)$$

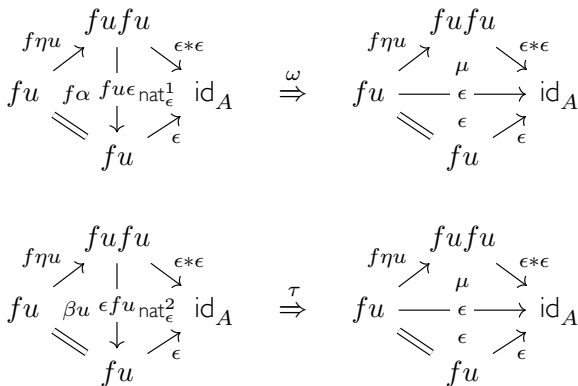
(*together with their quasi-inverses and the witnessing homotopies).

Theorem(R-Shulman). Given functors $u: A \rightarrow B$ and $f: B \rightarrow A$ between Segal types the type of quasi-transposing adjunctions $f \dashv u$ is equivalent to the type of quasi-diagrammatic adjunctions $f \dashv u$.

Coherent adjunction data



A half-adjoint diagrammatic adjunction consists of:



Theorem (R-Shulman). Given functors $u: A \rightarrow B$ and $f: B \rightarrow A$ between Segal types the type of transposing adjunctions $f \dashv u$ is equivalent to the type of half-adjoint diagrammatic adjunctions $f \dashv u$.

Uniqueness of coherent adjunction data



If $\eta: \mathbf{hom}_{B \rightarrow B}(\text{id}_B, uf)$ is a unit, then that adjunction is uniquely determined:

Theorem (R-Shulman). Given Segal types A and B , functors $u: A \rightarrow B$ and $f: B \rightarrow A$, and a natural transformation $\eta: \mathbf{hom}_{B \rightarrow B}(\text{id}_B, uf)$ the following are equivalent propositions:

- The type of $(\epsilon, \alpha, \beta, \mu, \omega, \tau)$ extending (f, u, η) to a half-adjoint diagrammatic adjunction.
- The propositional truncation of the type of $(\epsilon, \alpha, \beta)$ extending (f, u, η) to a quasi-diagrammatic adjunction.

Theorem (R-Shulman). Given the data $(f, u, \eta, \epsilon, \alpha)$ as in a quasi-diagrammatic adjunction, the following are equivalent propositions:

- The type of $(\beta, \mu, \omega, \tau)$ extending this data to a half-adjoint diagrammatic adjunction.
- The propositional truncation of the type of β extending this data to a quasi-diagrammatic adjunction.

Where does Rezk-completeness come in?



For **Rezk types** — the synthetic ∞ -categories — adjoints are literally unique, not just “unique up to isomorphism”:

Theorem (R-Shulman). Given a Segal type B , a Rezk type A , and a functor $u: A \rightarrow B$, the following types are equivalent propositions:

- The type of transposing left adjoints of u .
- The type of half-adjoint diagrammatic left adjoints of u .
- The propositional truncation of the type of quasi-diagrammatic left adjoints of u .



4

Discussion



- In an ∞ -cosmos, we prove that **there exists** a quasi-diagrammatic adjunction if and only if there exists a quasi-transposing adjunction. In simplicial HoTT, we prove the **types** of such are equivalent, which conveys more information (though I'm not exactly sure what).
- The ∞ -cosmos **Rezk** does not see Segal or ordinary types — because we've axiomatized the fibrant objects, rather than the full model category.
- It seems to be much easier to produce an ∞ -cosmos than to define a model of simplicial HoTT.
- But overall the experiences of working with either approach to the synthetic theory of ∞ -categories are strikingly similar — and I'm not sure I entirely understand why that is.

References



For more on homotopical trinitarianism, see:

Michael Shulman

- [Homotopical trinitarianism: a perspective on homotopy type theory](http://home.sandiego.edu/~shulman/papers/trinity.pdf), home.sandiego.edu/~shulman/papers/trinity.pdf

For more on the synthetic theory of ∞ -categories, see:

Emily Riehl and Dominic Verity

- [∞-category theory from scratch](https://arxiv.org/abs/1608.05314), arXiv:1608.05314
- [∞-Categories for the Working Mathematician](http://www.math.jhu.edu/~eriehl/ICWM.pdf), www.math.jhu.edu/~eriehl/ICWM.pdf

Emily Riehl and Michael Shulman

- [A type theory for synthetic ∞-categories](#), Higher Structures 1(1):116–193, 2017; arXiv:1705.07442

Thank you!