# Algebraic Topology in an Elementary Higher Topos

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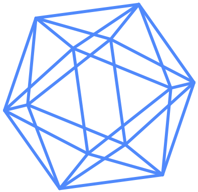

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# Intuition

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An **elementary higher topos** is an $(\infty, 1)$-category ...

1. ... that behaves like the $(\infty, 1)$-category of spaces.
2. ... that should be a model for homotopy type theory.
3. ... generalizes an elementary topos from classical category theory.

# Connections to Homotopy Type Theory ...

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1. An Elementary Higher Topos should be a model for Homotopy Type Theory.
2. Homotopy Type Theory should be the internal language of an Elementary Higher Topos.

# ... but we are not there yet

However, we still don't know how to make this argument precise as it is tricky to go from the **strictness** of a **type theory** to the **flexibility** of an $(\infty, 1)$-**category**.
We don't have such a general result, but we can focus on specific results in homotopy type theory and how they relate to elementary higher toposes.

# $(\infty, 1)$-Category Theory

An $(\infty, 1)$-category $\mathcal{C}$ has following properties:

1. It has **objects** $x, y, z, \ldots$
2. For any two objects $x, y$ there is a **mapping space** (Kan complex) $map_{\mathcal{C}}(x, y)$ with a notion of composition that holds only "up to homotopy".
3. Mapping spaces give us **homotopic maps** and **equivalences**.

   **Homotopy:** Two maps $f, g : x \to y$ are homotopic if they are homotopic in the space $map_{\mathcal{C}}(x, y)$.

   **Equivalence:** A map $f$ is an equivalence if there exist $g, h$ such that $fg$ and $hf$ are homotopic to identity maps.
4. This is a direct generalization of classical categories and isomorphisms and all categorical notions (limits, adjunction, ...) generalize to this setting.

# Core of an $(\infty, 1)$-category

## Notation

We denote the subcategory of equivalences by $\mathcal{C}^{core}$.

$\mathcal{C}^{core}$ is an $(\infty, 1)$-groupoid, which is an $(\infty, 0)$-category. An $(\infty, 0)$-category is a space, where we have:

1. Points in the space are the objects.
2. Paths in the space are the morphisms
3. 2-cells are homotopies
4. ...

# Examples

1. Spaces form an $(\infty, 1)$-category which we denote by $\mathcal{S}$paces.
2. For a cardinal $\kappa$, we denote the sub-category of $\kappa$-small spaces as $\mathcal{S}$paces$^\kappa$. Notice in this case $(\mathcal{S}\text{paces}^\kappa)^{core}$ is a space that is NOT $\kappa$-small.
3. $(\infty, 1)$-categories form a **large** $(\infty, 1)$-category denoted by $\mathcal{C}\text{at}_\infty$.
4. Classical categories are all $(\infty, 1)$-categories, in particular $\mathcal{S}$et is an $(\infty, 1)$-category.

# Subobjects

## Definition

Let $\mathcal{C}$ have finite limits. There is a functor

$$Sub(-) : \mathcal{C}^{op} \to \mathcal{S}et$$

that takes each object $c$ to the set of equivalence classes of subobjects of $c$ (mono maps into $c$).

## Example

In $\mathcal{S}$paces the subobjects are exactly the $(-1)$-*truncated maps* i.e. maps $f : X \to Y$ such that the following map is an equivalence

$$X \xrightarrow{\simeq} X \underset{Y}{\times} X$$

(alternatively $f$ is an equivalence on path-components).

# Subobject Classifiers

## Definition

A *subobject classifier* in $\mathcal{C}$ is an object $\Omega$ representing $Sub(-)$. So for each object $c$ we have

$$Sub(c) \cong map_{\mathcal{C}}(c, \Omega)$$

In particular, there is a *universal subobject* $t : 1 \to \Omega$ and for each subobject $i : c' \to c$ there is a pullback square

$$\begin{tikzcd} c' \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "t"] \\ c \arrow[r, "\chi_i"'] & \Omega \end{tikzcd}$$

## Example

In $\mathcal{S}$paces we have $\Omega = \{0, 1\}$.

# Generalizing Subobject classifiers

**Definition**

Let $\mathcal{C}$ have finite limits. There is a map

$$\mathcal{C}_{/-} : \mathcal{C}^{op} \to \mathcal{C}at_\infty$$

taking an object $c$ to the over-category $\mathcal{C}_{/c}$.

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## Definition

Let $S$ be a suitable subclass of maps. We define

$$((\mathcal{C}_{/-})^S)^{core} : \mathcal{C}^{op} \to \mathcal{C}at_\infty \xrightarrow{core} \mathcal{S}paces$$

taking an object $c$ to the space $((\mathcal{C}_{/c})^S)^{core}$, the full subspace of $(\mathcal{C}_{/c})^{core}$ generated by elements in $S$.

# How does $(\mathcal{C}_{/c})^{core}$ look like?

1. **Points (0-Simplices):**

$$\begin{tikzcd} & d \arrow[dl, "f \in S"] \\ c & \end{tikzcd}$$

2. **Paths (1-Simplices):**

$$\begin{tikzcd} d' \arrow[rr, "\simeq"] \arrow[dr, "f' \in S"'] & & d \arrow[dl, "f \in S"] \\ & c & \end{tikzcd}$$

3. **Two-Simplices:**

$$\begin{tikzcd} & d'' \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] \arrow[dd, dashed, "f'' \in S"] & \\ d' \arrow[rr, "\simeq"'] \arrow[dr, "f' \in S"'] & & d \arrow[dl, "f \in S"] \\ & c & \end{tikzcd}$$

# Universes

## Definition

Let $\mathcal{C}$ have finite limits and let $S$ be a subclass of maps. An object $\mathcal{U}^S$ is called a *universe* if it represents $(\mathcal{C}^S_{/-})^{core}$.

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## Example

Let $\kappa$ be a suitably large cardinal and $S$ the class of maps with $\kappa$-small fiber. Then

$$\mathcal{U}^\kappa = (\mathcal{S}\text{paces}^\kappa)^{core},$$

$$((\mathcal{S}\text{paces}_{/X})^\kappa)^{core} \simeq map_{\mathcal{S}\text{paces}}(X, \mathcal{U}^\kappa)$$

where $X$ is a space.

# Universal Fibration

For a universe $\mathcal{U}^S$ there is a universal fibration $p : \mathcal{U}^S_* \to \mathcal{U}^S$ such that for each map $f : Y \to X$ in $S$ there is a pullback square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}^S_* \arrow[d, "p"] \\
X \arrow[r] & \mathcal{U}^S
\end{tikzcd}$$

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## Example

Continuing our previous example in spaces we have

$$\mathcal{U}^\kappa_* = (\mathcal{S}\text{paces}^\kappa_*)^{core}$$

with $p : (\mathcal{S}\text{paces}^\kappa_*)^{core} \to (\mathcal{S}\text{paces}^\kappa)^{core}$ the forgetful map.

# Example: Objects via Universal Fibrations

A map $f_X : * \to \mathcal{S}paces^{\kappa}$ corresponds to a choice of $\kappa$-small space $X$. We can pull back the universal fibration along $f_X$:

$$\begin{tikzcd} X \simeq map(*, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathcal{S}paces^{\kappa}_{*})^{core} \arrow[d, "p"] \\ * \arrow[r, "f_X"'] & (\mathcal{S}paces^{\kappa})^{core} \end{tikzcd}$$

# Elementary Higher Topos

## Definition

We call an $(\infty, 1)$-category $\mathcal{E}$ an *elementary higher topos* if it satisfies following conditions:

1. It has finite limits and colimits.
2. It has a subobject classifier.
3. It is locally Cartesian closed.
4. There exists a chain of universes $\{\mathcal{U}^S\}$ such that each map is classified by a universe.

# Examples

## Example

As we have shown $\mathcal{S}$paces is an elementary higher topos.

## Example (Theorem 6.1.6.8, Higher Topos Theory, Lurie)

More generally every **higher topos** $\mathcal{X}$ is an elementary higher topos. This follows from the following two facts:

1. For a large enough cardinal $\kappa$ the descent condition implies that the functor:

$$((\mathcal{X}_{/-})^{\kappa})^{core} : \mathcal{X}^{op} \to \mathcal{S}\text{paces}$$

   takes colimits to limits.

2. The presentability implies that any such functor is represented by an object $\mathcal{U}^{\kappa}$.

# Towards Algebraic Topology

For the rest of the talk we want to see what kind of algebraic topological concepts we can prove in an arbitrary elementary higher topos (EHT).

# Definition of an NNO

## Definition

Let $\mathcal{E}$ be an EHT. A **natural number object** is an object $\mathbb{N}$ along with two maps $o : 1 \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ such that for all $(X, b, u)$

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "s"] \arrow[d, dashed, "\exists! f"] & \mathbb{N} \arrow[d, dashed, "\exists! f"] \\ 1 \arrow[ur, "o"] \arrow[dr, "b"'] & & \\ & X \arrow[r, "u"] & X \end{tikzcd}$$

the space of maps $f$ making the diagram commute is contractible.

# NNOs and Logical Constuctions

We can use NNOs in the classical setting to prove infinite results without assuming the existence of infinite colimits.

**Theorem (Theorem D5.3.5, Sketches of an Elephant, Johnstone)**

*Let $\mathcal{E}$ be an* **elementary** 1-**topos** *with natural number object. Then we can construct free finitary algebras (monoids, groups, ...).*

In an elementary topos the existence of an NNO is not a vacuous condition as there are elementary toposes without NNOs (such as finite sets).

# NNOs in an EHT

In the higher setting things are different

**Theorem**

*Every EHT has a natural number object.*

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**Theorem**

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The idea is to build an infinite object out of a finite one.

# Idea of Proof I

We use the fact from algebraic topology that $\pi_1(S^1) = \mathbb{Z}$.

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We use the fact from algebraic topology that $\pi_1(S^1) = \mathbb{Z}$.
We can take a coequalizer (in the EHT)

$$1 \underset{id}{\overset{id}{\rightrightarrows}} 1 \longrightarrow S^1$$

The object $S^1$ behaves similar to the circle in spaces. In particular we can take it's loop object.

# Idea of Proof II

$$\begin{tikzcd}
\Omega S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d] \\
1 \arrow[r] & S^1
\end{tikzcd}$$

$\Omega S^1$ behaves similar to the classical loop space of the circle: It comes with an automorphism $s : \Omega S^1 \to \Omega S^1$ and a map $o : 1 \to \Omega S^1$.

The smallest subobject of $\Omega S^1$ closed under $s$ and $o$ is immediately an NNO in the classical setting (the elementary topos of 0-truncated objects). Finally, we use an argument by Shulman to prove it is an NNO in the higher setting.

# The Three Aspects of an Elementary Higher Topos

It is interesting to observe how all three aspects of a topos are used in this proof:

1. First we use our knowledge of **spaces** to build the object $\Omega S^1$.
2. Then we use our knowledge of **elementary toposes** to show we have an NNO in the underlying elementary topos.
3. Finally, we use **homotopy type theory** to prove it is an NNO in the actual elementary higher topos.

# Sequential Diagrams

Using natural number objects, we can define sequential colimits in a topos. This is based on work of Rijke in homotopy type theory.

## Definition

A sequence of objects is a map $\{A_n\}_{n:\mathbb{N}} : \mathbb{N} \to \mathcal{U}$. This is equivalent to a map $p : \sum_{n:\mathbb{N}} A_n \to \mathbb{N}$.

## Definition

A sequential diagram

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$$

is a choice of map $\{f_n\}_{n:\mathbb{N}} : \sum_{n:\mathbb{N}} A_n \to \sum_{n:\mathbb{N}} A_{n+1}$ over $\mathbb{N}$.

# Sequential Colimits

## Definition

For a sequential diagram $\{(A_n, f_n)\}_{n:\mathbb{N}}$ we define the sequential colimit $A_\infty$ as the coequalizer

$$\sum_{n:\mathbb{N}} A_n \underset{id}{\overset{\{f_n\}_{n:\mathbb{N}}}{\rightrightarrows}} \sum_{n:\mathbb{N}} A_n \longrightarrow A_\infty$$

We will use sequential colimits to construct truncations.

# Truncated Objects

## Definition

A space $X$ is $n$-truncated if $\pi_k(X) = *$ for all $k > n$.

## Definition

An object $X$ in a $(\infty, 1)$-category is $n$-truncated if $map(Y, X)$ is an $n$-truncated space for all $Y$.

# Truncations

One amazing feature of spaces is the existence of truncations. We can take any space X and universally construct a truncated object $\tau_n X$.

**Theorem**

*There exists an adjunction*

$$\mathcal{S}paces \underset{i}{\overset{\tau_n}{\rightleftarrows}} \tau_n \mathcal{S}paces$$

*where $i$ is the inclusion.*

The original proof is via the *small object argument*, which does not generalize.

# Truncations in an Elementary Higher Topos

However, there is an alternative approach with the same result:

**Theorem**

*Let $\mathcal{E}$ be an EHT. Then there exists an adjunction*

$$\mathcal{E} \underset{i}{\overset{\tau_n}{\rightleftarrows}} \tau_n\mathcal{E}$$

*where $i$ is the inclusion.*

The idea for the proof comes from work of Egbert Rijke in the context of homotopy type theory.

# $(-1)$-Truncations

There are two steps

## Proposition

*For any object $A$, the sequential colimit of the diagram*

$$A \xrightarrow{inl} A * A \xrightarrow{inl} (A * A) * A \xrightarrow{inl} \ldots$$

*is the $(-1)$-truncation. Here $A * B = A \coprod_{A \times B} B$.*

## Remark

This result is the $(\infty, 1)$-categorical analogue of Theorem 3.3 of *The Join Construction* by Rijke. But, the proof uses a different approach. It is not completely clear how to translate the proofs from higher category theory to homotopy type theory (and vice versa).

# $n$-Truncations in Spaces

We can generalize the previous result to hold for any $n$ via induction. Here is the idea: In order to $(n+1)$-truncate a space $X$, it suffices to $n$-truncate all the loop spaces $\Omega_x X$ for each basepoint $x$ in $X$.

# $n$-Truncations in Spaces

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$$
\begin{tikzcd}
 & \tau_{n+1}(X) \arrow[dr, hook, bend left] & \\
X \arrow[ur, two heads, bend left] \arrow[r, hook] & (\mathcal{S}\text{paces}^{core})^X \arrow[r] & (\tau_n \mathcal{S}\text{paces}^{core})^X \\
x \arrow[r, mapsto] & Path(-,x) \arrow[r, mapsto] & \tau_n(Path(-,x))
\end{tikzcd}
$$

# $n$-Truncations in an Elementary Higher Topos

This argument directly generalizes to an elementary higher topos.

$$\begin{tikzcd} & \tau_{n+1}(X) \arrow[dr, hook, bend left] & \\ X \arrow[ur, two heads, bend left] \arrow[r, hook] & \mathcal{U}^X \arrow[r] & (\tau_n \mathcal{U})^X \end{tikzcd}$$

where $X \to \mathcal{U}^X$ is the map classifying $\Delta_X : X \to X \times X$.

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where $X \to \mathcal{U}^X$ is the map classifying $\Delta_X : X \to X \times X$.

**Remark**

Again, the proof has some similarities with the proof in homotopy type theory, but also some differences.

# The classical Blakers-Massey Theorem

One fascinating result in algebraic topology is the Blakers-Massey theorem.

## Theorem

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] \arrow[ur, phantom, "\ulcorner"] & W \end{tikzcd}$$

*Let the above be a pushout diagram in $\mathcal{S}paces$, such that $f$ is $m$-connected and $g$ is $n$-connected. Then the map $(f, g) : Z \to X \times_W Y$ is $(m + n)$-connected.*

We want to show that this result holds in an elementary higher topos as well.

# Functorial Factorization

## Definition

A **functorial factorization** is a choice of functors on the arrow categories $\mathcal{L} : \mathcal{E}^{\rightarrow} \to \mathcal{E}^{\rightarrow}$ and $\mathcal{R} : \mathcal{E}^{\rightarrow} \to \mathcal{E}^{\rightarrow}$ such that $f \simeq \mathcal{R}(f) \circ \mathcal{L}(f)$.

We call maps in the image of $\mathcal{L}$ the **left class** and in the image of $\mathcal{R}$ the **right class**.

# Modalities

## Definition

A **factorization system** is a functorial factorization such that the space of lifts of the following diagram is contractible

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r] \arrow[ur, dashed] & W \end{tikzcd}$$

where $f$ is in the left class and $g$ in the right class.

## Definition

A **modality** is a factorization system such that the left class is closed under base change.

# Blakers-Massey Theorem for Modalities in a Higher Topos

Fortunately, we have following general result:

**Theorem (Theorem 4.1.1 of Generalized Blakers-Massey Theorem by Anel, Biedermann, Finster and Joyal)**

*Let $\mathcal{E}$ be a presentable elementary higher topos. Let*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] & W \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

*be a pushout diagram such that the pushout product $\Delta f \square_Z \Delta g$ is in $\mathcal{L}$. Then the map $(f, g) : Z \to X \times_W Y$ is in $\mathcal{L}$ as well.*

# Classical Blakers-Massey Theorem in a Higher Topos

The class of $n$-truncated and $n$-connected maps form a modality, with $\mathcal{L}$ being $n$-connected maps and $\mathcal{R}$ the $n$-truncated maps. Combining the previous two results we get:

**Corollary**

*The classical Blakers-Massey theorem holds in a presentable elementary higher topos.*

# Blakers-Massey in an Elementary Higher Topos

Doubly fortunately, the proof of their theorem does not actually require the presentability condition.

## Theorem

*The generalized (and thus also the classical) Blakers-Massey theorem holds in an elementary higher topos.*

# Blakers-Massey in an Elementary Higher Topos

Doubly fortunately, the proof of their theorem does not actually require the presentability condition.

## Theorem

*The generalized (and thus also the classical) Blakers-Massey theorem holds in an elementary higher topos.*

## Remark

In a **presentable** elementary higher topos we can construct factorization systems out of sets of maps, which allows us to build new modalities. However, in an elementary higher topos, we cannot do that and we have to assume their existence to be able to use Blakers-Massey theorem.

# Further Algebraic Topology in an Elementary Higher Topos

What else can we do? Here are some further topics related to algebraic topology that can be studied in an EHT:

1. We have truncations and spheres, which means we can define homotopy groups. How does the homotopy groups of spheres compare with the classical homotopy groups?
2. Blakers-Massey gives us Freudenthal suspension theorem, which means we have stabilizations. How does the stablization compare to spectra?
3. Can we construct Eilenberg-MacLane objects in an elementary higher topos?