Axiomatizing Cubical Sets Models of Univalent Foundations

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Joint work with Ian Orton, Dan Licata & Bas Spitters

HoTT from the outside in

Why do I study models of univalent type theory?
(instead of just developing univalent foundations)

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- univalence
  as a concept, as opposed to a particular formal axiom, and its relation to other foundational concepts & axioms
- theorem-provers with user-defined
  higher inductive types
  from models to new type theories

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  from models to new type theories

This talk concentrates on the first point, but the second one is probably of more importance in the long term.

Neither point is directly motivated by applications to algebraic topology.

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(instead of just developing univalent foundations)

- univalence
- theorem-provers with user-defined higher inductive types

Wanted:

- simpler proofs of univalence for existing models
- new models
- [better understanding of HITs]

HoTT from the outside in

Some possible approaches to existing models:

- Direct calculations in set/type theory with presheaves
  [wood from the trees]
- Categorical algebra (theory of model categories)
  [strictness issues]

HoTT from the outside in

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- Direct calculations in set/type theory with presheaves.
- Categorical algebra (theory of model categories).
- Here: **categorical logic**

  In a version of type theory interpretable in any elementary topos with countably many universes $\Omega : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \cdots$, there are

  elementary axioms for $\begin{cases} \text{interval object } 0, 1 : 1 \rightrightarrows \mathbb{I} \\ \text{cofibrant propositions } \mathbf{Cof} \rightarrowtail \Omega \end{cases}$

  that suffice for a version of the model of univalence of **Coquand *et al*.**

CCHM Univalent Universe

C. Cohen, T. Coquand, S. Huber and A. Mörtberg, *Cubical type theory: a constructive interpretation of the univalence axiom* [arXiv:1611.02108]

Uses categories-with-families (CwF) semantics of type theory for the CwF associated with presheaf topos

$$\mathcal{E} = \mathbf{Set}^{\square^{\mathrm{op}}}$$

where $\square$ is the Lawvere theory of De Morgan algebras.

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Topos theory background

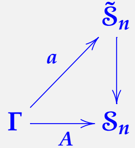
Elementary topos $\mathcal{E}$ = cartesian closed category with subobject classifier Ω (& natural number object)

Toposes are the category-theoretic version of theories in extensional impredicative higher-order intuitionistic predicate calculus.

Topos theory background

Elementary topos $\mathcal{E}$ = cartesian closed category with subobject classifier Ω (& natural number object)
& universes $\Omega : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \cdots$

Can make a category-with-families (CwF) out of $\mathcal{E}$ and soundly interpret [a form of] Extensional MLTT in it

Type Theory		CwF $\mathcal{E}$
context	Γ	object Γ
type (of size n) in context	$\Gamma \vdash_n A$	morphism $\Gamma \xrightarrow{A} \mathcal{S}_n$
typed term in context	$\Gamma \vdash a : A$	section 
judgemental equality	$\Gamma \vdash a = a' : A$	equality of morphisms
extensional identity types		cartesian diagonals

Axiomatic CCHM

Starting with any topos $\mathcal{E}$ satisfying some

axioms for $\begin{cases} \text{interval object } 0, 1 : \mathbf{1} \rightrightarrows \mathbb{I} \\ \text{cofibrant propositions } \mathbf{Cof} \rightarrowtail \Omega \end{cases}$

one gets a model of MLTT + univalence

by building a new CwF $\mathcal{F}$ out of $\mathcal{E}$:

- objects of $\mathcal{F}$ are the objects of $\mathcal{E}$
- families in $\mathcal{F}$: $\mathcal{F}_n(\Gamma) \triangleq \sum_{A:\Gamma\to\mathcal{S}_n} \mathbf{Fib}_n\, A$ where
  $\mathbf{Fib}_n\, A$ = set of CCHM fibration structures on $A : \Gamma \to \mathcal{S}_n$
- elements of $(A, \alpha) \in \mathcal{F}_n(\Gamma)$ are elements of A in $\mathcal{E}$

CCHM fibrations

Path functor: $\wp\Gamma \triangleq \mathbb{I} \to \Gamma$ (type of functions from $\mathbb{I}$ to Γ)

Extension relation: we identify each cofibrant proposition $\varphi : \mathbf{Cof}$ with the corresponding subterminal $\varphi \rightarrowtail \mathbf{1}$. For each function $f : \varphi \to \Gamma$ (partial element of Γ with domain φ) and each $x : \Gamma$, define

$$f \nearrow x \triangleq \forall u : \varphi,\ f\,u = x$$

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Fix on one of the universes $\mathcal{S} = \mathcal{S}_n$ in $\mathcal{E}$

Type of composition structures for a path of types $A : \wp\,\mathcal{S}$

$$\begin{aligned}\mathsf{Comp}\,A \triangleq\ & (\varphi : \mathsf{Cof})(f : (i : \mathbb{I}) \to \varphi \to A\,i) \to \\ & (\textstyle\sum a : A\,0,\ f\,0 \nearrow a) \to (\textstyle\sum a : A\,1,\ f\,1 \nearrow a)\end{aligned}$$

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Type of fibration structures for a family of types $A : \Gamma \to \mathcal{S}$

$$\mathbf{Fib}\, A \triangleq (p : \wp\Gamma) \to \mathbf{Comp}(A \circ p)$$

(Compare this with the direct, presheaf definition.)

CCHM Fibration structure

Type of composition structures for a path of types $A : \wp\,\mathcal{S}$

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Some simple properties of $\mathbb{I}$ and **Cof** enable one to prove that the existence of fibration structure is preserved under forming Σ-types, Π-types, (propositional) identity types,. . .

What about universes of fibrations?

We get them via "tinyness" of the interval. . .

Tiny interval

$\mathbb{I} \in \mathcal{E}$ is tiny if $(\_)^{\mathbb{I}}$ has a right adjoint $\surd(\_)$

$$\frac{\Gamma^{\mathbb{I}} \to \Delta}{\Gamma \to \surd\Delta} \quad \text{(natural bijection)}$$

preserving universe levels: $\Delta : \mathcal{S}_n \Rightarrow \surd\Delta : \mathcal{S}_n$

(notion goes back to Lawvere's work in synthetic differential geometry)

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When $\mathcal{E} = \mathbf{Set}^{\square^{\mathrm{op}}}$, the topos of cubical sets, the category $\square$ has finite products and the interval in $\mathcal{E}$ is representable: $\mathbb{I} = \square(\_, I)$.

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Hence the path functor $(\_)^{\mathbb{I}} : \mathbf{Set}^{\square^{\mathrm{op}}} \to \mathbf{Set}^{\square^{\mathrm{op}}}$ is $(\_ \times I)^*$

and so $(\_)^{\mathbb{I}}$ not only has a left adjoint $(\_ \times \mathbb{I})$, but also a right adjoint, given by right Kan extension (and hence preserving universe levels).

Tiny interval

Recall $\mathcal{F}_n(\Gamma) \triangleq \sum_{A : \Gamma \to \mathcal{S}_n} \mathsf{Fib}_n\, A$ = set of CCHM fibrations over an object $\Gamma \in \mathcal{E}$. This is functorial in Γ.

Theorem. If interval $\mathbb{I}$ is tiny, then $\mathcal{F}_n(\_) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ is representable:

$$\mathcal{U}_n \qquad\qquad (\mathsf{E}, \nu) \in \mathcal{F}_n(\mathcal{U}_n)$$

object — generic fibration

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Γ	(A, α)	$\in \mathfrak{F}_n(\Gamma)$
$\mathcal{U}_n$	(E, v)	$\in \mathfrak{F}_n(\mathcal{U}_n)$
object	generic fibration	

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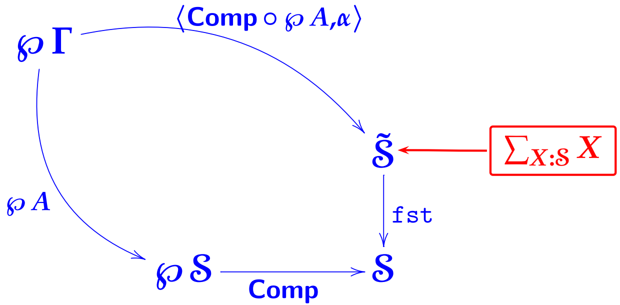
$$\begin{array}{ccccc}
\Gamma & & (A,\alpha) & \in & \mathcal{F}_n(\Gamma) \\
{\scriptstyle \exists!}\downarrow{\scriptstyle \ulcorner A,\alpha \urcorner} & & \uparrow & & \\
\mathcal{U}_n & & (\mathsf{E},\upsilon) & \in & \mathcal{F}_n(\mathcal{U}_n) \\
\text{object} & & \text{generic fibration} & &
\end{array}$$

Proof in **Licata-Orton-AMP-Spitters** FSCD 2018 [arXiv:1801.07664] generalizes unpublished work of **Coquand & Sattler** for the case $\mathcal{E}$ is a presheaf topos.

$$\Phi = (A, \alpha) \in \mathcal{F}(\Gamma) \cong \mathcal{E}(1, \sum_{A:\Gamma \to \mathcal{S}} \prod_{p:\wp\,\Gamma} (\textsf{Comp} \circ \wp\, A)\, p)$$



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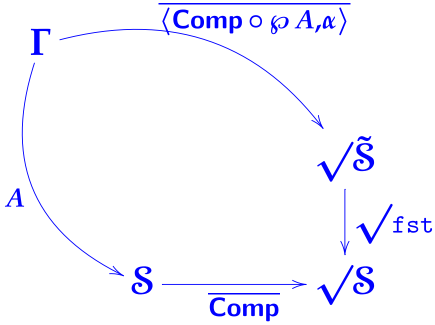

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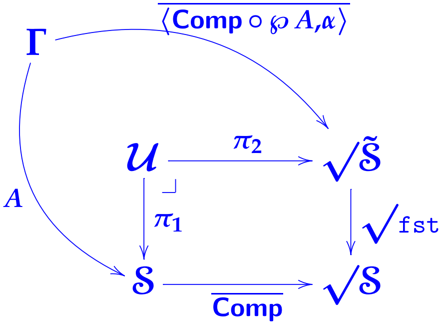

$\mathcal{U} \triangleq$ pullback of $\overline{\mathbf{Comp}}$ and $\sqrt{\mathtt{fst}}$

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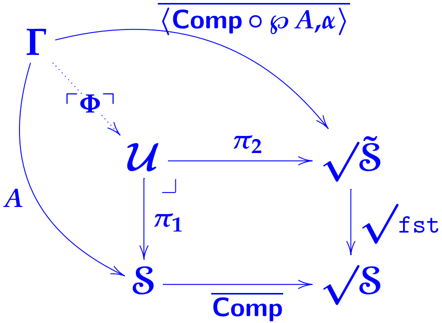

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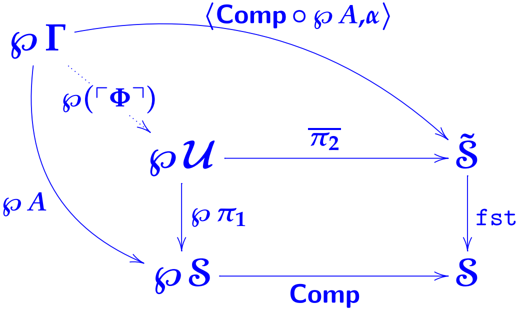

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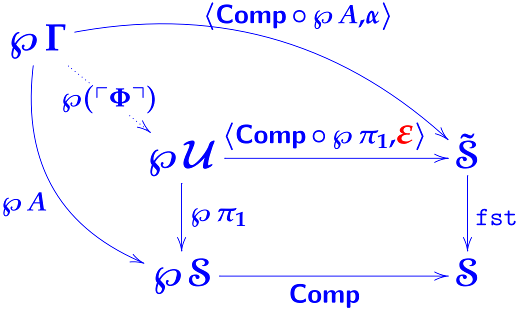

generic fibration $\mathbf{E} \triangleq (\mathcal{U} \xrightarrow{\pi_1} \mathcal{S}, \varepsilon)$

uniqueness of $\ulcorner\mathbf{\Phi}\urcorner$ follows from universal property of the pullback

Tiny interval

Theorem. The universes $(\mathcal{U}_n, \mathsf{E})$ of CCHM fibrations are closed under Π-types, propositional identity types and inductive types (e.g. Σ) if $\mathbb{I}$ has a weak form of binary minimum ("connection" structure) and Cof satisfies

$$\mathsf{false} \in \mathsf{Cof}$$
$$(\forall i, \varphi)\ \varphi \in \mathsf{Cof} \;\Rightarrow\; \varphi \vee i = 0 \in \mathsf{Cof}$$
$$(\forall i, \varphi)\ \varphi \in \mathsf{Cof} \;\Rightarrow\; \varphi \vee i = 1 \in \mathsf{Cof}$$

What about univalence of $(\mathcal{U}_n, \mathsf{E})$?

Univalence

Theorem. For any topos $\mathcal{E}$ with tiny $\mathbb{I}$ & **Cof** satisfying assumptions so far, there is a term of type

$$\prod_{u:\mathcal{U}_n} \mathsf{isContr}\left(\sum_{v:\mathcal{U}_n} (\mathsf{E}u \simeq \mathsf{E}v)\right)$$

if **Cof** is closed under $\forall i : \mathbb{I}$ and satisfies the isomorphism extension axiom:

$$\mathsf{iea} : \prod_{A:\mathcal{S}_n} \mathsf{Ext}\left(\sum_{B:\mathcal{S}_n} (A \cong B)\right)$$

In this case $\mathcal{U}_n$ is a fibration (over $\mathbf{1}$) and $(\mathcal{U}_n, \mathsf{E})$ is univalent.

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equivalent to the usual univalence axiom (given suitable properties of $\mathcal{U}_n$)

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$$\begin{aligned}
\mathbf{isContr}\, A &\triangleq \sum_{x:A} \prod_{x':A} (x \sim x') \\
x \sim x' &\triangleq \sum_{p:\mathbb{I}\to A} (p\,0 \equiv x \wedge p\,1 \equiv x') \\
\mathbf{Ext}\, A &\triangleq \prod_{\varphi:\mathbf{Cof}} \prod_{f:\varphi\to A} \sum_{x:A} (f \nearrow x) \\
A \cong B &\triangleq \sum_{f:A\to B} \sum_{g:B\to A} (g \circ f \equiv \mathbf{id} \wedge f \circ g \equiv \mathbf{id}) \\
A \simeq B &\triangleq \sum_{f:A\to B} \prod_{y:B} \mathbf{isContr}\left(\sum_{x:A} (f\,x \sim y)\right)
\end{aligned}$$

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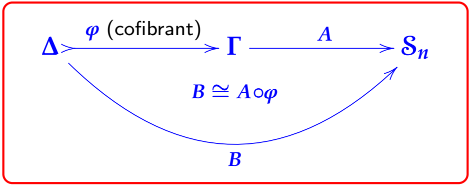

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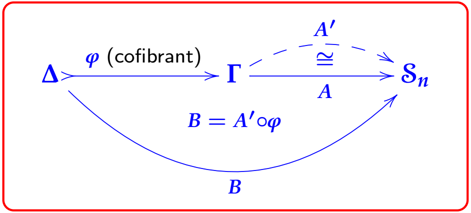

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In a presheaf topos $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, **Cof** has an **iea** if for each $X \in \mathbf{C}$ and $S \in \mathbf{Cof}(X) \subseteq \Omega(X)$, the sieve S is a decidable subset of $\mathbf{C}/X$.
(So with classical meta-theory, always have **iea** for presheaf toposes.)

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Proof is non-trivial! It combines results from:

Cohen-Coquand-Huber-Mörtberg TYPES 2015 [arXiv:1611.02108]
Orton-AMP CSL 2016 [arXiv:1712.04864]
Sattler 2017 [arXiv:1704.06911]
Licata-Orton-AMP-Spitters FSCD 2018 [arXiv:1801.07664]

Summary of axioms

- Elementary topos $\mathcal{E}$ with universes $\mathbf{\Omega} : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \cdots$
- "Interval" object $\mathbb{I}$ (in $\mathcal{S}_0$) which has distinct end-points & connection operation (& for convenience, a reversal operation) and which is tiny.
- Universe of "cofibrant" propositions **Cof** $\rightarrowtail \mathbf{\Omega}$ containing $i \equiv 0$ and $i \equiv 1$, is closed under $\_ \vee \_$ and $\forall(i : \mathbb{I})\_$, and satisfies the isomorphism extension axiom.

Then CCHM fibrations in $\mathcal{E}$ give a model of MLTT with univalent universes w.r.t. propositional identity types given by $\mathbb{I}$-paths.

(**Swan**: can have true, judgemental identity types if **Cof** is also a dominance.)

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T ... t
u ...

Problem! Tinyness cannot be axiomatized in MLTT, because it's a global property of morphisms of $\mathcal{E}$, not an internal property of functions – there is an internal right adjoint to $(\_)^{\mathbb{I}}$ only when $\mathbb{I} \cong \mathbf{1}$.

Next: can remove the use of impredicativity (Ω) and formalize within MLTT **plus...**

Tinyness: natural *bijection* between hom sets $\mathcal{E}(\Gamma^{\mathbb{I}}, \Delta)$ and $\mathcal{E}(\Gamma, \surd\Delta)$.

Tinyness: natural *bijection* between hom sets $\mathcal{E}(\boldsymbol{\Gamma}^{\mathbb{I}}, \boldsymbol{\Delta})$ and $\mathcal{E}(\boldsymbol{\Gamma}, \sqrt{\boldsymbol{\Delta}})$.

If one had a natural *isomorphism* of function types

$$(\boldsymbol{\Gamma}^{\mathbb{I}} \to \boldsymbol{\Delta}) \cong (\boldsymbol{\Gamma} \to \sqrt{\boldsymbol{\Delta}})$$

then

$$\sqrt{\boldsymbol{\Delta}} \cong (\mathbf{1} \to \sqrt{\boldsymbol{\Delta}}) \cong (\mathbf{1}^{\mathbb{I}} \to \boldsymbol{\Delta}) \cong (\mathbf{1} \to \boldsymbol{\Delta}) \cong \boldsymbol{\Delta}$$

naturally in $\boldsymbol{\Delta}$

Tinyness: natural *bijection* between hom sets $\mathcal{E}(\Gamma^{\mathbb{I}}, \Delta)$ and $\mathcal{E}(\Gamma, \surd\Delta)$.

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$$(\Gamma^{\mathbb{I}} \to \Delta) \cong (\Gamma \to \surd\Delta)$$

then

$$\surd\Delta \cong (1 \to \surd\Delta) \cong (1^{\mathbb{I}} \to \Delta) \cong (1 \to \Delta) \cong \Delta$$
naturally in Δ

so $\surd \cong \mathbf{id}$

so (taking left adjoints) $(\_)^{\mathbb{I}} \cong \mathbf{id}$ ($\cong (\_)^{1}$)

so $1 \cong \mathbb{I}$

Crisp Type Theory

Licata-Orton-AMP-Spitters [FSCD 2018]

intensional Martin-Löf Type Theory
+ uniqueness of identity proofs
+ Hofmann-style quotient types ($\Rightarrow$ function extensionality & disjunction for mere propositions)

+ **modality** for expressing global/local distinctions, inspired by

- Pfenning+Davis's judgemental reconstruction of modal logic [MSCS 2001]
- de Paiva+Ritter, *Fibrational modal type theory* [ENTCS 2016]
- Shulman's *spatial type theory* for real cohesive HoTT [MSCS 2017]

Crisp Type Theory

Dual context judgements:

$$\boldsymbol{\Delta} \mid \boldsymbol{\Gamma} \vdash a : A$$

crisp/global/external variables $x :: A$

cohesive/local/internal variables $x : A$

types in the crisp context $\boldsymbol{\Delta}$ and terms substituted for crisp variables $x :: A$ depend only on crisp variables

Crisp Type Theory

Dual context judgements:

$$\Delta | \Gamma \vdash a : A$$

Interpretation in the CwF associated with $\mathcal{E} = \mathbf{Set}^{\square^{\mathrm{op}}}$:

$\Delta \in \mathcal{E}$, $\Gamma \in \mathcal{E}(\flat\Delta)$, $A \in \mathcal{E}(\sum(\flat\Delta)\Gamma)$, $a \in \mathcal{E}(\sum(\flat\Delta)\Gamma \vdash A)$,

where $\flat : \mathcal{E} \longrightarrow \mathcal{E}$ is the limit-preserving idempotent comonad

$\flat A =$ the constant presheaf on the set of global sections of A.

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where $\flat : \mathcal{E} \longrightarrow \mathcal{E}$ is the limit-preserving idempotent comonad

$\flat A$ = the constant presheaf on the set of global sections of A.

This just follows from the fact that
$\square$ is a connected category
(since it has a terminal object)

Crisp Type Theory

Dual context judgements:

$$\Delta | \Gamma \vdash a : A$$

Some of the rules:

$$\frac{}{\Delta, x :: A, \Delta' | \Gamma \vdash x : A}$$

$$\frac{\Delta | \ \vdash a : A \qquad \Delta, x :: A, \Delta' | \Gamma \vdash b : B}{\Delta, \Delta'[a/x] | \Gamma[a/x] \vdash b[a/x] : B[a/x]}$$

$$\frac{\Delta | \ \vdash A : \mathcal{S}_m \qquad \Delta, x :: A | \Gamma \vdash B : \mathcal{S}_n}{\Delta | \Gamma \vdash (x :: A) \to B : \mathcal{S}_{m \vee n}} \qquad \frac{\Delta, x :: A | \Gamma \vdash b : B}{\Delta | \Gamma \vdash \lambda(x :: A), b : (x :: A) \to B}$$

$$\frac{\Delta | \Gamma \vdash f : (x :: A) \to B \qquad \Delta | \ \vdash a : A}{\Delta | \Gamma \vdash f\, a : B[a/x]}$$

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$$\frac{\Delta | \Gamma \vdash f : (x :: A) \to B \qquad \Delta | \ \vdash a : A}{\Delta | \Gamma \vdash f\, a : B[a/x]}$$

Experimental implementation: Vezzosi's `agda-flat`

Needed: congruence for functions f of a crisp variable x

$$\texttt{crispwrong} : \{A :: \mathcal{S}_m\}\{x\ y :: A\}\{B : \mathcal{S}_n\}(f : (x :: A) \to B) \to$$
$$(\_ : x \equiv y) \to f\,x \equiv f\,y$$
$$\texttt{crispwrong}\ f\ \texttt{refl} = \texttt{refl}$$

Agda-flat says: `Wrong modality to solve` y `with` x `when checking that the pattern refl has type` $x \equiv y$

(Here I write "$::$" for what in `agda-flat` must be written "$:\{\flat\}$".)

Needed: congruence for functions f of a crisp variable x

$$\texttt{crispcong} : \{A :: \mathcal{S}_m\}\{x\ y :: A\}\{B : \mathcal{S}_n\}(f : (x :: A) \to B) \to (\_ :: x \equiv y) \to f\,x \equiv f\,y$$
$$\texttt{crispcong}\ f\ \texttt{refl} = \texttt{refl}$$

Agda-flat is happy with this (and so are we?).

Needed: congruence for functions f of a crisp variable x

```
crispcong : {A :: S_m}{x y :: A}{B : S_n}(f : (x :: A) → B) →
    (_ :: x ≡ y) → f x ≡ f y
crispwrong f refl = refl
```

Not needed (but definable): the crisp modality ♭ on types

```
data ♭ (A : S_n) : S_n where
    in♭ : (_ :: A) → ♭A
```

Axioms for tinyness in Agda-flat

$$\surd : (A :: \mathcal{S}_n) \to \mathcal{S}_n$$

$$\mathtt{R} : \{A, B :: \mathcal{S}_n\}(f :: \wp A \to B) \to A \to \surd B$$

$$\mathtt{L} : \{A, B :: \mathcal{S}_n\}(g :: A \to \surd B) \to \wp A \to B$$

$$\mathtt{LR} : \{A, B :: \mathcal{S}_n\}\{f :: \wp A \to B\} \to \mathtt{L}(\mathtt{R}\, f) \equiv f$$

$$\mathtt{RL} : \{A, B :: \mathcal{S}_n\}\{g :: A \to \surd B\} \to \mathtt{R}(\mathtt{L}\, g) \equiv g$$

$$\mathtt{R}\wp : \{A, B, C :: \mathcal{S}_n\}(g :: A \to B)(f :: \wp B \to C) \to \mathtt{R}(f \circ \wp g) \equiv \mathtt{R} f \circ g$$

where $\wp(\_) \triangleq \mathbb{I} \to (\_)$.

For more, see doi.org/10.17863/CAM.22369

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets. because the path functor is fibered over $\mathcal{E}$ and we can use internal language to describe many of the constructions on the way to a univalent universe. . .

  . . . but not all of them: tinyness does not internalize! (so neither does our universe construction).

  Crisp Type Theory to the rescue.

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones.
  (E.g. recent work by Taichi Uemura.)

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones.
- Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!

  We find the use of an interactive theorem proving system (Agda-flat) invaluable for developing and checking the proof – e.g. see [doi.org/10.17863/CAM.21675]

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones.
- Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!

  Are there simpler models of univalence? (must be non-truncated to qualify for our attention)

  E.g. can one avoid Kan-filling in favour of a (weak) notion of path composition?

  Why only presheaf toposes? (issues with universes in sheaf toposes)

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones.
- Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!
- Further reading:

  D. R. Licata, I. Orton, A. M. Pitts and B. Spitters, *Internal Universes in Models of Homotopy Type Theory* [FSCD 2018].

Questions?