A cubical model for weak ω-categories

(joint work with Tim Campion and Chris Kapulkin)

Yuki Maehara

Macquarie University

HoTTEST October 2020

Globular ω-categories

	Shape	Compositions
0-cells	$\bullet$	none

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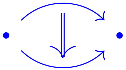
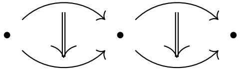
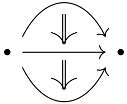
2-cells		hor.  vert. 

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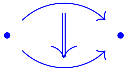
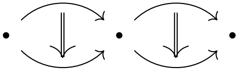
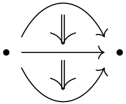
2-cells		hor.  vert. 
n-cells	n-dimensional globe	n ways

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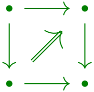
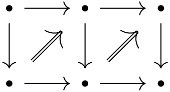
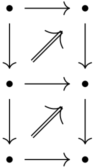
2-cells		hor.  vert. 

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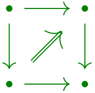
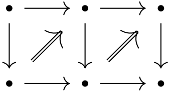
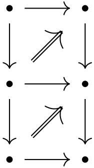
2-cells		hor.  vert. 
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Theorem (Al-Agl, Brown, Steiner)

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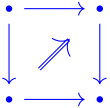

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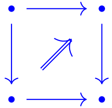
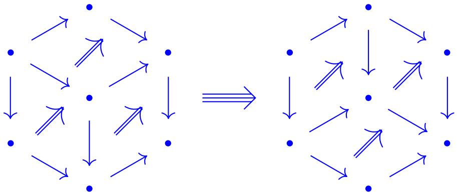
$\mathbb{I}^2 =$ 

$\mathbb{I}^3 =$ 

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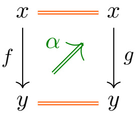

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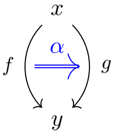

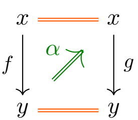

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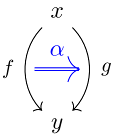

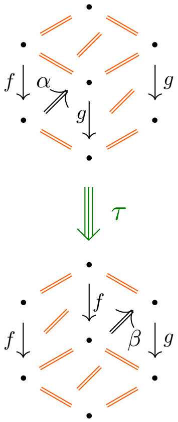

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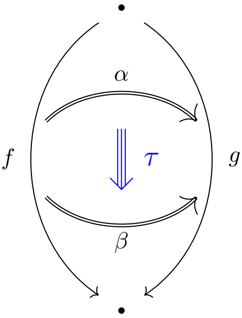

Lax Gray tensor product:

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$$\bullet \longrightarrow \bullet \quad \otimes \quad \begin{array}{c}\bullet\\ \downarrow\\ \bullet\end{array} \quad = \quad \begin{array}{ccc}\bullet & \longrightarrow & \bullet\\ \downarrow & \nearrow & \downarrow\\ \bullet & \longrightarrow & \bullet\end{array}$$

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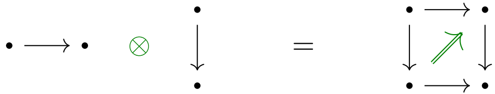

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$$\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$$

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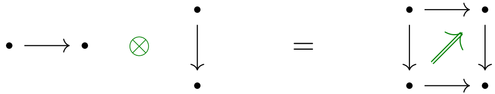

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The lax Gray tensor product is part of a unique biclosed monoidal structure on ω-$\underline{\mathrm{Cat}}$ such that $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$.

Gray tensor products on ω-$\underline{\mathrm{Cat}}$

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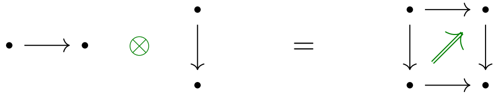

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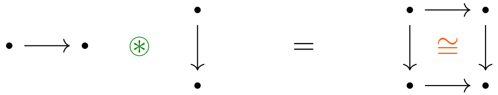
Pseudo Gray tensor product:



Cubical sets

Definition

Cubical sets (with connections) = presheaves on $\square$

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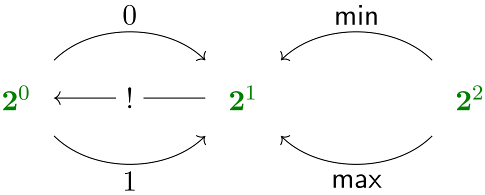

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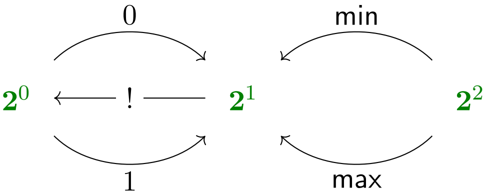

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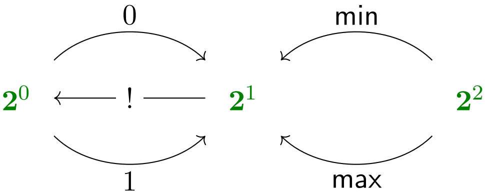

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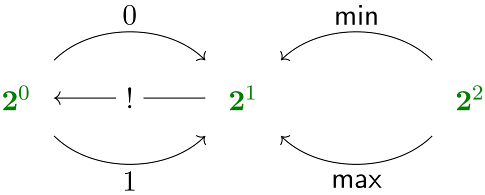

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Geometric product is Day convolution of $\mathbf{2}^m \otimes \mathbf{2}^n = \mathbf{2}^{m+n}$.

Certain cubes correspond to identities:

Cubical ω-categories via box filling

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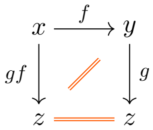

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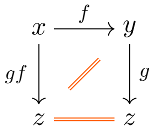

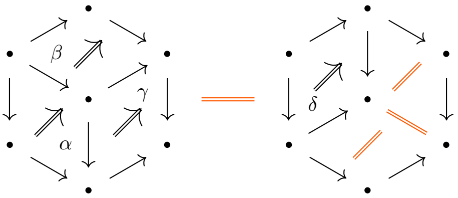

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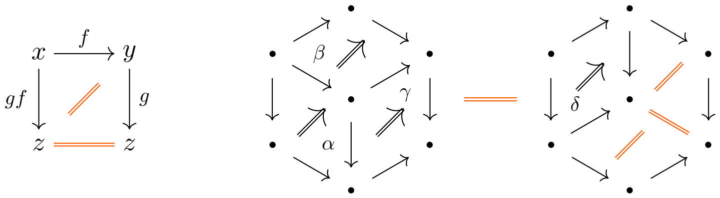

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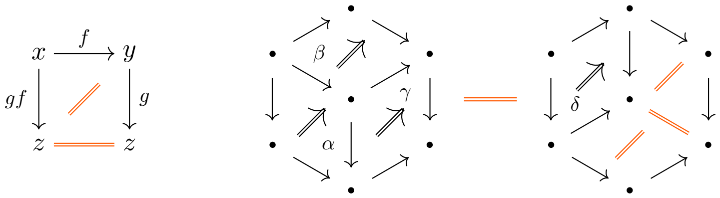

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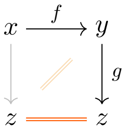

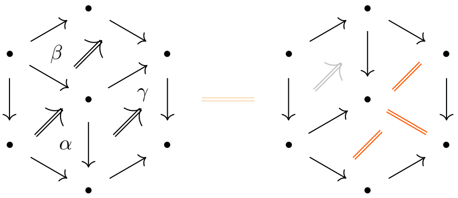

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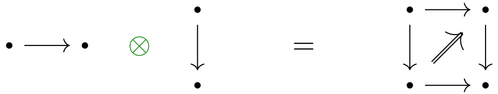
Lax Gray tensor product

Geometric product should underlie lax Gray tensor product.



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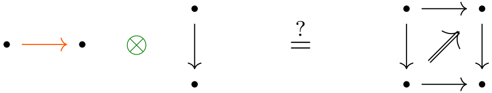

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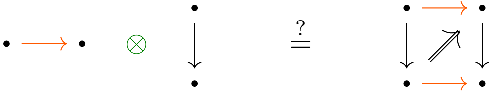

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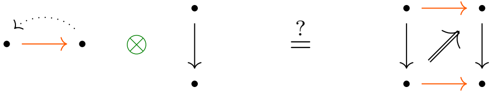

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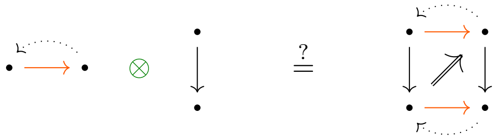

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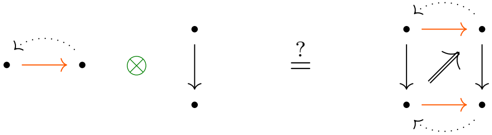

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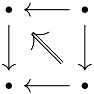

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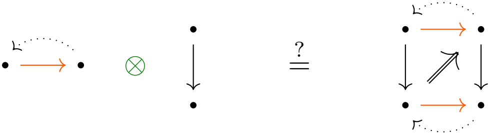

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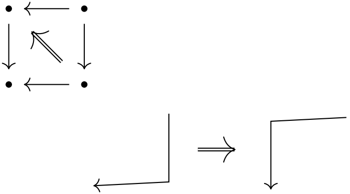

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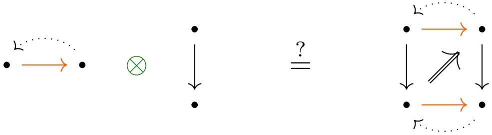

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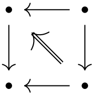

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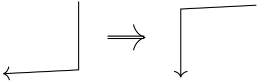

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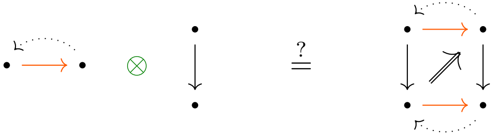

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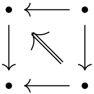

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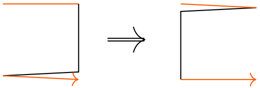

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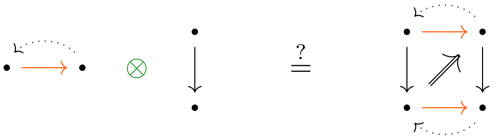

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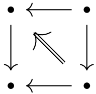

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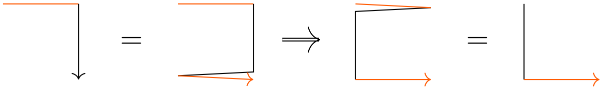

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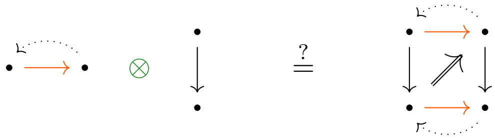
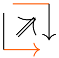
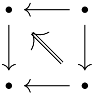
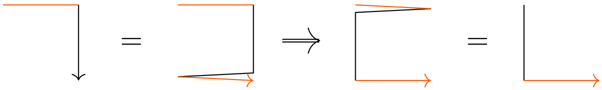


so we should have an inverse to 

 whiskered with $\longrightarrow$'s:



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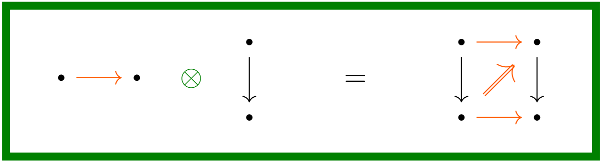
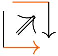
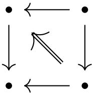


so we should have an inverse to 



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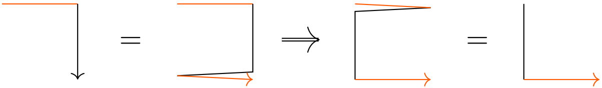

Gray tensor products

Definition

The lax Gray tensor product of marked cubical sets is

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Definition

In the pseudo Gray tensor product, the only unmarked cubes are:

- unmarked $\otimes$ 0-cube; and
- 0-cube $\otimes$ unmarked.

Model structure

Theorem

There is a model structure such that:

- $\{cofibrations\} = \{monos\}$
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Moreover it is monoidal wrt both Gray tensor products.

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The combinatorics is relatively easy!

Complicial sets

We have a simplicial precursor.

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Theorem (Verity)

$\{$*strict complicial sets*$\} \simeq \omega\text{-}\underline{\mathrm{Cat}}$

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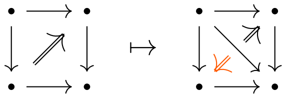

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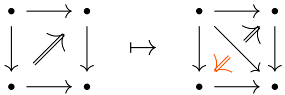

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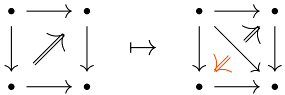

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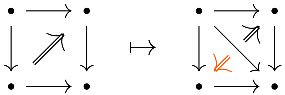

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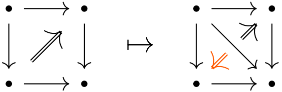

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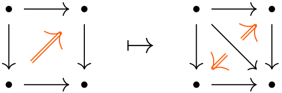

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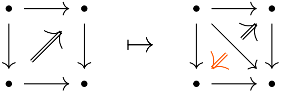

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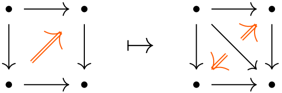

Theorem

It is left Quillen and preserves both Gray tensor products up to homotopy.

That's it!

Thank you!