Initiability for Martin-Löf Type Theory

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Stockholm University

HoTTEST, 10 September 2020

Some after-talk fixes + additions.
Video: https://youtu.be/1ogUFFUfU_M
**Initiality for dependent type theories**

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- Justifies categorical-algebraic definition of “models of DTT with XYZ-types” as “contextual cats with XYZ-structure”
- Packages the bureaucracy of interpreting syntax into such structures
- Should hold “robustly” for “all reasonable” type theories: not rely on “fragile” properties like normalisation
- Variations: could state with CwA’s, CwF’s, C-systems, etc.; with various different presentations of the type theory; with 2-categorical initiality; …
Status

▶ Thesis: Initiality is established
  ▶ Proven by Streicher (1991 book) for Calculus of Constructions; adapted+refined by Hofmann (1995 thesis) for DTT with $\Pi$, $\text{Id}$, $\mathbb{N}$.
  ▶ Proof extends straightforwardly + robustly to other type theories.
  ▶ (NB: Other presentations in literature (that I’m aware of) use techniques specific to certain type theories, don’t extend robustly; or use substantially different syntax; or handwave many details.)
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  ▶ What type theories is it even supposed to hold for? It fails for some!
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  ▶ “Experts” do understand what kinds of type theories it holds for, and how to extend Streicher–Hofmann proof.
  ▶ But: this understanding not clearly articulated anywhere, rigorously or even heuristically.
  ▶ Extension of proof mostly straightforward — minor tweaks needed, no substantial new ideas — but carefully making sure of this involves checking a lot of details.
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- But: this understanding not clearly articulated anywhere, rigorously or even heuristically.
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Solution proposals

Long-term solution

Define *general class of dependent type theories*; state and prove initiality for these.
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Define general class of dependent type theories; state and prove initiality for these.

▶ Define rigorously a general class of dependent type theories…
▶ …yielding as many specific theories of interest as possible…
▶ …modulo differences in presentation, as minor as possible.
▶ Define corresponding categorical-algebraic structures…
▶ …yielding the established definitions, as closely as possible…
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Various proposals: Bauer–Lumsdaine–Haselwarter; Brunerie; Uemura; Isaev, Capriotti.

See Peter’s HoTTTEST, June 2020: https://youtu.be/kQe0knDuZqg
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Short-term solution

**Just damn well prove it** for more non-trivial theories of interest!
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- Logical framework presentations:
  pre-Uemura: couldn’t see how they suffice for other motivations;
  linking the “naïve” and LF syntaxes: clearest proof I know (Hofmann) uses initiality for naïve syntax.
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  linking the “naïve” and LF syntaxes: clearest proof I know (Hofmann) uses initiality for naïve syntax.
▶ And various other options, e.g. QIITs. Generally: each has some advantages; but traditional “naïve” syntax still one important approach to understand.
Stockholm initiality formalisations

Goal

Prove initiality formally, for some specific type theory, ideally approaching “book HoTT”.


Small type theory at first: \( \Pi \) -types, a dependent family of base types. Key design criterion: robust extensibility. Avoid doing anything that wouldn’t extend to “arbitrary” constructors/rules.

“We can have this done within a week.” — PLL, 19 October 2018

Developments begun 22 October; Brunerie-de Boer initiality attained 19 November!
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Categories with families proof
Type theory under consideration

Raw syntax with binding: de Bruijn indices, well-scoped, fully annotated.

**Definition**

**Raw expressions**: family of sets $\text{Expr}^{\text{ty}}(n)$, $\text{Expr}^{\text{tm}}(n)$ inductively generated by **variables** and **constructors**:

- for $i \in \{1, \ldots, n\}$, have $x_i \in \text{Expr}^{\text{tm}}(n)$;
- for $A \in \text{Expr}^{\text{ty}}(n)$, $B \in \text{Expr}^{\text{ty}}(n+1)$, have $\Pi(A, B) \in \text{Expr}^{\text{ty}}(n)$;
- for $A \in \text{Expr}^{\text{ty}}(n)$, $B \in \text{Expr}^{\text{ty}}(n+1)$, $f, a \in \text{Expr}^{\text{tm}}(n)$, have $\text{app}(A, B, f, a) \in \text{Expr}^{\text{ty}}(n)$
- similar clauses for each constructor of the type theory

Basic operations (e.g. substitution) and lemmas all defined+proven recursively+inductively from these.
Judgements, derivability

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**Derivability**: inductively defined relation on judgements, by usual rules:

- structural rules;
- logical + congruence rules for the specific constructors.
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Note: no context or context-equality judgement, either primitive or used in rules. Can show: this doesn’t affect derivability.
### Categorical models

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- presheaf of terms: $\text{Tm}(\Gamma, A)$, for $A \in \text{Ty}(\Gamma)$;
- various operations, properties

(Closely equivalent: categories with attributes, split type-categories, …)

**Definition**

*Logical structure* on a CwA: operations for all constructors + rules of the type theory.
Environments; partial interpretation

**Definition**

**Environment** for scope $n$ in $C$:

object $\Gamma \in C$ and partial map $E : \{1, \ldots, n\} \rightarrow \sum_{A \in Ty(\Gamma)} Tm(\Gamma, A)$.

Idea: $\Gamma$ the interpretation of a context; $E$ gives its types and variables.
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## Definition

Given $C$, environment $(\Gamma, E)$ for scope $n$, and expression $e$ in scope $n$, get **partial interpretation** $\llbracket e \rrbracket^{\Gamma, E}$, suitable type or term, defined by recursion on $e$. 

Interpretation lemmas

Lemma

*Partial interpretation is stable under reindexing of environments.*
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Proof.

Structural induction on the expression.
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- No mention of equality on objects.
Coq formalisation
Background


Attempt foundered due to combination of several design choices making life hard:

- use of named variables
- use of setoids for the target models
- started with slightly overambitious type theory
- …

Very useful experience to build on — both the good and the bad aspects…
Design choices

This time round:

▶ Proof assistant: Coq; specifically, over UniMath. (Mainly: for a well-developed category theory library that both authors were familiar with.)
▶ Models: Categories with attributes, not assuming objects form a set (so, CwA a 2-category); for 1-categorical initiality, contextual categories as CwA’s plus contextuality axiom (implying objects a set).
▶ Variables in raw syntax: using de Bruijn indices. Raw syntax: well-scoped. These enable:
▶ All inductions purely structural, over either raw syntax or derivations. No size measures, auxiliary well-founded relations, etc.
▶ Context and context-equality judgements subsidiary, not primitive, don’t appear in derivations. Substitution admissible, not a primitive rule. These enable:
▶ Interpretation function (partial + totality): into arbitrary CwA’s. No use of equality on objects/contexts needed.
Experience

Good news

1. Interpretation function (partial + total): went very smoothly.
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**Bad news**

Specifically, interaction of 2 issues:

1. Syntactic CwA: dependently-typed (maps depend on objects), with objects quotiented (contexts, up to judgemental equality)
2. In UniMath’s quotients, the dependent eliminator doesn’t compute judgementally.

Together: ends up at least as painful as using setoids.
Status

1. Complete: “Pre-quotient” parts. Contains most mathematically interesting parts, and most useful for applications.
   ▶ definition of type theory, main syntactic lemmas;
   ▶ definition of suitably structured CwA’s, basic lemmas;
   ▶ interpretation function into suitable CwA’s (pre-quotient part of existence for initiality)
   ▶ functoriality of interpretation under CwA (pseudo-)maps (pre-quotient ingredient of uniqueness for initiality)

Around 4,000 lines of code (not including libraries).

2. Incomplete: “post-quotient” parts, i.e. assembling into the syntactic CwA and functors thereon. Mathematically mostly less interesting, but hard to formalise.
   ▶ Done: syntactic category; most of CwA structure thereon; some logical structure; most of underlying functor of interpretation map.
   ▶ Remaining: rest of CwA structure, and logical structure; interpretation as a structure-preserving map of CwA’s; uniqueness of the interpretation map.

Arround 3,000 lines of code, so far!
Over to Guillaume!