The Reedy diagrams model of dependent type theory

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Problem

Work in some small dependent type theory (e.g. Id, Σ, Π).

Suppose we have…

…some type expression T, containing an atomic type X; e.g.:

$$\mathsf{List}(\mathsf{X}^2) \quad \mathsf{isContr}(\mathsf{X}) \quad \mathsf{RingStruc}(\mathsf{X})$$

…some model $\mathbf{C}$ of type theory (e.g. simplicial sets, realisability, …) and two "types" A, B in $\mathbf{C}$.

Get two interpretations: $[\![T]\!]^{\mathsf{X} \mapsto A}$, $[\![T]\!]^{\mathsf{X} \mapsto B}$.

Question

Does an equivalence $e : A \simeq B$ induce an equivalence $[\![T]\!]^{\mathsf{X} \mapsto A} \simeq [\![T]\!]^{\mathsf{X} \mapsto B}$?

Answer: Univalence?

Similar to statement of univalence, but a bit different.
Univalence…

- …is a statement about a universe;
- …says: arbitrary constructions on that universe respect equivalence.

Here…

- …no universe assumed in **C**!
- …but T assumed definable: an actual expression of the type theory.

Must make use of type-theoretic definition of T somehow!

Model in equivalences

Idea: induct up on the definition/derivation of T. Show each step is invariant under equivalence.

But: we're in a dependent type theory! Derivation may involve not just closed types but dependent types, terms, contexts…

I.e. want new model of this type theory, whose "closed types" consist of a pair of closed types of $\mathbf{C}$ and an equivalence between them (in some sense).

I.e. want construction on models: $\mathbf{C} \mapsto \mathbf{C}^{\mathrm{Eqv}}$.

Span-equivalences

What notion of equivalence to use?

$$\vdash A\ \mathsf{type} \qquad \vdash B\ \mathsf{type} \qquad x{:}A,\ y{:}B \vdash R(x,y)\ \mathsf{type}$$

A (type-valued) relation between A and B…

$$x{:}A \vdash \mathsf{isContr}\left(\sum(y{:}B)\, R(x,y)\right)$$
$$y{:}B \vdash \mathsf{isContr}\left(\sum(x{:}A)\, R(x,y)\right)$$

…forming a one-to-one correspondence.

Call this a Reedy span-equivalence; without the second part, just a Reedy span. So want:

- $\mathbf{C}^{\mathrm{Eqv}}$, model whose types are Reedy span-equivalences in $\mathbf{C}$;
- $\mathbf{C}^{\mathrm{Eqv}} \subseteq \mathbf{C}^{\mathrm{Span}}$, whose types are Reedy spans in $\mathbf{C}$—a "relations" model).

Categories with Attributes

Use categorical/algebraic notion of model of type theories:

Definition

A category with attributes (CwA) is:

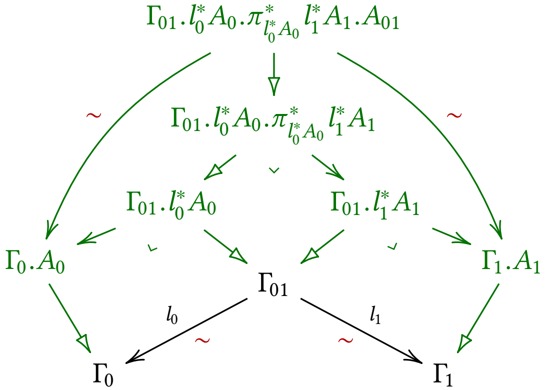
- a category $\mathbf{C}$ [sometimes assumed: with terminal object $\diamond$];
- a functor $\mathrm{Ty} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$;
- for each $A \in \mathrm{Ty}(\Gamma)$, an object $\Gamma.A$ and map $\pi_A : \Gamma.A \longrightarrow \Gamma$;
- for each $A \in \mathrm{Ty}(\Gamma)$ and $f : \Delta \longrightarrow \Gamma$,

  a map $f.A$ giving pullback $$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{f.A} & \Gamma.A \\ {\scriptstyle \pi_{f^*A}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_A} \\ \Delta & \xrightarrow{f} & \Gamma, \end{array}$$ functorially in f.

Further: equip CwA's with logical structure, i.e. algebraic operations/axioms corresponding to the logical rules of DTT (Id, Σ, Π, …)

CwA of span-equivalences

$\mathbf{C}^{\mathrm{Span}}$, $\mathbf{C}^{\mathrm{Eqv}}$ have contexts and types given by:



- I.e. Reedy span(-equivalence)s as defined syntactically above,
- expressed diagramatically in $\mathbf{C}$,
- relativised to over a general span(-equivalence) as context.

Σ-types in span(-equivalence)s

Input to Σ-types:

$$\vdash A \text{ type} \qquad x{:}A \vdash B(x) \text{ type}$$

In spans (working syntactically for readability):

$$\vdash A_0 \text{ type} \qquad \vdash A_1 \text{ type} \qquad x_0{:}A_0,\ x_1{:}A_1 \vdash A_{01}(x_0, x_1) \text{ type}$$

$$x_0{:}A_0 \vdash B_0 \text{ type} \qquad x_1{:}A_1 \vdash B_1 \text{ type}$$

$$x_0{:}A_0,\ x_1{:}A_1,\ x_{01}{:}A_{01}(x_0, x_1),\ y_0{:}B_0(x_0),\ y_1{:}B_1(x_1) \\ \vdash B_{01}(x_0, x_1, x_{01}, y_0, y_1) \text{ type}$$

Define $\Sigma\,(x{:}A)\,B$ as:

$$\vdash \Sigma\,(x_0{:}A_0)\,B_0(x_0) \text{ type} \qquad \vdash \Sigma\,(x_1{:}A_1)\,B_1(x_1) \text{ type}$$

$$z_0 : \Sigma\,(x_0{:}A_0)B_0(x_0),\ z_1 : \Sigma\,(x_0{:}A_0)\,B_0(x_0) \\ \vdash \Sigma\,(x_{01} : A_{01}(\mathsf{pr}_1(z_0), \mathsf{pr}_1(z_1)))\,B_{01}(x_{01}, \mathsf{pr}_2(z_0), \mathsf{pr}_2(z_1)) \text{ type}$$

Moreover: this span is an equivalence if A, B both were.

Exercise: similarly, give the definition of Π-types in spans.

Reedy diagrams on inverse categories

Definition

- Inverse category: no infinite descending chain of non-identity morphisms

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

- Ordered inverse category: ordering on objects of each coslice, satisfying certain conditions.
- Homotopical category: equipped with distinguished class of maps, “equivalences”.

Examples, non-homotopical: the span category; the opposite of the semi-simplicial category.

Example, homotopical: the equivalence-span category, i.e. the span category with all maps equivalences.

Fact: every inverse category admits an ordering.

Reedy diagrams on inverse categories

Definition

Suppose $\mathcal{I}$ an ordered inverse cat, $\mathbf{C}$ a CwA, $\Gamma : \mathcal{I} \longrightarrow \mathbf{C}$ a diagram.

Reedy type A over $\mathcal{I}$:

- a diagram $(\Gamma.A) : \mathcal{I} \longrightarrow \mathbf{C}$ over Γ,
- in which each object arises from a type A_i over a matching object $M_i A$.

Suppose $\mathcal{I}$ homotopical. A diagram $\Gamma : \mathcal{I} \longrightarrow \mathbf{C}$ is homotopical if it sends equivalences to equivalences.
Have CwA's $\mathbf{C}^{\mathcal{I}}$, $\mathbf{C}^{\mathcal{I}}_h$.

Example: Reedy spans, Reedy span-equivalences.

Orderings are used just to construct $M_i A$ as context extension.

Summary

Theorem

$\mathbf{C}$ *a CwA with* Id*-types,* $\mathcal{I}$ *an ordered homotopical inverse category. Then:*

1. $\mathbf{C}^{\mathcal{I}}$ *carries* Id*-types; if* $\mathbf{C}$ *carries 1- and* Σ*-types, so does* $\mathbf{C}^{\mathcal{I}}$*.*
2. *If* $\mathbf{C}$ *carries extensional* Π*-types, and additionally all maps of* $\mathcal{I}$ *are equivalences, then* $\mathbf{C}^{\mathcal{I}}$ *carries extensional* Π*-types.*
3. *A CwA map* $F : \mathbf{C} \longrightarrow \mathbf{D}$ *induces a CwA map* $F^{\mathcal{I}} : \mathbf{C}^{\mathcal{I}} \longrightarrow \mathbf{D}^{\mathcal{I}}$*, preserving whatever logical structure* F *preserved, functorially in* F*.*
4. *Any homotopical discrete opfibration* $f : \mathcal{I} \longrightarrow \mathcal{J}$ *induces a map* $\mathbf{C}^f : \mathbf{C}^{\mathcal{J}} \longrightarrow \mathbf{C}^{\mathcal{I}}$*, preserving all logical structure, and functorially in* f*.*
5. *If* $f : \mathcal{I} \longrightarrow \mathcal{J}$ *as above is moreover injective, then* $\mathbf{C}^f$ *is a local fibration; and if* f *is a homotopy equivalence, then* $\mathbf{C}^f$ *is a local equivalence.*

Application: Homotopy theory of type theories

Long-term goal: some precise version of "HoTT is the internal logic of elementary ∞-toposes" (and similar statements for fragments of HoTT vs. lex and lccc ∞-categories).

More precise goal: construct $(\infty, 1)$-equivalance $\mathrm{DTT}_{\mathrm{HoTT}} \simeq_\infty \mathrm{ElemTop}_\infty$, for some suitable $(\infty, 1)$-categories of DTT's and elementary ∞-toposes; similarly $\mathrm{DTT}_{\mathrm{Id},\Sigma} \simeq_\infty \mathrm{Lex}_\infty$, etc.

Analogous to established statements for IHOL/toposes, etc. Pragmatic interpretation: "something holds in suitable infinity-categories exactly when you can prove it in type theory".

First step: give tractable construction of suitable $(\infty, 1)$-categories of dependent type theories.

Given in Kapulkin–Lumsdaine, *The homotopy theory of type theories*, arXiv:1610.00037; see also Isaev, *Model structures on categories of models of type theories*, arXiv:1607.07407.

Contextual categories

Definition

A CwA is $\mathbf{C}$ *contextual* if it has a distinguished terminal object $\diamond$, s.t. every object of $\mathbf{C}$ is uniquely expressible as $\diamond.A_1.\cdots.A_n$.

Take $\mathrm{DTT}_{\mathbf{T}}$ to be (1-)category of contextual categories equipped with logical structure for the rules of $\mathbf{T}$.

Inclusion $\mathrm{DTT}_{\mathbf{T}} \longrightarrow \mathbf{CwA}_{\mathbf{T}}$ has right adjoint, sending CwA $\mathbf{C}$ to $\mathbf{C}(\diamond)$:

- objects: "context extensions" $(A_1, \ldots, A_n)$ over $\diamond$;
- maps, types, structure: inherited from $\mathbf{C}$.

Why not use CwA's for $\mathrm{DTT}_{\mathbf{T}}$? Type theory can't reason about arbitrary contexts of a CwA.

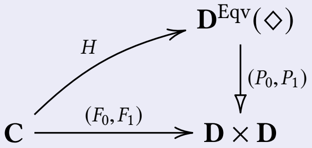
Why not use contextual cats throughout? Many constructions much simpler with CwA's (eg contexts in diagram models). E.g. for $\mathbf{C}^{\mathrm{Span}}(\diamond)$ given directly, see Tonelli 2013, *Investigations into a model of type theory based on the concept of basic pair*.

Path objects as Reedy diagrams

Key technical tool: Right homotopy, with $\mathbf{C}^{\mathrm{Eqv}}(\Diamond)$ as path-objects.

Definition

$F_0, F_1 : \mathbf{C} \longrightarrow \mathbf{D}$ in $\mathrm{DTT}_{\mathrm{Id},\Sigma(,\Pi_{ext})}$ are right homotopic ($F_0 \sim_r F_1$) if they factor jointly through $\mathbf{D}^{\mathrm{Eqv}}(\Diamond)$:



Problem: not an equivalence relation! E.g. no reflexivity map $\mathbf{D} \longrightarrow \mathbf{D}^{\mathrm{Eqv}}(\Diamond)$ in $\mathrm{DTT}_{\mathbf{T}}$.

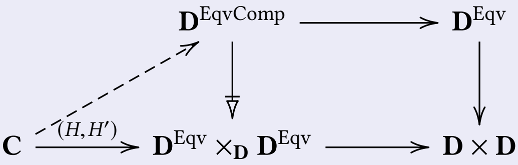
Example: transitivity of path-objects

Proposition

Right homotopy is an equivalence relation on $\mathrm{DTT}(\mathbf{C}, \mathbf{D})$, *when* $\mathbf{C}$ *is cofibrant.*

Proof.

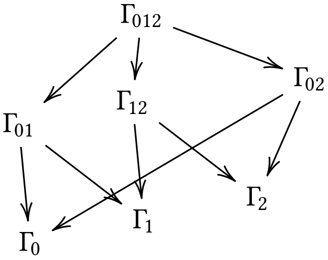
Construct a suitable CwA $\mathbf{D}^{\mathrm{EqvComp}}$ with a trivial fibration $\mathbf{D}^{\mathrm{EqvComp}} \twoheadrightarrow \mathbf{D}^{\mathrm{Eqv}} \times_{\mathbf{D}} \mathbf{D}^{\mathrm{Eqv}}$:



□

Example: transitivity of path objects

$\mathbf{D}^{\mathrm{EqvComp}}$: CwA of homotopical Reedy types on the category



with all maps equivalences.

Payoff

Theorem (Kapulkin–Lumsdaine 2016)

There is a left semi-model structure on $\mathrm{DTT}_{\mathrm{Id},\Sigma(,\Pi_{\mathrm{ext}})}$, *with equivalences the type-theoretic equivalences.*

(Heuristically, expect this to extend to $\mathrm{DTT}_{\mathrm{HoTT}}$, for suitable definition thereof.)

This gives precise statement of the "internal language" conjectures for these type theories. In fact, now proven in the finitely-complete case:

Theorem (Kapulkin–Szumiło 2017)

There is an $(\infty,1)$*-equivalence* $\mathrm{DTT}_{\mathrm{Id},1,\Sigma)} \longrightarrow \mathrm{Lex}_\infty$.

Kapulkin, Szumiło, *Internal language of finitely complete* $(\infty,1)$*-categories*, arXiv:1709.09519.

Bonus: exercise solution, Π-types in span(-equivalence)s

Input to Π-types is same as for Σ-types:

$$\vdash A \text{ type} \qquad x{:}A \vdash B(x) \text{ type}$$

In spans:

$$\vdash A_0 \text{ type} \qquad \vdash A_1 \text{ type} \qquad x_0{:}A_0,\ x_1{:}A_1 \vdash A_{01}(x_0, x_1) \text{ type}$$

$$x_0{:}A_0 \vdash B_0 \text{ type} \qquad x_1{:}A_1 \vdash B_1 \text{ type}$$

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Define $\Pi\,(x{:}A)\,B$ as:

$$\vdash \Pi\,(x_0{:}A_0)\,B_0(x_0) \text{ type} \qquad \vdash \Pi\,(x_1{:}A_1)\,B_1(x_1) \text{ type}$$

$$f_0 : \Pi\,(x_0{:}A_0) B_0(x_0),\ f_1 : \Pi\,(x_0{:}A_0)\,B_0(x_0) \vdash \Pi\,(x_0{:}A_0)\,(x_1{:}A_1)\,(x_{01}{:}A_{01}),\ B_{01}(x_{01}, \mathsf{app}(f_0, x_0), \mathsf{app}(f_1, x_1)) \text{ type}$$