The language of a model category

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Theorem (Freyd, Blanc...)

A first order formula ϕ "in the language of categories", i.e. not involving equalities between objects is:

- *Invariant under isomorphisms.*

If ϕ depends on some parameters (in a category $\mathcal{C}$), then replacing the values of these parameters by isomorphic ones (in a consistent way) do not change the validity of ϕ.

Theorem (Freyd, Blanc...)

A first order formula ϕ "in the language of categories", i.e. not involving equalities between objects is:

- *Invariant under isomorphisms.*
- *Invariant under equivalence.*

If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, then the validity of ϕ for some parameter chosen $\mathcal{C}$ is equivalent to the validity of ϕ for the image by F of these parameters.

Example

$$\mathrm{SubTerminal}(X) := \forall Z : \mathrm{Ob}, \forall g, h : \mathrm{Hom}(Z, X), g = h$$

$$\mathrm{Mono}(X, Y, f : X \to Y) := \forall Z : \mathrm{Ob}, \forall g, h : \mathrm{Hom}(Z, X), (fg = fh \Rightarrow g = h)$$

General Goal of the talk:

To each model category $\mathcal{M}$ we will attach a language with similar properties. The theorem above will corresponds to the case of the "folk" model structure on the category of categories.

Remark: I will work in a classical meta-theory, and I will hide some set theoretical difficulties to keep the talk simple.

Definition

A (Quillen) model category is a (co)complete category $\mathcal{C}$ endowed with three classes of maps Fibrations, cofibrations, weak equivalences, such that:

- Weak equivalences satisfies 2-out-of-3.
- (Cofibrations, trivial Fibrations) and (Trivial cofibrations, fibrations) are weak factorization systems.

Remark: This a framework for categorical homotopy theory: one can make sense of "homotopies" between maps from a cofibrant object to a fibrant object, defines a homotopy category. A model category can be seen as a presentation of a (co)complete $(\infty, 1)$-category.

Remark: Everything I will say also work for left/right semi-model categories or weak model categories (if you have heard about them).

Very Quick recall: a Cartmell algberaic theory, or dependently typed algebraic theory, is a kind of multi-sorted algebraic theory which allows for its sort (called type) to depend on parameters living in other types.

Example

The theory of categories can be seen as having two types: a sort Ob of objects and for each pair of object $X, Y : \mathrm{Ob}$ a type of morphisms $\mathrm{Hom}(X, Y)$.

Composition is an operation:

$$X, Y, Z : \mathrm{Ob}, f : \mathrm{Hom}(X, Y), g : \mathrm{Hom}(Y, Z) \vdash g \circ f : \mathrm{Hom}(X, Z)$$

Cartmell theories have the same expressive power as essentially algebraic theories or partial Horn theories (i.e. same categories of models).

But the category of models of a Cartmell theory will come with certain additional structures that will be very important in the rest of the talk.

To a Cartmell theory $\mathbb{T}$, there is an associated contextual category $\mathrm{Con}_{\mathbb{T}}$ (i.e. a category with certain maps called "display maps", encoding dependent types, that admit strictly functorial pullbacks, encoding re-indexation of dependent types).

Models of $\mathbb{T}$ are the same as functors $\mathrm{Con}_{\mathbb{T}} \to \mathrm{Set}$ that preserve pullback of display maps.

In this talk we will also need to consider *an infinitary version of Cartmell theories*, where operations can have infinite arities and types depend on infinitely many variables. Everything still works the same, but now models of $\mathbb{T}$ correspond to functors $\mathrm{Con}_{\mathbb{T}} \to \mathrm{Set}$ preserving pullbacks and transfinite compositions of display maps.

We now add a layer of first order logic:

We fix $\mathbb{T}$ a (possibly infinitary) Cartmell generalized algebraic theory. Given a context Γ in $\mathbb{T}$ one defines inductively the "set" $\mathcal{F}(\Gamma)$ of first order formulas in context Γ as follows:

- $\bot, \top \in \mathcal{F}(\Gamma)$.
- If ϕ is a formula in context Γ, then $\neg\phi$ is in $\mathcal{F}(\Gamma)$.
- If $(\phi_i)_{i \in I}$ are formulas in context Γ, then $\bigcap \phi_i, \bigcup \phi_i$ are in $\mathcal{F}(\Gamma)$.
- If $\Gamma' = (\Gamma, x_1 : T_1, \dots)$ and ϕ is a formula in context Γ' then $(\exists x_1, x_2, \dots, \phi)$ and $(\forall x_1, \dots, \phi)$ are formulas in context Γ.

Considered up to some logical rules that I will not list but for example: the set of formulas in context Γ is a small-complete boolean algebra.

Remark: We are *not* including *equality*.
i.e. if x, y are terms of type A in a context Γ, then $x = y$ is in general not a formula in context Γ.
"Bootstrapping" is done using $\bot, \top$ and quantifiers: if A is a type in context Γ, then $\exists x : A \in \mathcal{F}(\Gamma)$.

One can create a way to talk about equalities of terms of type A, If its introduction rule is $\Gamma \vdash A$: Type, one can add to $\mathbb{T}$ a new type:

$$\Gamma, x, y : A \vdash (x =_A y) : \text{Type}$$

Which we will call an "equality predicate for A". Together with some obvious axioms:

$$\Gamma, x : A \vdash r_x : x =_A x \qquad\qquad \Gamma, x, y : A, t : x =_A y \vdash x \equiv y$$

$$\Gamma, x, y : A, t : x =_A y \vdash t \equiv r_x$$

Adding this type does not change the category of models of $\mathbb{T}$, and we can then use $x = y \in \mathcal{F}$ as a notational shortcut for $(\exists t : x =_A y)$.

Example

Take $\mathbb{T}$ to be the Cartmell theory of categories, i.e. with basic types:

$$\vdash \mathrm{Ob} : \mathrm{Type} \quad X, Y : \mathrm{Ob} \vdash \mathrm{Hom}(X, Y) : \mathit{Type}$$

together with an equality predicate on $\mathrm{Hom}(X, Y)$.
Formulas in $\mathbb{T}$ are exactly the formulas in the language of categories without equalities on objects refered to in theorem of the first slide.

Example

Makkai's FOLDS corresponds exactly to the special case of a Cartmell theory $\mathbb{T}$ with only type introduction axioms (no terms or type equality axioms), with some specific types having an equality predicate, and no type depends on a type that has an equality predicate (except the equality predicate itself of course).

Small Digression:

Given $\mathbb{T}$ a Cartmell theory (possibly infinitary), there is a *weak factorization system* on the category $\mathcal{M}$ of models of $\mathbb{T}$ such that:

- The left class, called cofibrations are the retracts of morphisms $A \to B$ where B is obtained by freely adding elements (of various types) to A.
- The right class, called trivial fibrations, are the maps with the "term lifting property". I.e. the $f : X \to Y$ such that for any type axiom $\Gamma \vdash A :$ Type in the theory, if one has $x \in X(\Gamma)$ and $y = (f(x), t) \in Y(\Gamma, A)$ then there is a lift $x' = (x, k) \in X(\Gamma, A)$ such that $f(k) = t$.

It is cofibrantly generated (with one generator per type axiom in the theory).

Example

If $\mathbb{T}$ is the theory of categories mentioned before, cofibrations are the injective on object functors and trivial fibrations are the fully faithful and surjective on objects functors. These are exactly the cofibrations and trivial fibrations of the "folk" model structure on Cat.

Example

In the case of a FOLDS signature, categories of models are presheaves over the directed category of basic types, cofibrations are exactly the maps that are injective on type that do not have equality predicate. Trivial fibrations were considered by Makkai in his definition of equivalences: equivalences are spans of trivial fibrations.

One can give a more conceptual approach to the notion of formula. Given a Cartmell theory $\mathbb{T}$, and let $\mathcal{C}$ be its category of contexts (possibly infinite):

Definition

A **complete boolean algebra over** $\mathcal{C}$ is a functor $\mathcal{B} : \mathcal{C}^{op} \to \text{Bool}$ where Bool is the category of (possibly large) small-complete boolean algebras:

- If $p : X \twoheadrightarrow Y$ is a display map then $p^* : \mathcal{B}(Y) \to \mathcal{B}(X)$ has a left adjoint $\exists_p : \mathcal{B}(X) \to \mathcal{B}(Y)$.
- For each pullback square in $\mathcal{C}$:

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{g} & Y \end{array}$$

  where p (and hence p') is a display map, one has:

$$g^* \exists_p = \exists_{p'} f^*$$

Theorem

Given $\mathbb{T}$ a Cartmell theory, and $\mathcal{C}$ its category of contexts, the functor sending each Γ to $\mathcal{F}(\Gamma)$ the set of first order formulas in context Γ, with functoriality given by reindexing is the initial complete boolean algebra over $\mathcal{C}$.

Example

Given X a model of $\mathbb{T}$, i.e. a functor $X : \mathcal{C} \to \text{Set}$ that preserves pullback along display maps. One defines $\mathcal{P}X$ to be the complete boolean algebra over $\mathcal{C}$:

$$\mathcal{P}X(\Gamma) = \mathcal{P}(X(\Gamma)) = \{S \subset X(\Gamma)\}$$

There is in particular a unique morphism $\mathcal{F} \to \mathcal{P}X$. It sends any formula ϕ in context Γ, to the set of elements of $X(\Gamma)$ which validate ϕ.

Remark : A morphism $f : X \to Y$ of $\mathbb{T}$-models induces a pullback map $\mathcal{P}Y \to \mathcal{P}X$, which is functorial with respect to the contravariant functoriality, but is in general not compatible with the covariant $\exists_p$ functoriality. In fact one has:

Lemma

Given $f : X \to Y$ a morphism of $\mathbb{T}$-models the induced map $f^* : \mathcal{P}Y \to \mathcal{P}X$ is a morphism of complete boolean algebras over $\mathcal{C}$ if and only if f is a trivial fibration (i.e. has the term lifting property).

This has a very nice consequence:

Theorem (Makkai)

If $f : X \to Y$ is a trivial fibration of $\mathbb{T}$-models and ϕ is a formula in context Γ, then $x \in X(\Gamma)$ satisfies ϕ if and only if $f(x) \in Y(\Gamma)$ satisfies ϕ.

Indeed the previous result shows that trivial fibrations preserve the interpretation of first order formulas:

$$\begin{array}{ccc} \mathcal{F} & & \\ {\scriptstyle !}\downarrow & \searrow^{!} & \\ \mathcal{P}Y & \xrightarrow[f^*]{} & \mathcal{P}X \end{array}$$

We now assume that the category of $\mathbb{T}$-Mod carries a model structure whose cofibrations and trivial fibrations are as defined above. Then:

- Elements of $X(\Gamma)$ can be seen as morphisms $\mathbb{F}\Gamma \to X$, where $\mathbb{F}\Gamma$ is the (cofibrant) object freely generated by the context Γ.
- In particular one can ask whether two elements of $X(\Gamma)$ are homotopic or not (especially if X is fibrant).

Theorem

Under these assumptions:

- *If X is fibrant, $x, y \in X(\Gamma)$ and $\phi \in \mathcal{F}(\Gamma)$ then if $x \sim y$ one has $\phi(x) \Leftrightarrow \phi(y)$.*
- *If $f : X \to Y$ is a weak equivalence between fibrant $\mathbb{T}$-models and $x \in X(\Gamma)$, then $\phi(x) \Leftrightarrow \phi(f(x))$.*

Most model categories we use in practice are of the form above. But we can (as promised) make sense of this for an arbitrary model category $\mathcal{M}$:

- One can always build an infinitary Cartmell theory such that $\mathrm{Con}_{\mathbb{T}} \simeq \mathrm{Cof}(\mathcal{M})^{op}$ and use the first order language of $\mathbb{T}$.
- Even better: the purely categorical definition of formulas (as an initial boolean algebra) make sense independently of the choice of a Cartmell theory.
- With these definitions, everything I have mentioned so far can be formulated and proved directly in terms of $\mathcal{M}$.

In this point of view "contexts" are cofibrant objects, $\mathcal{F}$ is a functor $\mathcal{F} : \mathrm{Cof}(\mathcal{M}) \to \mathrm{Bool}$ with left adjoints along cofibrations, "models" are general objects of $\mathcal{M}$, and $x \in X(\Gamma)$ just means that $x : \Gamma \to X$.

Last question: Invariance of the first order logic along Quillen adjunction and Quillen equivalences ?

For a Quillen adjunction:

$$L : \mathcal{M} \leftrightarrows \mathcal{N} : R$$

One can inductively construct:

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{M}}(\Gamma) & \to & \mathcal{F}_{\mathcal{N}}(L\Gamma) \\ \phi & \mapsto & L\phi \end{array}$$

such that if $x : L\Gamma \to X$ then

$$(L\phi)(x) \Leftrightarrow \phi(\tilde{x})$$

where $\tilde{x}$ is the map $\tilde{x} : \Gamma \to RX$ obtained from x.

To go further one needs to consider the quotient $\mathcal{F}(\Gamma) \twoheadrightarrow \mathcal{F}^h(\Gamma)$ by the equivalence relation $\phi \sim \psi$ iff ϕ and ψ have equal interpretation in all fibrant objects:

$$\mathcal{F}^h(\Gamma) := \mathcal{F}(\Gamma)/\sim$$

- If $f : \Gamma \to \Gamma'$ is a weak equivalence between cofibrant objects then $\mathcal{F}^h\Gamma \simeq \mathcal{F}^h\Gamma'$.
- If $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ is a Quillen equivalence then $\mathcal{F}^h_{\mathcal{M}}(\Gamma) \simeq \mathcal{F}^h_{\mathcal{N}}(L\Gamma)$.

The proof I have of this last result is surprisingly hard, and very inexplicit (!).

Example

For the *Joyal* model structure on simplicial sets, this produces a "language of quasi-categories":

- One can talk about simplicies of a quasi-category satisfying specified boundary conditions.
- One cannot talk about equality of simplicies in general (other than by specifying boundary condition).

This language is automatically invariant under "isormophisms" (in a quasi-category) and categorical equivalences of quasi-categories. Most notions developed for quasi-category are naturally written in this language.

Example

For the *projective model structure on chain complexes*, one can talk about chains $x \in C_n$, the group operations on them, cycles $x \in Z_n$, and more generally, for any chain t, the type of all x such that $\partial x = t$, but in general not the equality of chains, only whether $x - x'$ is a boundary or not.

Example

The category of pairs of categories with a functor $F : C_1 \to C_2$ has two model structures whose equivalences are pairs of equivalences. One where all object are fibrants and the other where fibrant objects are (iso)fibrations.

- The language of the first one only allows to use the language of C_1 and C_2 separately.
- The language of the second one allows in addition to consider the type of objects $x \in C_1$ such that $F(x) = y$ for each object $y \in C_2$.
- The fact that the two model structures are Quillen equivalent shows that one can always translate a formula of the second types into an equivalent formula of the first type. (Example: Grothendieck fibrations VS Street fibrations).