From algebraic weak factorisation systems to models of type theory

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Mini-courses

- Charles Rezk
- Peter LeFanu Lumsdaine

Special lectures

- Benedikt Ahrens
- André Joyal
- Emily Riehl

Tutorials

- Paige North
- Nima Rasekh
- Karol Szumilo

Topic: models of type theory

Several issues

- standard type constructors (Π-types, Σ-types, ...)
- intensional **Id**-types
- associativity of substitution
- interaction of substitution with type formers

Optional requirements

- models may be defined constructively
- 'homotopical' models should support the types-as-spaces idea

Today

- method for obtaining models via awfs

Outline

Part I: Context and motivation

- Models of type theory
- Two goals

Part II: Comprehension categories

- Modelling type theory
- The right adjoint splitting

Part III: Algebraic weak factorisation systems

- Type-theoretic awfs's
- Examples

References

N. Gambino and C. Sattler
The Frobenius condition, right properness and uniform fibrations
Journal of Pure and Applied Algebra, 2018

M. F. Larrea
Models of type theory from algebraic weak factorisation systems
PhD thesis, University of Leeds, 2019

All results below, unless otherwise stated, are due to Marco Larrea.

Part I: Context and motivation

How can we construct models of type theory?

It is useful to isolate three kinds of structures:

(1) **Raw structures**, i.e. mathematical structures occurring in practice, e.g.
- Quillen model categories
- weak factorisation systems

(2) **Intermediate structures**, i.e. a packaging of the above which mirrors syntax, e.g.
- comprehension categories
- categories with a universe

(3) **Models** = genuine on-the-nose models, e.g.
- split comprehension categories
- contextual categories

Constructing a model of type theory typically involves two steps:

$$(1)\text{ Raw structure} \Longrightarrow (2)\text{ Intermediate structure} \Longrightarrow (3)\text{ Model}$$

Models of type theory?

Many options of model are possible. How should we choose?

Criteria: good notions should

- be supported by a general theory
- facilitate step (1) $\Rightarrow$ (2)
- support a theorem for the step (2) $\Rightarrow$ (3)

Comprehension categories

Today, we work with

- comprehension categories as intermediate structures
- split comprehension categories as models

As we will see, these satisfy the criteria above.

In particular, there are three ways of splitting comprehension categories:

(1) right adjoint splitting (Bénabou, Hoffmann)
(2) left adjoint splitting (Lumsdaine and Warren)
(3) splitting via universe (Voevodsky)

General method for constructing models

Step 1

- Isolate once and for all what structure on a comprehension category we need in order for a splitting to produce a model.

Step 2

- Find examples of such a structure.

Note: quite a lot is already known for Step 1

- for left adjoint splitting, see Lumsdaine-Warren
- for the right adjoint splitting, the result can be extracted from Hoffmann and Warren (see later)
- for the universe splitting, the result can be translated from work of Voevodsky

Today's seminar

- Review what structure on a comprehension category is necessary in order for the right adjoint splitting to give a model of type theory.
- Describe how natural examples of such a structure can be found.

Key notion: algebraic weak factorisation system.

Executive summary: The 'algebraic' aspect of awfs makes it possible to satisfy the assumptions necessary to make the right adjoint splitting work.

Note:

- This is an idea that goes back to Richard Garner (cf. comments in Michael Warren's thesis, Chapter 2, page 34)
- see also [van den Berg and Garner 2012].

Part II: Models via comprehension categories

Fibrations

A functor $p : \mathbb{E} \to \mathbb{C}$ is said to be a **fibration** if whenever we have

$$\begin{array}{c} \mathbb{E} \\ \downarrow{\scriptstyle p} \\ \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} & & A \\ & & \vdots \\ \Delta & \xrightarrow[\sigma]{} & \Gamma \end{array}$$

we get a Cartesian map

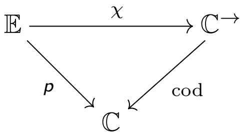
$$\begin{array}{c} \mathbb{E} \\ \downarrow{\scriptstyle p} \\ \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} A[\sigma] & \longrightarrow & A \\ \vdots & & \vdots \\ \Delta & \xrightarrow[\sigma]{} & \Gamma \end{array}$$

Note Cartesian here means universal in a suitable way.

Note. All fibrations today will be assumed to be cloven.

Comprehension categories

A **comprehension category** has the form

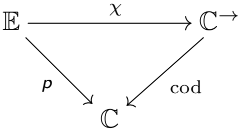


where

- p is a fibration
- $\mathbb{C}$ has pullbacks, so cod is a fibration
- χ sends Cartesian squares to pullback squares.

Comprehension categories: some intuition

We think of



as follows:

- $\mathbb{C}$ is a category of contexts
- For $\Gamma \in \mathbb{C}$, $\mathbb{E}_\Gamma$ is the category of types A in context Γ
- Functor χ maps a type A in context Γ to the 'display map'

$$\chi_A : \Gamma.A \to \Gamma$$

- cod models substitution in contexts
- p models substitution in types

Split comprehension categories

Without further assumptions, we only have

$$A[\sigma][\tau] \cong A[\sigma \circ \tau]\,, \qquad A[1_\Gamma] \cong A$$

When these are identities, we have a **split** comprehension category.

Note:

- These are hard to find ‘in nature’
- We have splitting procedures

Today, we focus on the so-called **right adjoint splitting**.

The right adjoint splitting

For a comprehension category $(\mathbb{C}, p, \chi)$, we let $\mathbb{E}^R$ be the category with

- **Objects:** pairs $(A, A[-])$, where $A \in \mathbb{E}$ and $A[-]$ is a function mapping $\sigma : \Delta \to \Gamma$ to a Cartesian arrow

$$\begin{array}{ccc} A[\sigma] & \xrightarrow{\sigma^*} & A \\ \vdots & & \vdots \\ \Delta & \xrightarrow[\sigma]{} & \Gamma \end{array}$$

- **Maps:** $f : (A, A[-]) \to (B, B[-])$ are maps $f : B \to A$ in $\mathbb{E}$.

We then obtain a split comprehension category

$$\begin{array}{ccccc} \mathbb{E}^R & \longrightarrow & \mathbb{E} & \xrightarrow{\chi} & \mathbb{C}^{\to} \\ & {\scriptstyle p^R}\searrow & & \swarrow{\scriptstyle \mathrm{cod}} & \\ & & \mathbb{C} & & \end{array}$$

Pseudo-stable **Id**-types

Let $(\mathbb{C}, p, \chi)$ be a comprehension category.

Definition. A **pseudo-stable choice of Id-types** consists of a choice, for each $\Gamma \in \mathbb{C}$ and $A \in \mathbb{E}_\Gamma$, of

- $\mathbf{Id_A} \in \mathbb{E}_{\Gamma.A.A}$
- reflexivity maps r_A
- elimination maps j_A
- for $\sigma : \Delta \to \Gamma$ in $\mathbb{C}$ and every Cartesian $f : B \to A$ over σ, in $\mathbb{E}$, we have a Cartesian map

  $$\mathrm{Id}_f : \mathrm{Id}_B \to \mathrm{Id}_A$$

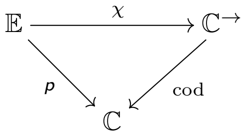
  over $\delta_f : \Delta.B.B \to \Gamma.A.A$, suitably functorial and coherent with reflexivity and elimination maps.

Note. Elimination maps are operations selecting diagonal fillers.

Similar definitions can be given for Π-types and Σ-types.

A coherence theorem

Theorem. Let



be a comprehension category equipped with pseudo-stable choices of Σ, Π and **Id**-types.

Then its right adjoint splitting $(\mathbb{C}^R, p^R, \chi^R)$ is a split comprehension category equipped with strictly stable choices of Σ, Π and **Id**-types.

Note. It remains to find examples of comprehension categories with pseudo-stable choices of Σ, Π and **Id**-types.

Problem: Weak factorisation systems and model categories do not give rise to examples, as elimination maps are not given by operations.

Part III: Algebraic weak factorisation systems

Issues

Fundamental distinction:

- satisfaction of a property
- the existence of additional structure.

Examples:

- categories with finite products
- fibrations.

Sometimes ignoring this distinction is not harmful.

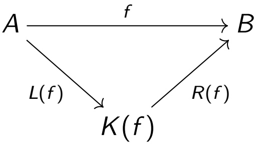
But sometimes things become more subtle:

- choices are unique up to higher and higher homotopies
- coherence issues
- constructivity issues.

Algebraic weak factorisation systems

Recall that in a weak factorization system $(\mathbf{L}, \mathbf{R})$, we often ask for

- **functorial factorizations**, i.e. functors such that



  gives the required factorization.

In an **algebraic weak factorization system**, we ask also that

- L has the structure of a comonad,
- R has the structure of a monad,
- a distributive law between L and R.

Grandis and Tholen (2006), Garner (2009).

L-maps and **R**-maps

Given an awfs (L, R) on a category $\mathbb{C}$, the comonad and the monad

$$L : \mathbb{C}^{\to} \to \mathbb{C}^{\to} , \qquad R : \mathbb{C}^{\to} \to \mathbb{C}^{\to}$$

are in particular a copointed and pointed endofunctors, respectively.

So we can consider the categories

$$L\text{-Map} , \qquad R\text{-Map}$$

of coalgebras and algebras for the copointed and pointed endofunctors.

These replace the standard classes of left and right maps in a wfs.

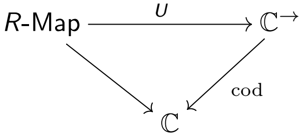
Note: There are forgetful functors

$$L\text{-Map} \to \mathbb{C}^{\to} , \qquad R\text{-Map} \to \mathbb{C}^{\to}$$

So being a left map or a right map is a **structure**, not a property.

From awfs's to comprehension categories

Proposition. Let (L, R) be an awfs on a category $\mathbb{C}$. Then



is a comprehension category.

This has always a choice of pseudo-stable Σ-types.

Type-theoretic awfs's

Definition. A **type-theoretic awfs** consists of an awfs (L, R) equipped with

- a stable functorial choice of path objects, i.e. factorisations

$$\begin{array}{ccc} X & \xrightarrow{\quad \delta_f \quad} & X \times_Y X \\ & {\scriptstyle r_f} \searrow \quad \nearrow {\scriptstyle p_f} & \\ & \mathcal{P}_f & \end{array}$$

  such that r_f is an L-map, p_f is an R-map, satisfying stability and functoriality conditions.

- a functorial Frobenius structure, i.e. a lift of the pullback functor so that the pullback of an L-map along an R-map is an L-map.

Note: The Frobenius property is necessary to model Π-types.

From type-theoretic awfs to comprehension categories

Theorem. Let (L, R) be a type-theoretic awfs. Then the associated comprehension category

$$\begin{array}{ccc} R\text{-Map} & \xrightarrow{\quad U \quad} & \mathcal{E}^{\to} \\ & \searrow \qquad \swarrow_{\mathrm{cod}} & \\ & \mathcal{E} & \end{array}$$

is equipped with pseudo-stable choices of Σ-, Π-, and **Id**-types.

So by the earlier coherence theorem, we are left with the question of finding examples of type-theoretic awfs's.

An easy example comes from the category of groupoids: the right maps are the normal isofibrations.

Examples of type-theoretic awfs's

Let

- $\mathcal{E}$ be a presheaf category
- $I \in \mathcal{E}$ an interval object with connections.

E.g. Simplicial sets and cubical sets.

From [Gambino and Sattler 2017], we know

- an awfs $(\mathbf{C}, \mathbf{F}_t)$ such that $\mathbf{C}$-Map is the category of monomorphisms and pullback squares,
- an awfs $(\mathbf{C}_t, \mathbf{F})$ such that F-Map is a category of uniform fibrations à la Bezem-Coquand-Huber
- the awfs $(\mathbf{C}_t, \mathbf{F})$ has the Frobenius property.

Building on this, Larrea showed

Theorem. $(\mathbf{C}_t, \mathbf{F})$ is a type-theoretic awfs.

Key step: showing that the 'reflexivity map' $r_f : X \to \mathcal{P}(f)$ is a $\mathbf{C}_t$-map.

Summary

Type-theoretic weak factorisation systems give rise to models of type theory with Σ-types, Π-types and **Id**-types.

(1) Type-theoretic awfs

$\Longrightarrow$ (2) Comprehension categories with pseudo-stable ...

$\Longrightarrow$ (3) Comprehension categories with strictly stable ...

Examples

- presheaf categories (e.g. **SSet** and **CSet**) provide many examples of type-theoretic awfs.