A survey of constructive models of univalence

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Presheaf models

We explore how to build *presheaf* models of univalence

Inspired from Voevodsky's simplicial set model

But in a constructive metatheory

With a cumulative hierarchy of universes $\mathcal{U}_0,\ \mathcal{U}_1, \ldots$

Metatheory: can be CZF or "extensional" type theory (NuPrl)

or more primitive (nominal extension of type theory)

Content of the talk

Describe the models, parametrised by two presheaves $\mathbb{I}$ and $\mathbb{F}$

Relation with Cisinski's work (2006) on model structures on presheaf models

Describe a model structure on these presheaf models

Some applications of having a model in a constructive metatheory

Presheaf models

Base category $\mathcal{C}$, objects $I, J, K, \ldots$

Two parameters: interval/segment $\mathbb{I}$ and $\mathsf{Cofib} : \Omega \to \Omega$

Cofib defines a subpresheaf $\mathbb{F}$ of Ω

$\Omega(I)$ *class* of sieves (not a set in a predicative setting), subobject classifier

Classically one can take $\mathbb{F} = \Omega$

Interval $\mathbb{I}$

Used to represent the notion of paths

Two global elements 0 and 1

$0 \neq 1$

Constructively, we should assume that each $\mathbb{I}(K)$ has decidable equality

We can ask for *connections* structure

$x \wedge 0 = 0 \wedge x = 0 \qquad x \wedge 1 = 1 \wedge x = x$

$x \vee 0 = 0 \vee x = x \qquad x \vee 1 = 1 \vee x = 1$

Cofibrant propositions $\mathbb{F}$

If $\varphi : \Omega$ we define the subsingleton $[\varphi]$, subpresheaf of 1

$[\varphi](K) = \{0 \mid 1_K \in \varphi(K)\}$ where $\varphi(K)$ sieve on K

Cofib : $\Omega \to \Omega$ determines a subpresheaf $\mathbb{F} \subseteq \Omega$

We think of $[\psi]$ as a dependent type over $\psi : \mathbb{F}$

A mono $A \to B$ classified by $B \to \mathbb{F}$ is called a *cofibration*

Cofibrant propositions $\mathbb{F}$

Axioms explored by

Orton, Pitts
Axioms for Modelling Cubical Type Theory in a Topos 2017

(1) Cofib(1)

(2) if Cofib(φ) and Cofib(ψu) for $u : [\varphi]$ then Cofib$(\exists(u : [\varphi])\psi u)$

This expresses that Cofib defines a *dominance*: the identity map is a cofibration and cofibrations are closed by composition

(3) Cofib closed by $\vee$

Cofibrant propositions $\mathbb{F}$

Axioms mixing $\mathbb{I}$ and Cofib

(4) $\mathsf{Cofib}(i = 0)$ and $\mathsf{Cofib}(i = 1)$

(5) $\mathsf{Cofib}(\forall(i : \mathbb{I})\psi i)$ if $\forall(i : \mathbb{I})\mathsf{Cofib}(\psi i)$

Cofibrant propositions $\mathbb{F}$

Escardo
Synthetic topology of data types and classical spaces, 2004

In the setting of synthetic topology a cofibration corresponds to an *open* map inclusion and

$\mathsf{Cofib}(\forall(i:\mathbb{I})\psi i)$ if $\forall(i:\mathbb{I})\mathsf{Cofib}(\psi i)$

states that $\mathbb{I}$ is *compact*

which is appropriate since $\mathbb{I}$ is supposed to formally represent $[0,1]$!

Another formulation of compactness is: the path functor $X \mapsto X^{\mathbb{I}}$ preserves cofibration

Cofibrant propositions

If for each K the set $\mathbb{I}(K)$ has decidable equality one can require

(4*) $\forall(i\ j : \mathbb{I})\mathsf{Cofib}(i = j)$ instead of (4) $\forall(i : \mathbb{I})\mathsf{Cofib}(i = 0) \wedge \mathsf{Cofib}(i = 1)$

(4*) plays a crucial role for the cartesian cubical set model

This expresses that $\Delta : \mathbb{I} \to \mathbb{I}^2$ is a cofibration

Classically one can take $\mathbb{F} = \Omega$ and all monos are cofibrations

In all cases we looked at, $\mathbb{F}(I) =$ all decidable sieves is also a possible choice

Model of type theory

Two parameters $\mathbb{I}$ and $\mathbb{F}/\mathsf{Cofib}$ we build a model of type theory with Paths

Presheaves form a model of dependent type theory

$\Gamma \vdash A$ means Γ presheaf and A presheaf on the category of elements of Γ

$\Gamma \vdash a : A$ means a global section of A

$\sigma : \Delta \to \Gamma$ means natural transformation

$(\rho, u) : \Gamma.A$ if $\rho : \Gamma$ and $u : A\rho$

Projection $\Gamma.A \to \Gamma$

Model of type theory

We can use usual syntax of dependent type theory

$\mathsf{Type}_n(\Gamma)$ set of $\mathcal{U}_n$ presheaves on the category of elements of Γ

Type_n is *representable* by a presheaf V_n, universe of size n

Model of type theory

Given Γ presheaf and $\gamma : \Gamma^{\mathbb{I}}$ define $const(\gamma)$ to mean $\forall (i : \mathbb{I}) \gamma(0) = \gamma(i)$

If $\Gamma \vdash A$ and $\Gamma \vdash a_0 : A$ and $\Gamma \vdash a_1 : A$ then we define $\Gamma \vdash \mathsf{Path}\ A\ a_0\ a_1$

$\omega : (\mathsf{Path}\ A\ a_0\ a_1)\rho$ if $\omega : (A\rho)^{\mathbb{I}}$ and $\omega(0) = a_0\rho$ and $\omega(1) = a_1\rho$

Fibration structure

$\Gamma \vdash A$ *fibration*

We have an extension operation, given $\gamma : \Gamma^{\mathbb{I}}$ and $\psi : \mathbb{F}$ and a partial section

$\Pi(i : \mathbb{I})[\psi \vee i = 0] \to A\gamma(i)$

extends it to a total section

$\Pi(i : \mathbb{I})A\gamma(i)$

and we can extend any element $\Pi(i : \mathbb{I})[\psi \vee i = 1] \to A\gamma(i)$ to a total section

This can be expressed internally as a type $\mathsf{Fib}(\Gamma, A)$ of fibration *structures*

Model of type theory

If $\psi : \mathbb{F}$ then $\psi \vee i = 0$ (resp. $\psi \vee i = 1$) define a *cofibration* of codomain $\mathbb{I}$

This can be seen as a generalized open box

ψ is the "tube" of the box and $i = 0$ (resp. $i = 1$) the given lid

$\Gamma \vdash A$ fibration iff $\Gamma.A \to \Gamma$ has the right lifting property w.r.t. any open box

Involves only the interval $\mathbb{I}$ and the notion of partial elements

(no notion of higher dimensional cubes)

Model of type theory

What we want is the *transport* property

$A\gamma(0) \leftrightarrow A\gamma(1)$

This expresses that path equal elements share the same "properties"

Model of type theory

We define $\Gamma \vdash \Sigma\ A\ B$ if $\Gamma \vdash A$ and $\Gamma.A \vdash B$

We should prove that $\Sigma\ A\ B$ has a transport function if A and B have

This is the case if we have *path lifting* property

This extends an element in $A\gamma(0)$ (resp. $A\gamma(1)$) to a section

$\Pi(i : \mathbb{I})A\gamma(i)$

Model of type theory

Should have that $\Gamma \vdash \mathsf{Path}\ A\ a\ b$ has a path lifting operation if A has one

This is the case if we refine the path lifting property to the extension property of any

$\Pi(i:\mathbb{I})[\psi \vee i = 0] \to A\gamma(i)$

to a total section $\Pi(i:\mathbb{I})A\gamma(i)$

This motivates (internally) the refinement of the path lifting property

Eilenberg

This refinement can be seen as a formulation of the following principle of *homotopy extension property* emphasized by Eilenberg (1939)

Proposition: *If A is a subpolyhedra of B, given two homotopic functions $f_0\ f_1 : A \to X$ and an extension $f_0' : B \to X$ of f_0 there is an extension f_1' of f_1 homotopic to f_0*

Proofs of basic results about homotopy "can be obtained quite neatly by repeated, and sometimes tricky, use of this general lemma" (Bourbaki's notes on homotopy by Eilenberg, 1951)

The notion of cofibrations represents the notion of subpolyhedra

Model of type theory

For closure under dependent *products* there are two alternatives (so far)

In both cases, the idea is to reduce path lifting to transport

(A_1) we add a connection structure to $\mathbb{I}$

(A_2) we *refine* the path lifting property to the extension property of any

$\Pi(i:\mathbb{I})[\psi \vee i = k] \rightarrow A\gamma(i)$

to a total section $\Pi(i:\mathbb{I})A\gamma(i)$ for an *arbitrary* given k in $\mathbb{I}$

instead of simply requiring it for $k = 0$ or $k = 1$

Model of type theory

In both cases (A_1) and (A_2), we can show closure under dependent product

$\psi \vee i = 0$ and $\psi \vee i = 1$ generalized form of open boxes

The solution (A_2) uses yet a more general form $\psi \vee i = k$

Angiuli, Brunerie, C. Favonia, Harper, Licata
Cartesian cubical type theory, 2017

Equivalence extension property

From "compactness" of $\mathbb{I}$ follows the *equivalence extension property*

Given

$\sigma : \Delta \rightarrowtail \Gamma$ cofibration

$\Gamma \vdash A$ fibration

$\Delta \vdash e : \mathsf{Equiv}\ T\ A\sigma$ with $\Delta \vdash T$ fibration

we can find $\Gamma \vdash \tilde{e} : \mathsf{Equiv}\ \tilde{T}\ A$ such that $\tilde{T}\sigma = T$ and $\tilde{e}\sigma = e$ and $\Gamma \vdash \tilde{T}$ fibration

Equivalence extension property

This equivalence extension property is crucial to show univalence in the form

$\mathsf{isContr}(\Sigma(X : U_n)\mathsf{Equiv}\ X\ A)$

and crucial to show that the universe of fibrant types is fibrant

Model of type theory

We get a model of Π, Σ, Path

Do we get a model of type theory?

Two issues

-elimination rule J for Path holds only up to Path equality

-universes

Model of type theory

-elimination rule J for Path holds only up to Path equality

Nothing requires the path lifting of a constant path to be constant!

This is solved by (an adaptation of) Swan's idea

(ψ, ω) : Id $A\ a\ b$ if $\psi : \mathbb{F}$ and ω : Path $A\ a\ b$ such that $\psi \Rightarrow const(\omega)$

If (ψ, ω) : Id $A\ a\ b$ and $A \vdash P$ and $u : Pa$ then we can extend the map

$w : \Pi(i : \mathbb{I})[\psi \vee i = 0] \to P(\omega\ i) \qquad w\ i\ x = u$

to a section $s : \Pi(i : \mathbb{I})P(\omega\ i)$ which satisfies $s\ 1 = u$ if $\psi = 1$

Model of type theory

-universes: we can build universes of fibrant objects (of a given size) provided the path functor $X \mapsto X^{\mathbb{I}}$ has a right adjoint

This can even be done internally using suitable modalities

Licata, Orton, Pitts, Spitters
Internal Universes in Models of Homotopy Type Theory 2018

Typically this property is *not* satisfied for *simplicial sets* and $\mathbb{I} = \Delta^1$

Universes

$\mathsf{FType}_n(\Gamma)$ is the set of pairs A, c with A in $\mathsf{Type}_n(\Gamma)$ and c in $\mathsf{Fib}(\Gamma, A)$

If the path functor has a right adjoint, this is *representable*

U_n represents the universe of fibrant types

From the *equivalence extension property* follows both

U_n is *fibrant* and U_n is *univalent*

Summary

We had two parameters $\mathbb{I}$ and $\mathbb{F}$

Variation in the choice of anodyne maps (generating trivial cofibrations)

(1) $\psi \vee i = 0$ and $\psi \vee i = 1$ then $\mathbb{I}$ is required to have connections

(2) $\psi \vee i = j$ then we require $\mathsf{Cofib}(i = j)$

In all cases, the path functor $X \mapsto X^{\mathbb{I}}$ should have a right adjoint

Summary

The last condition holds if $\mathbb{I}$ is representable and $\mathcal{C}$ is cartesian

$\mathcal{C}$ can be Lawvere theory of distributive lattices, de Morgan algebra, Boolean algebra, $\ldots$

$\mathcal{C}$ can be Lawvere theory of two constants $0, 1$ (cartesian cubical sets) but then we should have $\mathsf{Cofib}(i = j)$

We can always take $\mathbb{F}(I)$ decidable sieves on I

Summary

Everything works with $\mathcal{C} = \Delta$ and $\mathbb{I} = \Delta^1$ and $\mathbb{F}(K)$ decidable sieves

but the last condition on the path functor $X \mapsto X^{\mathbb{I}}$ having a right adjoint

We cannot define a universe of fibrant types with the definition which requires a fibration as a *structure* and not as a *property*

Question: can we find a refinement of the notion of fibration structure which also works with simplicial set?

Simplicial sets

If the path functor $X \mapsto X^{\mathbb{I}}$ has a right adjoint

Then also $X \mapsto X^{\mathbb{I} \times \mathbb{I}}$

And so $X \mapsto X^{\mathbb{I} \times \mathbb{I}}$ would preserve colimits

But this is not the case for $\mathbb{I} = \Delta^1$

Pointwise fibrant

In the case $\Gamma = 1$, a family $\Gamma \vdash A$ is a type

It is *fibrant* if we can extend any partial path $\Pi(i : \mathbb{I})[\psi \vee i = 0] \to A$ to a total path in $A^{\mathbb{I}}$ (and similarly with $i = 1$)

In general $\Gamma \vdash A$ fibration is not the same as: each $A\rho$ fibrant

If $\Gamma \vdash A$ is pointwise fibrant and we have (generalized "transport")

Given $\gamma : \Gamma^{\mathbb{I}}$ and $\psi : \mathbb{F}$ and $\psi \Rightarrow const(\gamma)$ then any *constant* partial section $\Pi(i : \mathbb{I})[\psi \vee i = 0] \to A\gamma(i)$ can be extended to a total section

Similarly with $\Pi(i : \mathbb{I})[\psi \vee i = k] \to A\gamma(i)$ for any $k : \mathbb{I}$

Pointwise fibrant

We can decompose a fibration structure into a

-pointwise fibrant structure

-transport structure

This is needed to interpret Higher Inductive Types

Question: can we adapt this interpretation of Higher Inductive Types to the simplicial set model?

Relation to Cisinski's work

Cisinski
Les préfaisceaux comme modèles des types d'homotopie 2006

Cisinski works in a classical metatheory and takes $\mathbb{F} = \Omega$

We look at the case where the functorial cylinder is given by the interval

He analyses model structures where anodyne maps are generated open boxes inclusion (i.e. defined by $\psi \vee i = 0$ and $\psi \vee i = 1$)

He describes then a model structure where the fibrant *objects* are the same as the one we have described

Complete Cisinski's model structure

But his notion of fibrant *map* is much more subtle

First define a map $A \to B$ to be a *weak equivalence* iff for any X fibrant the map $X^B \to X^A$ is a homotopy equivalence

Then define trivial cofibration as mono and weak equivalence

Then define fibration map by orthogonality

Cisinski called "naive" fibration our notion of fibration (only orthogonal to open boxes inclusion)

The model structure is said to be *complete* if the two notions coincide: a map is fibrant as soon as it is orthogonal to open boxes inclusion

Model structure

In our framework we can define a map to be

-trivial cofibration left lifting property w.r.t. any fibration (non *absolute* definition)

We then show (in a constructive framework) the factorization of any map $A \to B$

$A \rightarrowtail E \overset{\sim}{\twoheadrightarrow} B$ cofibration-trivial fibration

$A \overset{\sim}{\rightarrowtail} F \twoheadrightarrow B$ trivial cofibration- fibration

Factorization cofibration-trivial fibration

The factorization of $\alpha : A \to B$ in

$A \; \rightarrowtail \; F \overset{\sim}{\twoheadrightarrow} B$

is direct and can be done internally

$(b, \psi, u) : F$ if $b : B$ and $\psi : \mathbb{F}$ and $u : A$ and $\psi \Rightarrow \alpha(u) = b$

Factorization trivial cofibration-fibration

The factorization of $\alpha : A \to B$ in

$A \overset{\sim}{\rightarrowtail} E \ \twoheadrightarrow \ B$

is more subtle

It can be done by an inductive process, which does not involve quotient (nor ordinals)

Model structure

Using the fact that we have (fibrant) universes we get

Key property: *any fibration can be extended along a trivial cofibration*

Indeed, given a fibration $A : \Delta \to U_n$ and a trivial cofibration $\Delta \overset{\sim}{\rightarrowtail} \Gamma$ *since* U_n *is fibrant* we can extend A to a map $\Gamma \to U_n$

Model structure

Define now a map $A \to B$ to be a weak equivalence iff the map $A \overset{\sim}{\rightarrowtail} F \twoheadrightarrow B$ is a *trivial* fibration

(Remember that "trivial fibration" is well defined)

We can show

-trivial cofibration $=$ cofibration and weak equivalence so it is absolute after all

-trivial fibration $=$ fibration and weak equivalence

and we get a Quillen model structure

Model structure

Since this model structure has the same notion of *fibrant objects* and *cofibration* as in Cisinski model structure it *coincides* (by a result of Joyal) with Cisinski model structure

See *Categorical Homotopy Theory*, Riehl, Theorem 15.3.1 or nlab

Hence we get a new class of *complete* Cisinski model structures

Model of type theory

We have a class of models with two parameters

$\mathbb{I}$ and $\mathbb{F}$

Model of type theory

How do these models compare?

Huber's *canonicity theorem*

A closed term in Nat (using univalence) has the same value in all these models

Open problems

Classical meta framework

In the case of distributive lattices, or cartesian cubical sets and $\mathbb{F} = \Omega$ there is a canonical map $\hat{\mathcal{C}} \rightarrow \hat{\Delta}$

Is this a Quillen equivalence?

Do these models validate excluded middle?

Open problems

Note that any topological space has a fibrant structure in the case of distributive lattices, de Morgan algebras, cartesian cubical sets

Note also that simplicial sets satisfy all laws of “distributive lattice” cubical sets (connections) for type theory without universes

Checking the model

Using NuPrl as a metalanguage

E.g. we define $\mathcal{C}$ as a NuPrl type and we define internally what is a presheaf (in a given universe)

We check that we get a model of type theory (Bickford)

In Agda, postulating some axioms (Orton-Pitts)

Checking the model

These models only justify syntax with *explicit* substitutions

CZF justifies syntax with *implicit* substitution

Application 1: consistency strength

The metatheory can be CZF extended with a hiararchy of universes as in

Aczel
On Relating Type Theory and Set Theory, 1998

or it can be NuPrl, as done by Bickford

In both cases, these systems are known to have the same proof theoretic strength as dependent type theory with Π, Σ, W and a hieararchy of universes

The axiom of univalence and propositional truncation does not add any proof theoretic strength to type theory

E.g. provably total functions $\mathbb{N} \rightarrow \mathbb{N}$ are the same

Application 2: impredicative universe

Uemuara
Cubical Assemblies and the Independence of the propositional resizing axiom 2018

We work in an *extensional* type theory with an impredicative universe

We get a model of type theory with an impredicative universe

Awodey, Frey, Speight
Impredicative Encodings of (Higher) Inductive Types 2018

Question: can one use this model to show consistency of *Church's Thesis* (formulated with existence) together with univalence and propositional truncation?

Application 3: consistency with uniform continuity

We start with $\mathcal{C}$ a category of cubes with $\mathbb{I}$ and $\mathbb{F}$ for which we have a model of type theory

We let $\mathcal{D}$ be the category of Boolean *spaces*

Opposite of the category of Boolean *algebras*

Covering by partition of unity

This is a Boolean version of the Zariski site

Application 3: consistency with uniform continuity

We can take as our base category $\mathcal{C} \times \mathcal{D}$

Objects are now $I|B$ where I object of $\mathcal{C}$ and B Boolean algebra

$\mathbb{I}'(J|B) = \mathbb{I}(J)$

$\mathbb{F}'(J|B)$ can be $\mathbb{F}(J)$ or all decidable sieves on $J|B$

Application 3: consistency with uniform continuity

On this model we have a type of *covering* Cov and (internally) a family D_c of *left exact modalities* in the sense of

Rijke, Shulman, Spitters
Modalities in homotopy type theory, 2017

The objects modal for all these modalities form a model ("stack" model) of type theory with univalence

In this model all functions $2^N \to N$ are uniformely continuous

Application 4: countable choice

Similarly we can adapt the usual sheaf models to show that countable choice is not provable in type theory with univalence and propositional truncation

Some references

Cisinski
Les préfaisceaux comme modèles des types d'homotopie 2006

Licata, Orton, Pitts, Spitters
Internal Universes in Models of Homotopy Type Theory 2018

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