Liquid Tensor Experiment

Joint work with

- Peter Scholze

The Lean community

- Adam Topaz
- Riccardo Brasca
- Patrick Massot
- Scott Morrison
- Kevin Buzzard
- Bhavik Mehta
- Filippo A.E. Nuccio
- Andrew Yang
- Damiano Testa
- Heather Macbeth
- Mario Carneiro
- many others

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Check the main theorem of liquid vector spaces

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…on a computer

Credit: https://spongebob.gavinr.com/

Credit: https://twitter.com/Jcrudess/status/1338922029278441483/photo/1

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Peter Scholze

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tracks include: *Solid Resolution Theory*,
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2022 Jul 14: complete formal verification of
main theorem of liquid vector spaces

Why liquid vector spaces?

1. Embeds real/complex manifolds into a unifying category of geometric objects: *analytic spaces*

2. Makes algebraic methods available in functional analysis

Topological algebraic problem

$$\mathbb{R}^\delta \to \mathbb{R}$$

Condensed sets

A *condensed set* is a functor

$$\text{Profinite}^{\text{op}} \to \text{Set}$$

satisfying a certain
sheaf condition.

"Yoneda" gives: $\text{Top} \to \text{Cond}(\text{Set})$

Condensed sets (2)

CompHaus is equivalent to
qcqs objects in $\mathrm{Cond}(\mathrm{Set})$.

Weakly Hausdorff compactly generated X
are roughly the same as
quasi-separated condensed sets.

Condensed abelian groups

$\mathrm{Cond}(\mathrm{Ab})$ is very nice:

abelian category satisfying
(AB3), (AB4), (AB5), (AB3*),
and even (AB4*) and (AB6).

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Examples: discrete rings, p-adic rings

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– An analytic ring has an analytic spectrum

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- Examples: schemes, formal schemes, Berkovich spaces, adic spaces
- What about real/complex manifolds?

Functional analysis

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- $\mathbb{R}[S]$ is the space of linear combinations of Dirac measures
- We want to complete it
- First candidate: $\mathcal{M}(S)$, the space of signed Radon measures

Functional analysis

- What we want:

$$\begin{array}{ccc} S & \xrightarrow{\;f\;} & V \\ \big\downarrow & \overset{\exists!}{\nearrow} & \\ \mathcal{M}(S) & & \end{array} \qquad \mu \mapsto \int_S f(s)\,\mathrm{d}\mu$$

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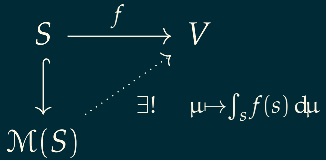

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– But wait! For which kind of spaces V?

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- The formalism of analytic rings forces non-locally-convex topological vector spaces into the picture
- V needs to be a p-Banach space for $p < 1$: a complete TVS whose topology is induced by a p-norm:

$$||\lambda v|| = |\lambda|^p ||v||$$

Functional analysis

– The entropy function

$$\ell^1 \to \ell^2$$
$$(x_n)_n \mapsto (x_n \log |x_n|)_n$$

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- We need to use use $\mathcal{M}_{p'}(S)$ instead: the subspace of signed Radon measures satisfying some $\ell^{p'}$-convergence condition

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Then

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geqslant 1$.

Liquid vector spaces

Fix a real parameter $0 < p \leqslant 1$

Then p-liquid vector spaces
form a full subcat $\mathrm{Liq}_p \subset \mathrm{Cond}(\mathbb{R})$
with good properties ...

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- If $p' \leqslant p$, then $\mathrm{Liq}_p \subset \mathrm{Liq}_{p'}$
- Examples:
  Banach spaces, nuclear Fréchet spaces
- Liquid tensor product is compatible with
  topological tensor product of nuclear Fréchets

Liquid vector spaces

Summary:

Liquid analytic ring structure on $\mathbb{R}$

The Experiment

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– "proof has some very unexpected features … very much of arithmetic nature"

– "I think this may be my most important theorem to date"

– "I think nobody else has dared to look at the details, and so I still have some small lingering doubts"

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Let $0 < p' < p < 1$ be real numbers,
let S be a profinite set,
and let V be a p-Banach space.

Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S.

Then

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

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First target (Thm 9.4 of Analytic.pdf)

Fix $0 < r < r' < 1$.

For any m, there exists a k and c_0
such that for all profinite sets S
and r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V
the system of complexes

$$C_c^\bullet \colon \hat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leqslant c})^{T^{-1}} \to \hat{V}(\overline{\mathcal{M}}_{r'}(S)^2_{\leqslant \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leqslant k$-exact in degrees $\leqslant m$ for $c \geqslant c_0$.

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target 1	100% on May 29, 2021
target 2	100% on Jul 14, 2022

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- Alternative to Breen–Deligne resolutions

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- Answer to Question 9.9 of `Analytic.pdf`
- Alternative to Breen–Deligne resolutions
- Effort to conceptualize parts of the proof

Theorem (Breen–Deligne)

There exists a functorial resolution of an abelian group A of the form

$$\cdots \to \bigoplus_{j=1}^{n_j} \mathbb{Z}[A^{r_{i,j}}] \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

where all n_j and $r_{i,j}$ are natural numbers.

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where all n_j and $r_{i,j}$ are natural numbers.

Proof uses homotopy theory.

Lemma

The MacLane Q'-construction is a functorial *complex* of an abelian group A of the form

$$Q'(A): \qquad \cdots \to \mathbb{Z}[A^{2^i}] \cdots \to \mathbb{Z}[A^4] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

with the property that:

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if $\mathrm{Ext}^i(Q'(A), B) = 0$ for all $i \geqslant 0$,

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with the property that:

if $\operatorname{Ext}^i(Q'(A), B) = 0$ for all $i \geqslant 0$,

then $\operatorname{Ext}^i(A, B) = 0$ for all $i \geqslant 0$.

Human-friendly proof?

Can we find a proof that does not need computer verification?

I will present some thoughts

(j/w Reid Barton)

Back to technical theorem 9.4

The technical key ingredient
involved a chain complex of objects:

$$\mathbb{R}_{\geqslant 0}^{\mathrm{op}} \to \mathrm{NormAbGrp}$$

The theorem claims that this complex
is "exact" in some sense.

Logics

The presence of functors

$$\mathbb{R}_{\geqslant 0}^{\mathrm{op}} \to \mathrm{NormAbGrp}$$

and the metric on objects in NormAbGrp
suggests that we can try to
mix *sheaf semantics* and *continuous logic*.

Sheaf semantics

Suppose we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with $A, B, C : \mathbb{R}_{\geqslant 0}^{\mathrm{op}} \to \mathrm{NormAbGrp}$.

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Then $\forall b : B, g(b) = 0 \vdash \exists a : A, f(a) = b$
interpretes to

$$\exists k \geqslant 1, \forall s \gg 0, \ldots$$

$$\forall b \in B_{ks}, g(b) = 0 \implies \exists a \in A_s, f(a) = b|_s$$

Continuous logic

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Then $\forall b : B, g(b) = 0 \vdash \exists a : A, f(a) = b$
interpretes to

$$\exists K, \ldots$$

$$\forall b \in B, \forall \varepsilon > 0, \exists a \in A, \|f(a) - b\| \leqslant K\|g(b)\| + \varepsilon$$

Normed exactness

Combine the sheaf semantics and continuous logic

Result:
“exactness” interpretes to “normed exactness”
as needed for the chain complex in theorem 9.4.

Upshot

We can “compile” the proofs
of homological algebra results
into the “normed-and-sheafy“ setting.

Examples: snake lemma, long exact sequence,
spectral sequence

A diamond “modality”?

At one point in the global proof,
this strategy fails.

We need to make a certain norm estimate.

And it is not simply the interpretation
of an internal reasoning step.

We think it is related to some modal operator.
But there are some speedbumps. WIP!

Other input

The proof is certainly not “a formality”.

At crucial points one needs:

- Combinatorics (Gordan’s lemma)
- Results about cohomology of profinite sets

Lessons

Lessons (1)

State of the art maths

can be formalized

in a reasonable amount of time

Lessons (2)

Proof *assistant*

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Proof *assistant*

Lean showed to be a powerful tool

for managing complex proofs

Partial lessons (3)

What makes the proof tick?

Why does it pass through arithmetic?

In which logic should the proof work?