A Foundation for Synthetic Algebraic Geometry

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Related work

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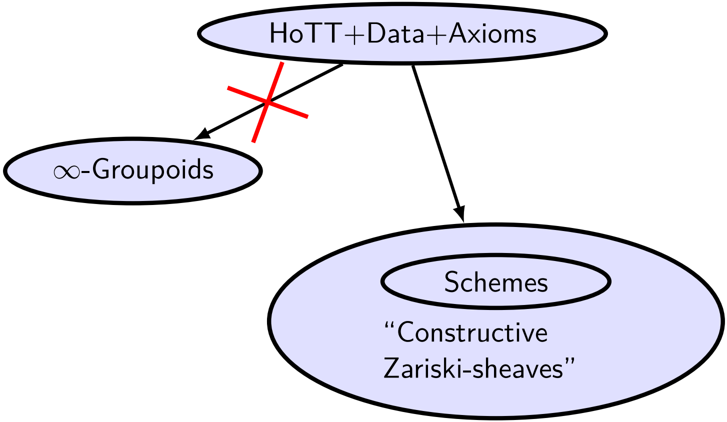
This work is part of a larger project with many collaborators. A lot of the things we have figured out are on github:

`https://github.com/felixwellen/synthetic-zariski/`

In addition to the authors this also contains contributions of

- Peter Arndt
- Ingo Blechschmidt
- Hugo Moeneclaey
- Marc Nieper-Wißkirchen
- David Wärn



* Schemes = quasi-compact, quasi-separated schemes of finite type

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Definition

(i) An R-algebra is finitely presented (fp) if it is merely $R[X_1, \dots, X_n]/(P_1, \dots, P_l)$.

(ii) $\mathrm{Spec}(A) := \mathrm{Hom}_{R\text{-Alg}}(A, R)$ is the *spectrum* of an fp R-algebra A.

(iii) Any X such that there is an A with $X = \mathrm{Spec}(A)$ is called *affine scheme*.

Classical vs synthetic

How can we make the functor

$$A \mapsto \operatorname{Spec}(A)$$

fully faithful?

Classical algebraic geometry	Synthetic algebraic geometry
Endow $\operatorname{Spec}(A)$ with additional structure: - Zariski topology - structure sheaf $\mathcal{O}_{\operatorname{Spec}(A)}$	**Axiom (SQC)**[1]**.** The map $A \to R^{\operatorname{Spec} A}$ $a \mapsto (\varphi \mapsto \varphi(a))$ is an equivalence for any finitely presented R-algebra A.

[1] "Synthetic Quasi-Coherence", due to Ingo Blechschmidt

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$$A \xrightarrow{\sim} R^{\mathrm{Spec}\,A}$$

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- $\operatorname{Spec}(R/(r)) = (r = 0)$. Thus: if $r \neq 0$, then r is invertible.
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Axiom: The ring R is local.

- If $r_1, \ldots, r_n : R$ are not all zero, then some r_i is invertible.

An affine scheme

Let $f : A$.

$$D(f) := \mathrm{Spec}(A_f) = \mathrm{Spec}(A[X]/(fX-1))$$
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Any subset which is merely a finite union of $D(f)$s is called *global-open*. Let $f_1, \ldots, f_n : A$. Then $\mathrm{Spec}(A) = \bigcup_i D(f_i)$ if and only if $(f_1, \ldots, f_n) = (1)$.

Closed and open propositions

For $r_1, \ldots, r_n : R$ we have the propositions

$$V(r_1, \ldots, r_n) :\equiv (r_1 = \cdots = r_n = 0),$$

$$D(r_1, \ldots, r_n) :\equiv (r_1 \neq 0 \vee \ldots \vee r_n \neq 0).$$

Then define:

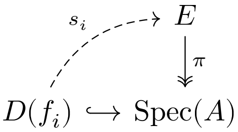
$$\text{closedProp} :\equiv \sum_{p:\text{hProp}} \exists r_1, \ldots, r_n. (p = V(r_1, \ldots, r_n))$$

$$\text{openProp} :\equiv \sum_{p:\text{hProp}} \exists r_1, \ldots, r_n. (p = D(r_1, \ldots, r_n))$$

A *closed subtype* of X is a map $X \to \text{closedProp}$.
An *open subtype* of X is a map $X \to \text{openProp}$.

Zariski-local choice

Axiom (Zariski-local choice):
For every surjective π, there merely exist local sections s_i



with $f_1, \ldots, f_n : A$ such that $(f_1, \ldots, f_n) = (1)$.

Alternative formulation:
Axiom (Zariski-local choice):
Let $B : \mathrm{Spec}(A) \to \mathcal{U}$ be such that $(x : \mathrm{Spec}(A)) \to \|B(x)\|$. Then there merely are $n : \mathbb{N}$, $f_1, \ldots, f_n : A$ such that $(f_1, \ldots, f_n) = (1)$ and $s_i : (x : D(f_i)) \to B(x)$.

Pointwise-global principle

Theorem

Let $f : A$.

(a) *A global-open $U \subseteq D(f)$ is global-open in $\operatorname{Spec}(A)$*

(b) *A subset $U \subseteq \operatorname{Spec}(A)$ is open if and only if it is global-open.*

Proof-Idea.

Let $U \subseteq \operatorname{Spec}(A)$ be open.

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Proof-Idea.

Let $U \subseteq \mathrm{Spec}(A)$ be open. That means we have

$$t : \prod_{x:\mathrm{Spec}(A)} \left\| \sum_{n:\mathbb{N}} \sum_{r_1,\dots,r_n:R} U(x) = (r_1 \neq 0 \vee \dots \vee r_n \neq 0) \right\|$$

By something called "boundedness", we can assume we have a global "$n : \mathbb{N}$" and by Zariski-choice we have

$$s_i : (x : D(f_i)) \to \sum_{r_1,\dots,r_n:R} U(x) = (r_1 \neq 0 \vee \dots \vee r_n \neq 0)$$

□

Schemes

A type X is a *scheme* if there exist $U_1, \dots, U_n : X \to \mathrm{openProp}$ such that $X = \bigcup_i U_i$ and every U_i is an affine scheme.

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Example. *Projective* n*-space*:

$$\begin{aligned} \mathbb{P}^n &:= \{x : R^{n+1} \mid x \neq 0\}/\approx \;\text{ where } (x \approx y) := \exists \lambda : R. \lambda x = y \\ &= \{\,\text{submodules } L \subseteq R^{n+1} \text{ such that } \|L = R^1\|\,\} \end{aligned}$$

is a scheme, since

$$U_i([x]) := (x_i \text{ is invertible})$$

is an open affine cover.

Line bundles

The type

$$\mathrm{Lines} := \sum_{L:R\text{-Mod}} \|L = R^1\|$$

has a wild group structure:

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A *line bundle* on X is a map $X \to \mathrm{Lines}$.
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The *Picard group* of X is

$$\mathrm{Pic}(X) := \|X \to \mathrm{Lines}\|_{\mathsf{set}}.$$

(In fact, $\mathrm{Lines} = K(R^\times, 1)$ and $\mathrm{Pic}(X) = H^1(X, R^\times)$.)

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- $\prod$-type.
- $\|\_\|_{\mathrm{set}}$ is a modality.
- Homotopy group: $H^n(X, A) = \pi_k(\prod_{x:X} K(A, n+k))$.

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Non-trivial for $X : \mathrm{Set}$ because:

$X =$ Pushout of sets $U \leftarrow Y \to V$,

Then a "cohomology class" $X \to K(A, 1)$ is given by:

- Maps $f : U \to K(A, 1)$, $g : V \to K(A, 1)$.
- And $h : (x : Y) \to f(x) = g(x)$, which is essentially a map $Y \to A$, if U and V don't have higher cohomology...

Zariski-Choice and Cohomology

Let $X = \mathrm{Spec}(A)$ and $M : X \to R$-Mod such that $((x : D(f)) \to M_x) = ((x : X) \to M)_f$, then

$$H^1(X, M) = 0$$

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So for $t_{ij}(x) :\equiv s_j(x)^{-1} \cdot s_i(x)$ we have $t_{ij} + t_{jk} = t_{ik}$. By algebra, we get $u_i : (x : D(f_i)) \to M_x$ with $t_{ij} = u_i - u_j$. Then the $\tilde{s}_i :\equiv s_i - u_i$ glue to a global trivialization.

Results of the larger project

- Vanishing of $H^n(\mathrm{Spec}(A), M)$ for $n > 0$ and Čech-Cohomology.
- Serre's theorem on Affineness: If all $H^1(X, M)$ vanish, then X is affine.
- Smooth schemes are locally standard smooth.
- Closed subsets of $\mathbb{P}^n$ are compact as defined by Martín Escardó in synthetic topology.
- $\mathbb{P}^\infty$ is $\mathbb{A}^1$-equivalent to $BR^\times$.
- A subcanonical candidate for the synthetic fppf-topology.
- Stacks...

Thank you!

Thank you!

If you want to know more:

- There is a workshop in Gothenburg planned for
  11-15 March 2024.
- The github page mentioned above:
  `https://github.com/felixwellen/synthetic-zariski/`
- A formalization project:
  `https://github.com/felixwellen/synthetic-geometry/`