$\mathcal{C}$ category (AKS)

Problem : Define a type $Psh(\mathcal{C})$ of "homotopy coherent" type-valued presheaves on $\mathcal{C}$.

[Interpretation of $Psh(\mathcal{C})$ in the simplicial model matches (up to w.e.) the correct ∞-groupoid of presheaves on $\mathcal{C}$]

- $\mathcal{C}$ groupoid $\quad Psh(\mathcal{C}) :\equiv |\mathcal{C}| \to \mathcal{U}$
- $\mathcal{C}$ free on a graph $\quad Psh(\mathcal{C}) :\equiv$ graph morphisms
- $\mathbb{G} = 0 \rightrightarrows 1 \rightrightarrows 2 \rightrightarrows 3 \rightrightarrows \cdots$ $\qquad \omega = 0 \to 1 \to 2 \to 3 \to \cdots$

  $Psh(\mathbb{G})$ = globular types

$\mathcal{C}$ direct category : $\deg : |\mathcal{C}| \to \mathbb{N}$

$\underset{f : x \to y}{f \neq id} \quad \deg(x) < \deg(y)$

$\mathcal{C}^{\leq n} \qquad Psh(\mathcal{C}^{\leq n+1}) = \left(X : Psh(\mathcal{C}^{\leq n})\right) \times \left(\boxed{\partial X} \to \mathcal{U}\right)$

∂ mutually defined

Δ_+

Idea of Finster's defn.

Apply this template for $\mathcal{C} = \Phi$ (opetopes)

Polynomials

(i) type-theoretic

(ii) categorical

(iii) indexed

$\boxed{I \quad \text{sorts}}$

(i) $Op : I \to U$
operations by output

$Param : \{j : I\} \to \underset{t}{\underline{Op(j)}} \to \underline{I} \to U$

(i') $Arity : \{j : I\} \to Op(j) \to U$

$Sortof : Arity\ a \to I$

(ii) $I \xleftarrow{s} A' \xrightarrow{p} A \xrightarrow{t} I$

A = operations

A' = operations with a marked input

p = forgets marking

t = output sort

s = input sort of the marking

$A = (j : I) \times Op\ j$

$A' = (i\,j : I) \times (a : Op\ j) \times Param(a, i)$

$p(i, j, a, x) = (j, a)$

$t(j, a) = j$

$s(i, j, a, x) = i$

(iii) $P : \underbrace{\{X : U\}}_{\text{space of inputs}} \to \underbrace{(X \to I)}_{\text{input sorts}} \to \underbrace{I}_{\text{output sort}} \to U$

(iii) $\to$ (ii)

$A = (X : U) \times (i : X \to I) \times (j : I) \times P(i, j)$

$A' = ((X, i, j, a) : A) \times X$

$t(X, i, j, a) = j$

$s(X, i, j, a, x) = i(x)$

1) Morphisms of polynomials

(ii)

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \arrow[dr] & \\
I & B' \arrow[l] \arrow[r] & B \arrow[r] & I
\end{tikzcd}$$

(iii) $P, Q : \{X : U\} \to (X \to I) \to I \to U_1$

morph. $P \to Q \equiv \forall X, i, j\ ,\ P(i,j) \to Q(i,j)$

2) Polynomials form an ∞-topos

$$\mathrm{Poly}_U(I) = U_1 \Big/ \{X:U\} \times (X \to I) \times I \qquad (\Sigma \text{ model})$$

<u>Products</u> $\quad P \times Q\,(i,j) = P(i,j) \times Q(i,j)$

$Op^P,\ Param^P\ ,\ Op^Q,\ Param^Q$

$$Op^{P\times Q}(j) = (a : Op^P(j)) \times (b : Op^Q(j)) \times ((i : I) \to Param(a,i) \simeq Param(b,i))$$

<u>Observation</u>

$$\begin{tikzcd}
 & X' \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \arrow[dr] & \\
I & A' \arrow[l] \arrow[r] & \underline{A} \arrow[r] & I
\end{tikzcd}$$

$$P \equiv I \leftarrow A' \to \underline{A} \to I$$

$$\mathrm{Poly}(I)/P \simeq U/A \simeq \mathrm{Poly}(J)/Q$$

$$Q \equiv J \leftarrow A'' \to \underline{A} \to J$$

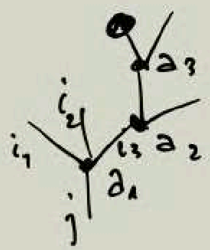
Trees

P polynomial

$tr(P) = P\text{-trees}$

$P \equiv I \leftarrow A' \rightarrow A \rightarrow I$

$W : I \rightarrow \mathcal{U}$



Arity $a_1 \simeq$ branching of the corr. node

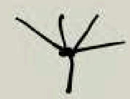
$tr'(P) = P\text{-trees}$ with a marked leaf

$tr^\bullet(P) = P\text{-trees}$ - - - - - - - node

$P \rightsquigarrow P^*$ (= free monad on P)

$$I \leftarrow tr'(P) \rightarrow tr(P) \rightarrow I$$

$P \rightsquigarrow \underline{P^* \times P}$ $\qquad I \leftarrow \partial' P \rightarrow \boxed{\partial P} \rightarrow I$



$$\begin{tikzcd}
 & \partial^\bullet P \arrow[r] \arrow[dl] \arrow[d] & \partial P \arrow[d] \arrow[dr] & \\
A & tr^\bullet(P) \arrow[l] \arrow[r] & tr(P) & A
\end{tikzcd}$$

$BD(P)$, $P^* \times P$ have the same operations

Magma. P polynomial $M : P^* \rightarrow P$.

Lemma: $BD(P)$ is a magma (flatten, bd-frm ...)

$P, M : P^* \rightarrow P$

(P,M) coherent $:\equiv \begin{cases} M \text{ subdivision invariant} \\ (P/\!\!/M, M') \text{ coherent} \end{cases}$

$M : P^* \rightarrow P$

$$P^* \rightarrow P^* \times P$$
$$\Downarrow$$
$$P/\!\!/M \rightarrow BD(P)$$
$$(P/M)^* \rightarrow BD(P)^* \rightarrow BD(P)$$

M sub. inv :

$$\begin{tikzcd}
 & P/\!\!/M \arrow[d] \\
(P/M)^* \arrow[ur, dashed, "M'"] \arrow[r] \arrow[ur, phantom, "\cong"] & BD(P)
\end{tikzcd}$$

- Opetopic tower :
  - P_i polynomials
    sorts $P_{i+1} \equiv$ operations of P_i
  - $P_{i+1} \xrightarrow[\pi_i]{} BD(P_i)$
- Opetopic type
  - opetopic tower
  - witness for the square

$$P_{i+2} \to BD(P_{i+1})$$
$$\wr$$
$$\overline{P_{i+2}} \to P^*_{i+1} \times P_{i+1}$$

$$\begin{tikzcd}
\overline{P_{i+2}} \arrow[r] \arrow[d] & P^*_{i+1} \arrow[d] \\
P_{i+1} \arrow[r] & BD(P_i)
\end{tikzcd}$$

- Segal condition $\overline{P_{i+1}} \to P_i^*$ is an equivalence

<u>Thm</u> : opetopic Segal type $\simeq$ polynomial monad

$$\boxed{P^*_{i+1} \times_{BD(P_i)} P_{i+1}}$$

$$\begin{tikzcd}
\overline{P_{i+3}} \arrow[r] \arrow[d] & P^*_{i+2} \arrow[d] \\
P_{i+2} \arrow[r] & \underline{BD(P_{i+1})}
\end{tikzcd} \sim \boxed{P^*_{i+1} \times P_{i+1}}$$