

# From Higher Groups to Homotopy Surfaces

Ulrik Buchholtz

TU Darmstadt

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- 1 Higher Groups
  - Basics
  - Lean Formalization
  - Beginning Higher Group Theory

- 2 Homotopy Surfaces

# Outline

- 1 Higher Groups
  - Basics
  - Lean Formalization
  - Beginning Higher Group Theory
- 2 Homotopy Surfaces

# Definitions

Recall the basic setup:

- *pointed connected* types  $B$  may be viewed as presenting higher groups, with *carrier*  $\Omega B := (\text{pt} =_B \text{pt})$ , and group structure induced from the identity types.

$$\begin{aligned}\infty\text{-Group} &:= (G : \text{Type}) \times (BG : \text{Type}_{\text{pt}}^{>0}) \times (G \simeq \Omega BG) \\ &\simeq (G : \text{Type}_{\text{pt}}) \times (BG : \text{Type}_{\text{pt}}^{>0}) \times (G \simeq_{\text{pt}} \Omega BG) \\ &\simeq \text{Type}_{\text{pt}}^{>0}\end{aligned}$$

$$\begin{aligned}n\text{-Group} &:= (G : \text{Type}_{\text{pt}}^{<n}) \times (BG : \text{Type}_{\text{pt}}^{>0}) \times (G \simeq_{\text{pt}} \Omega BG) \\ &\simeq \text{Type}_{\text{pt}}^{>0, \leq n}\end{aligned}$$

# Stability

The more deloopings the merrier! (Recall that Eckmann-Hilton implies that double loop spaces are homotopy commutative.)

$$(n, k)\text{GType} := (G : \text{Type}_{\text{pt}}^{\leq n}) \times (B^k G : \text{Type}_{\text{pt}}^{\geq k}) \times (G \simeq_{\text{pt}} \Omega^k B^k G) \\ \simeq \text{Type}_{\text{pt}}^{\geq k, \leq n+k}$$

$$(n, \omega)\text{GType} := \lim_k (n, k)\text{GType} \\ \simeq (B^- G : (k : \mathbb{N}) \rightarrow \text{Type}_{\text{pt}}^{\geq k, \leq n+k}) \\ \times ((k : \mathbb{N}) \rightarrow B^k G \simeq_{\text{pt}} \Omega B^{k+1} G).$$

Infinite loop types in this way are precisely connective spectra ( $n = \infty$ ).

# Periodic table

Periodic table of  $k$ -tuply groupal  $n$ -groupoids.

$k \setminus n$	0	1	2	...	$\infty$
0	pointed set	pointed groupoid	pointed 2-groupoid	...	pointed $\infty$ -groupoid
1	group	2-group	3-group	...	$\infty$ -group
2	abelian group	braided 2-group	braided 3-group	...	braided $\infty$ -group
3	— " —	symmetric 2-group	syllaptic 3-group	...	syllaptic $\infty$ -group
4	— " —	— " —	symmetric 3-group	...	?? $\infty$ -group
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\omega$	— " —	— " —	— " —	...	connective spectrum

# Lean formalization: Truncatedness

Many aspects have now been formalized in Lean (jww van Doorn, Rijke):

## Theorem

*Let  $X : \text{Type}_{\text{pt}}^{\geq k}$  be a  $(k - 1)$ -connected, pointed type for some  $k \geq 0$ , and let  $Y : X \rightarrow \text{Type}_{\text{pt}}^{\leq n+k}$  be a fibration of  $(n + k)$ -truncated, pointed types for some  $n \geq -1$ . Then the type of pointed sections,  $(x : X) \rightarrow_{\text{pt}} Y x$ , is  $n$ -truncated.*

## Corollary

*Let  $k \geq 0$  and  $n \geq -1$ . If  $X$  is  $(k - 1)$ -connected, and  $Y$  is  $(n + k)$ -truncated, then the type of pointed maps  $X \rightarrow_{\text{pt}} Y$  is  $n$ -truncated. In particular,  $\text{hom}_{(n,k)}(G, H)$  is an  $n$ -type for  $G, H : (n, k)\text{GType}$ .*

## Corollary

*The type  $(n, k)\text{GType}$  is  $(n + 1)$ -truncated.*

# Lean Formalization: Categorical equivalences

## Theorem

*We have the following equivalences of 1-categories (for  $k \geq 2$ ):*

$$(0, 0)\text{GType} \simeq \text{Set}_{\text{pt}};$$

$$(0, 1)\text{GType} \simeq \text{Group};$$

$$(0, k)\text{GType} \simeq \text{AbGroup}.$$



# Lean Formalization: Operations

decategorification  $\text{Decat} : (n, k)\text{GType} \rightarrow (n - 1, k)\text{GType}$   
 $\langle G, B^k G \rangle \mapsto \langle \|G\|_{n-1}, \|B^k G\|_{n+k-1} \rangle$

discrete categorification  $\text{Disc} : (n, k)\text{GType} \rightarrow (n + 1, k)\text{GType}$   
 $\langle G, B^k G \rangle \mapsto \langle G, B^k G \rangle$

looping  $\Omega : (n, k)\text{GType} \rightarrow (n - 1, k + 1)\text{GType}$   
 $\langle G, B^k G \rangle \mapsto \langle \Omega G, B^k G \langle k \rangle \rangle$

delooping  $B : (n, k)\text{GType} \rightarrow (n + 1, k - 1)\text{GType}$   
 $\langle G, B^k G \rangle \mapsto \langle \Omega^{k-1} B^k G, B^k G \rangle$

forgetting  $F : (n, k)\text{GType} \rightarrow (n, k - 1)\text{GType}$   
 $\langle G, B^k G \rangle \mapsto \langle G, \Omega B^k G \rangle$

stabilization  $S : (n, k)\text{GType} \rightarrow (n, k + 1)\text{GType}$   
 $\langle G, B^k G \rangle \mapsto \langle SG, \|\Sigma B^k G\|_{n+k+1} \rangle,$   
where  $SG = \|\Omega^{k+1} \Sigma B^k G\|_n$









# Lean Formalization: (de)categorification

Decat  $\dashv$  Disc with Decat  $\circ$  Disc = id:

$k \setminus n$	0	1	2	...	$\infty$
0	pointed set	$\rightleftarrows$ pointed groupoid	$\rightleftarrows$ pointed 2-groupoid	...	pointed $\infty$ -groupoid
1	group	$\rightleftarrows$ 2-group	$\rightleftarrows$ 3-group	...	$\infty$ -group
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\omega$	— " —	— " —	— " —	...	connective spectrum

# Lean Formalization: (de)looping

$B \dashv \Omega$  with  $\Omega \circ B = \text{id}$ :

$k \setminus n$	0	1	2	...	$\infty$
0	pointed set	 pointed groupoid	 pointed 2-groupoid	...	pointed $\infty$ -groupoid
1	group	 2-group	 3-group	...	$\infty$ -group
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\omega$	— " —	— " —	— " —	...	connective spectrum

# Lean Formalization: Stabilization

We also have:

- $S \dashv F$

## Lemma (Wedge connectivity)

*If  $A : \text{Type}_{\text{pt}}$  is  $n$ -connected and  $B : \text{Type}_{\text{pt}}$  is  $m$ -connected, then the map  $A \vee B \rightarrow A \times B$  is  $(n + m)$ -connected.*

## Theorem (Freudenthal)

*If  $A : \text{Type}_{\text{pt}}^{>n}$  with  $n \geq 0$ , then the map  $A \rightarrow \Omega \Sigma A$  is  $2n$ -connected.*

## Theorem (Stabilization)

*If  $k \geq n + 2$ , then  $S : (n, k)\text{GType} \rightarrow (n, k + 1)\text{GType}$  is an equivalence, and any  $G : (n, k)\text{GType}$  is an infinite loop space.*

# Examples

- $B\mathbb{Z} = \mathbb{S}^1$ , other free groups on pointed sets, free abelian groups.
- Automorphism groups  $\text{Aut } a := (a = a)$  for  $a : A$  with  $B\text{Aut } a := \text{im}(a : 1 \rightarrow A) = (x : A) \times \|a = x\|_{-1}$ .
- Fundamental  $n$ -group of  $(A, a)$ ,  $\Pi_n(A, a)$ , with corresponding delooping  $B\Pi_n(A, a) = \|B\text{Aut } a\|_n$ .
- Symmetric groups  $S_n := \text{Aut}([n])$ , where  $BS_n = B\text{Aut}([n])$  is the type of all (small)  $n$ -element sets. Colimit  $S_\infty$ .
- Generally, if  $G : \text{Group}$ , we can take  $BG$  to be the type of  $G$ -torsors.
- $\mathbb{S}^1 = B\mathbb{Z}$  has delooping  $B^2\mathbb{Z}$ , which we can take to be the type of oriented circles.
- $\mathcal{G}_n := \text{Aut}(\mathbb{S}^{n-1})$  and  $\mathcal{F}_n := \text{Aut}(\mathbb{S}_{\text{pt}}^n)$ . Colimits  $\mathcal{G} \simeq \mathcal{F}$ . Orientation preserving versions too.
- $\vdots$
- With cohesion, we should get  $BO(n)$ ,  $BU(n)$ , etc.

# Actions

A  $G$ -action on  $a : A$  is simply a homomorphism  $G \rightarrow \text{Aut } a$ .

A  $G$ -type is a function  $X : BG \rightarrow \text{Type}$ . Here we can form the

invariants  $X^{hG} := (z : BG) \rightarrow X(z)$ , also known as the *homotopy fixed points*, and the

coinvariants  $X_{hG} := (z : BG) \times X(z)$ , which is also known as *homotopy orbit space* or the *homotopy quotient*  $X // G$ .

Right and left adjoints to  $A \mapsto A^{\text{triv}}$  for  $A : \text{Type}$ .

## Proposition

Let  $f : H \rightarrow G$  be a homomorphism of higher groups with delooping  $Bf : BH \rightarrow_{\text{pt}} BG$ , and let  $\alpha : \text{hom}(X, Y)$  be a map of  $G$ -types. By composing with  $f$  we can also view  $X$  and  $Y$  as  $H$ -types, in which case we get a homotopy pullback square:

$$\begin{array}{ccc} X_{hH} & \longrightarrow & Y_{hH} \\ \downarrow & & \downarrow \\ X_{hG} & \longrightarrow & Y_{hG}. \end{array}$$

# Canonical actions

Every group  $G$  carries two canonical actions on itself:

the right action  $G : BG \rightarrow \text{Type}$ ,  $G(x) = (\text{pt} = x)$ , and the  
the adjoint action  $G^{\text{ad}} : BG \rightarrow \text{Type}$ ,  $G^{\text{ad}}(x) = (x = x)$  (by  
conjugation).

We have  $1 // G = BG$ ,  $G // G = 1$  and  $G^{\text{ad}} // G = LBG := (\mathbb{S}^1 \rightarrow BG)$ ,  
the free loop space of  $BG$ .

## Corollary

*If  $f : H \rightarrow G$  is a homomorphism of higher groups, then  $G // H$  is  
equivalent to the homotopy fiber of the delooping  $Bf : BH \rightarrow_{\text{pt}} BG$ ,  
where  $H$  acts on  $G$  via the  $f$ -induced right action.*

# Projective spaces

Consider the sequence of actions  $GM^n : BG \rightarrow \text{Type of } G$  given by

$$GM^{-1}(x) := 0$$

$$GM^{n+1}(x) := (\text{pt} = x) * GM^n(x) = G(x) * GM^n(x)$$

i.e., the iterated joins of the right action with itself ( $M$  is for Milnor). The types  $GM^n(\text{pt})$  are at least  $(n+1)(k+2) - 2$ -connected if  $G$  is  $k$ -connected. Then the colimit  $GM^\infty(\text{pt}) = \varinjlim GM^n(\text{pt})$  is contractible, so  $GM^\infty // G = 1 // G = BG$ .

We define the *projective spaces* for  $G$  to be  $GP^n := GM^n // G$ . Thus,  $GP^{-1} = 0$ ,  $GP^0 = 1$ ,  $GP^1 = \Sigma G$ , etc. (in general,  $GP^{n+1}$  is the mapping cone on the inclusion  $GM^n(\text{pt}) \rightarrow GP^n$ ).

The *real and complex projective spaces* are  $\mathbb{R}P^n := \mathcal{G}_1 P^n$  and  $\mathbb{C}P^n := S\mathcal{G}_2 P^n$ .



# Orbit Stabilizer Theorem

Let  $X$  be a  $G$ -type.

## Definition

Given  $x : X$ , the *stabilizer* of  $x$  is the group  $H_x$  with delooping

$BH_x := \text{im}(1 \rightarrow X \rightarrow X // G)$ .

The *orbit* of  $x$  is the type  $G \cdot x := (y : X) \times \|\langle \text{pt}, x \rangle = \langle \text{pt}, y \rangle\|_{-1}$ .

## Theorem (Orbit Stabilizer Theorem)

Given  $x : X$ ,  $G // H_x \simeq G \cdot x$ .

## Proof.

The map  $BH_x \rightarrow_{\text{pt}} BG$  is the projection from

$(z : BG) \times (y : X z) \times \|\langle \text{pt}, x \rangle = \langle z, y \rangle\|_{-1}$  to  $BG$ , so the fiber over  $\text{pt}$  is  $G \cdot x$ . □

# Central extensions

The cohomology of a higher group  $G$  is simply the cohomology of its delooping  $BG$ . Indeed, for any spectrum  $A$ , we define

$$H_{\text{Grp}}^k(G, A) := \|\|BG \rightarrow_{\text{pt}} B^k A\|_0.$$

Of course, to define the  $k$ 'th cohomology group, we only need the  $k$ -fold delooping  $B^k A$ .

If  $A : (\infty, 2)\text{GrpType}$  is a braided  $\infty$ -group, then we have the second cohomology group  $H_{\text{Grp}}^2(G, A)$ , and an element  $c : BG \rightarrow_{\text{pt}} B^2 A$  gives rise to a *central extension*

$$BA \rightarrow BH \rightarrow BG \xrightarrow{c} B^2 A,$$

## Example

The central extension  $1 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 1$  is classified by the map  $BC_n \rightarrow B^2\mathbb{Z}$  that sends an  $n$ -element set with a cyclic ordering to the canonical oriented circle obtained by gluing.

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# Motivation

## From “Open Problems” on the HoTT Wiki

‘Similarly to the torus, consider the projective plane, Klein bottle, . . . as discussed in the book (sec 6.6). Show that the Klein bottle is not orientable. (This requires defining “orientable”.)’

# Motivation

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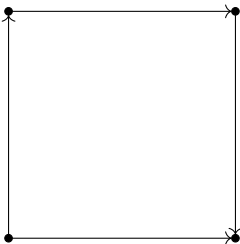
‘Similarly to the torus, consider the projective plane, Klein bottle, ... as discussed in the book (sec 6.6). Show that the Klein bottle is not orientable. (This requires defining “orientable”.)’

## Attempted solution

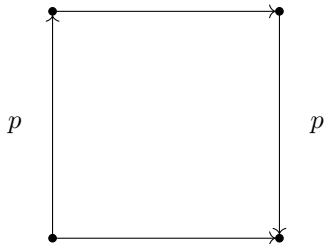
We can define the tangent bundle  $\tau : \text{KB} \rightarrow B\mathcal{G}_2$ . Looking at cohomology with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients, we see that  $\tau$  doesn't lift to  $B\mathcal{S}\mathcal{G}_2$ .

*This is not satisfactory!*

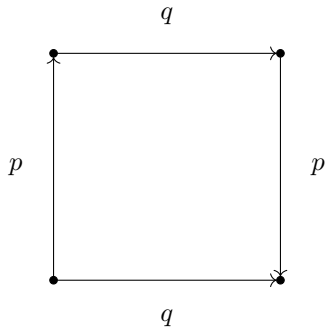
# Tangent bundle of the Klein bottle



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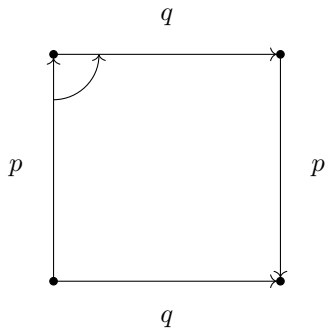


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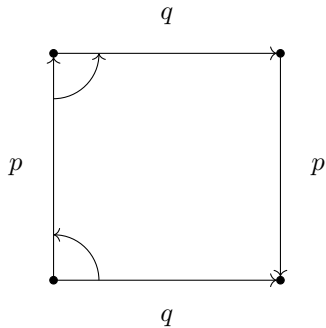




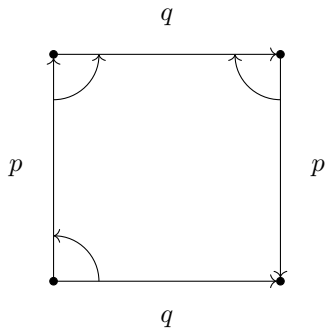
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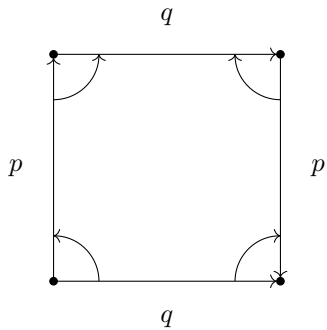
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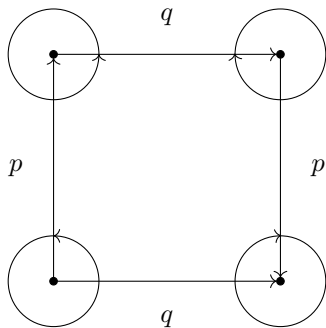
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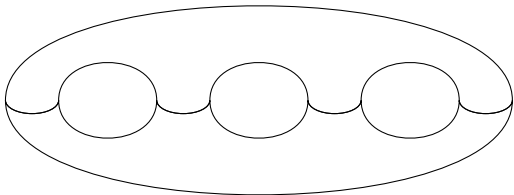
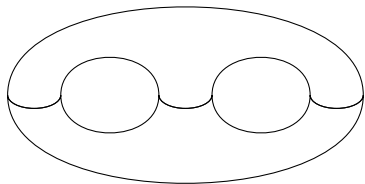
$$\tau : \text{KB} \rightarrow B\mathcal{G}_2$$

# A real solution

Desiderata:

- Define the type of surfaces,  $\text{Surf}$ .
- Define the type of oriented surfaces,  $\text{OredSurf}$  with a forgetful map  $\text{OredSurf} \rightarrow \text{Surf}$ .
- Then the type of orientable surfaces,  $\text{ObleSurf}$ , is the image.
- Prove that  $\text{OredSurf} \rightarrow \text{ObleSurf}$  is a principal  $\mathbb{Z}/2\mathbb{Z}$ -bundle.
- $\vdots$
- Prove the classification of surfaces theorem. (Constructively?).

## Some surfaces



# Poincaré Duality Surfaces

## Definition

A  $n$ -dimensional Poincaré duality type is a type  $X : \text{FinType}$  (i.e., merely equivalent to a finite complex) together with an *orientation class*  $w : X \rightarrow B\mathcal{G}_1$ , and a class  $[X] : H_n(X; \mathbb{Z}_w)$  called the *fundamental class* such that the cap product map

$$- \cap [X] : H^i(X; \mathbb{Z}_{\pi_1 X}) \rightarrow H_{n-i}(X; \mathbb{Z}_{\pi_1 X_w})$$

is an isomorphism.

## Theorem

For any  $X$ , the type of pairs  $(w, [X])$  making  $X$  an  $n$ -dimensional Poincaré duality type is contractible.

## Theorem (Eckmann-Müller-Linnell)

Every 2-dimensional Poincaré duality type in classical homotopy theory is equivalent to a closed surface.



# Haves and needs

Luckily, we have most of the components already:

- We have the type `FinCell` of finite cell complexes with realization map `FinCell`  $\rightarrow$  `Type` with image `FinType`.
- We know that (so far: unparametrized) cohomology of a finite complex can be computed via cellular cohomology (jww Favonia).
- We have the cup product and thus also cap product maps for integral coefficients, and hence with a little more work for free abelian coefficients over a family of finite sets.

We still need:

- Extend the work on cellular cohomology to parametrized versions and to homology.
- Construct the usual surfaces. (Perhaps Hurewicz would be handy to construct the fundamental classes.)
- Prove the classification theorem. In particular, we need to go from a 2-dim. PD type  $X$  to its Spivak stable normal bundle, and prove Spivak's theorem on spherical fibrations and get a tangent bundle. Prove basic theorems establishing that orientations of the tangent bundle correspond to orientation of  $X$ .

# Outlook

- Proposal for another big formalization project.
- Is it better to define surfaces using spectrum-level PD?
- Generalize to surfaces with boundary in order to study braid groups and mapping class groups, and . . . homological stability?
- In smooth/analytic cohesion, relate to smooth/analytic surfaces.
- Brown's theorem on computability of homotopy groups of finite complexes?

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