Computer-generated proofs for the monoidal structure of the smash product

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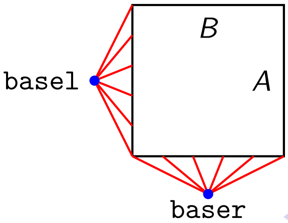
HoTTEST

The smash product as a higher inductive type

Definition

Given two pointed types $(A, \star_A)$ and $(B, \star_B)$, their smash product $A \wedge B$ is defined as the higher inductive type with constructors:

$$
\begin{aligned}
\mathtt{proj} &: A \times B \to A \wedge B, \\
\mathtt{basel} &: A \wedge B, \\
\mathtt{baser} &: A \wedge B, \\
\mathtt{pushl} &: (a : A) \to \mathtt{proj}(a, \star_B) = \mathtt{basel}, \\
\mathtt{pushr} &: (b : B) \to \mathtt{proj}(\star_A, b) = \mathtt{baser}.
\end{aligned}
$$



1-coherent monoidality

Goal

We want to prove (in book HoTT) that the smash product is a 1-coherent symmetric monoidal product on pointed types.[1]

This means that:

- The smash product is functorial (on pointed maps).
- There is a natural involution $\sigma_{A,B} : A \wedge B \to B \wedge A$.
- There is a natural equivalence $\alpha_{A,B,C} : (A \wedge B) \wedge C \to A \wedge (B \wedge C)$.
- It satisfies the hexagon and pentagon coherences.
- It has a unit with a triangular coherence.

This is used in particular to prove that the cup product on cohomology is associative.

[1] see pages 88 and 89 of my PhD thesis

Basic idea

All we have to do is to define various functions:

$$
\begin{aligned}
(x : A \wedge B) \to P(x) &\quad \text{(6 of them)} \\
(x : (A \wedge B) \wedge C) \to P(x) &\quad \text{(4 of them)} \\
(x : A \wedge (B \wedge C)) \to P(x) &\quad \text{(2 of them)} \\
(x : ((A \wedge B) \wedge C) \wedge D) \to P(x) &\quad \text{(1 of them)}
\end{aligned}
$$

where $P(x)$ is either constant or an equality $f(x) = g(x)$.

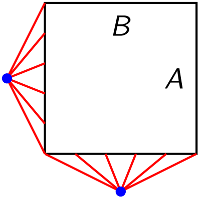
We define them by (iterated) induction on the smash product.

- In the (iterated) `proj` case, we know what to do.
- In the other cases, we "just" need to do some complicated path algebra.

Recursion rule

Given a type C, in order to define a map $f : A \wedge B \to C$, we need to define five terms/functions $f_{\mathtt{proj}}$, $f_{\mathtt{basel}}$, $f_{\mathtt{baser}}$, $f_{\mathtt{pushl}}$ and $f_{\mathtt{pushr}}$ such that:

$$
\begin{aligned}
f &: A \wedge B \to C \\
f(\mathtt{proj}(a,b)) &:\equiv f_{\mathtt{proj}}(a,b) && (f_{\mathtt{proj}} : A \times B \to C) \\
f(\mathtt{basel}) &:\equiv f_{\mathtt{basel}} && (f_{\mathtt{basel}} : C) \\
f(\mathtt{baser}) &:\equiv f_{\mathtt{baser}} && (f_{\mathtt{baser}} : C) \\
\mathsf{ap}_f(\mathtt{pushl}(a)) &:= f_{\mathtt{pushl}}(a) && (f_{\mathtt{pushl}} : (a : A) \to f_{\mathtt{proj}}(a, \star_B) =_C f_{\mathtt{basel}}) \\
\mathsf{ap}_f(\mathtt{pushr}(b)) &:= f_{\mathtt{pushr}}(b) && (f_{\mathtt{pushr}} : (b : B) \to f_{\mathtt{proj}}(\star_A, b) =_C f_{\mathtt{baser}})
\end{aligned}
$$



Example 1: commutativity

$$
\begin{aligned}
\sigma_{A,B} &: A \wedge B \to B \wedge A \\
\sigma_{A,B}(\mathtt{proj}(a, b)) &:= \mathtt{proj}(b, a) \\
\sigma_{A,B}(\mathtt{basel}) &:= \blacksquare^0 \\
\sigma_{A,B}(\mathtt{baser}) &:= \blacksquare^0 \\
\mathsf{ap}_{\sigma_{A,B}}(\mathtt{pushl}(a)) &:= \blacksquare^1 \\
\mathsf{ap}_{\sigma_{A,B}}(\mathtt{pushr}(b)) &:= \blacksquare^1
\end{aligned}
$$

Example 1: commutativity

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\sigma_{A,B}(\mathtt{basel}) &:= \mathtt{baser} \\
\sigma_{A,B}(\mathtt{baser}) &:= \mathtt{basel} \\
\mathsf{ap}_{\sigma_{A,B}}(\mathtt{pushl}(a)) &:= \mathtt{pushr}(a) \\
\mathsf{ap}_{\sigma_{A,B}}(\mathtt{pushr}(b)) &:= \mathtt{pushl}(b)
\end{aligned}
$$

Pointed maps

Definition

Given two pointed types $(A, \star_A)$ and $(A', \star_{A'})$, a pointed map from A to A' is a pair $(f, \star_f)$ where

$$f : A \to A'$$
$$\star_f : f(\star_A) = \star_{A'}$$

Example 2: functoriality

We have two pointed maps $f : A \to A'$ and $g : B \to B'$.

$$
\begin{aligned}
(f \wedge g) &: A \wedge B \to A' \wedge B' \\
(f \wedge g)(\mathtt{proj}(a, b)) &:= \mathtt{proj}(f(a), g(b)) \\
(f \wedge g)(\mathtt{basel}) &:= \mathtt{basel} \\
(f \wedge g)(\mathtt{baser}) &:= \mathtt{baser} \\
\mathsf{ap}_{f \wedge g}(\mathtt{pushl}(a)) &:= \blacksquare^1 \\
\mathsf{ap}_{f \wedge g}(\mathtt{pushr}(b)) &:= \blacksquare^1
\end{aligned}
$$

Example 2: functoriality

We have two pointed maps $f : A \to A'$ and $g : B \to B'$.

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\begin{aligned}
(f \wedge g) &: A \wedge B \to A' \wedge B' \\
(f \wedge g)(\mathtt{proj}(a, b)) &:= \mathtt{proj}(f(a), g(b)) \\
(f \wedge g)(\mathtt{basel}) &:= \mathtt{basel} \\
(f \wedge g)(\mathtt{baser}) &:= \mathtt{baser} \\
\mathsf{ap}_{f \wedge g}(\mathtt{pushl}(a)) &:= \blacksquare^1 \\
\mathsf{ap}_{f \wedge g}(\mathtt{pushr}(b)) &:= \blacksquare^1
\end{aligned}
$$

The two holes have type

$$\mathtt{proj}(f(a), g(\star_B)) = \mathtt{basel} \qquad \mathtt{proj}(f(\star_A), g(b)) = \mathtt{baser}$$

... with rewriting

We have

$$\begin{aligned} &\mathtt{proj}(f(a), g(\star_B)) \\ &\rightsquigarrow \mathtt{proj}(f(a), \star_{B'}) \quad \text{via} \quad \star_g \text{ in the second argument of } \mathtt{proj} \\ &\rightsquigarrow \mathtt{basel} \quad \text{via} \quad \mathtt{pushl}(f(a)) \end{aligned}$$

Therefore we can fill the first hole with

$$\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g) \bullet \mathtt{pushl}(f(a))$$

. . . with rewriting

We have

$$\begin{aligned}
&\mathtt{proj}(f(a), g(\star_B)) \\
&\rightsquigarrow \mathtt{proj}(f(a), \star_{B'}) \quad \text{via} \quad \star_g \ \text{ in the second argument of } \mathtt{proj} \\
&\rightsquigarrow \mathtt{basel} \quad \text{via} \quad \mathtt{pushl}(f(a))
\end{aligned}$$

Therefore we can fill the first hole with

$$\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g) \bullet \mathtt{pushl}(f(a))$$

Similarly for the second hole:

$$\begin{aligned}
&\mathtt{proj}(f(\star_A), g(b)) \\
&\rightsquigarrow \mathtt{proj}(\star_{A'}, g(b)) \quad \text{via} \quad \star_f \ \text{ in the first argument of } \mathtt{proj} \\
&\rightsquigarrow \mathtt{baser} \quad \text{via} \quad \mathtt{pushr}(g(b))
\end{aligned}$$

Proof-relevant rewriting

$$
\begin{array}{rcll}
f(\star_A) & \rightsquigarrow & \star_{A'} & \text{via} \quad \star_f \\
\mathtt{proj}(a, \star_B) & \rightsquigarrow & \mathtt{basel} & \text{via} \quad \mathtt{pushl}(a) \\
\mathtt{proj}(\star_A, b) & \rightsquigarrow & \mathtt{baser} & \text{via} \quad \mathtt{pushr}(b) \\
\mathtt{basel} & \rightsquigarrow & \mathtt{proj}(\star_A, \star_B) & \text{via} \quad \mathtt{pushl}(\star_A) \\
\mathtt{baser} & \rightsquigarrow & \mathtt{proj}(\star_A, \star_B) & \text{via} \quad \mathtt{pushr}(\star_B) \\
\mathtt{proj}(\star_A, \star_B) & \not\rightsquigarrow & &
\end{array}
$$

$$
\begin{array}{rcll}
f(u) & \rightsquigarrow & f(u') & \text{via} \quad \mathsf{ap}_f(p) \\
 & & & (\text{if } u \rightsquigarrow u' \text{ via } p) \\
u & \rightsquigarrow & u'' & \text{via} \quad p \bullet p' \\
 & & & (\text{if } u \rightsquigarrow u' \text{ via } p \text{ and } u' \rightsquigarrow u'' \text{ via } p')
\end{array}
$$

Squares

We use squares and cubes in the sense of [LB15][2] .

Definition

The type

$$\texttt{Square} : \{A : \texttt{Type}\}\{a, b, c, d : A\}$$
$$(p : a = b)(q : c = d)(r : a = c)(s : b = d) \to \texttt{Type}$$

is defined as the inductive family with one constructor

$$\texttt{ids} : \texttt{Square}(\texttt{idp}, \texttt{idp}, \texttt{idp}, \texttt{idp})$$

[2] D. Licata, G. Brunerie, *A Cubical Approach to Synthetic Homotopy Theory*, LICS 2015

Application of a homotopy to a path

Given a dependent function (where $g, h : A \to B$)

$$f : (x : A) \to g(x) =_B h(x)$$

and a path

$$p : a =_A a'$$

we have

$$\mathsf{ap}_f^+(p) : \mathtt{Square}(\mathsf{ap}_g(p), \mathsf{ap}_h(p), f(a), f(a'))$$

$$\begin{tikzcd}
g(a) \arrow[r, dash, "f(a)"] \arrow[d, dash, "\mathsf{ap}_g(p)"'] & h(a) \arrow[d, dash, "\mathsf{ap}_h(p)"] \\
g(a') \arrow[r, dash, "f(a')"'] & h(a')
\end{tikzcd}$$

Induction rule (into an identity type)

Given a type C and two functions $g, h : A \wedge B \to C$, in order to define a map

$$f : (x : A \wedge B) \to g(x) =_C h(x),$$

we need

$$\begin{aligned}
f(\mathtt{proj}(a,b)) &: g(\mathtt{proj}(a,b)) =_C h(\mathtt{proj}(a,b)) \\
f(\mathtt{basel}) &: g(\mathtt{basel}) =_C h(\mathtt{basel}) \\
f(\mathtt{baser}) &: g(\mathtt{baser}) =_C h(\mathtt{baser}) \\
\mathsf{ap}_f^+(\mathtt{pushl}(a)) &: \mathsf{Square}(\mathsf{ap}_g(\mathtt{pushl}(a)), \mathsf{ap}_h(\mathtt{pushl}(a)), \\
&\qquad\qquad f(\mathtt{proj}(a, \star_B)), f(\mathtt{basel})) \\
\mathsf{ap}_f^+(\mathtt{pushr}(b)) &: \mathsf{Square}(\mathsf{ap}_g(\mathtt{pushr}(b)), \mathsf{ap}_h(\mathtt{pushr}(b)), \\
&\qquad\qquad f(\mathtt{proj}(\star_A, b)), f(\mathtt{baser}))
\end{aligned}$$

Example 2: naturality of commutativity

We have two pointed maps $f : A \to A'$ and $g : B \to B'$.

$$\sigma\text{-}\mathtt{nat}_{f,g} : (x : A \wedge B) \to \sigma_{A',B'}((f \wedge g)(x)) = (g \wedge f)(\sigma_{A,B}(x))$$

$$\begin{aligned}
\sigma\text{-}\mathtt{nat}_{f,g}(\mathtt{proj}(a,b)) &:= \mathtt{idp}_{\mathtt{proj}(g(b),f(a))} \\
\sigma\text{-}\mathtt{nat}_{f,g}(\mathtt{basel}) &:= \mathtt{idp}_{\mathtt{baser}} \\
\sigma\text{-}\mathtt{nat}_{f,g}(\mathtt{baser}) &:= \mathtt{idp}_{\mathtt{basel}} \\
\mathsf{ap}^{+}_{\sigma\text{-}\mathtt{nat}_{f,g}}(\mathtt{pushl}(a)) &:= \blacksquare^2 : \mathsf{Square}(\mathsf{ap}_{\sigma_{A',B'} \circ (f \wedge g)}(\mathtt{pushl}(a)), \\
&\qquad\qquad \mathsf{ap}_{(g \wedge f) \circ \sigma_{A,B}}(\mathtt{pushl}(a)), \\
&\qquad\qquad \mathtt{idp}_{\mathtt{proj}(g(\star_B),f(a))}, \\
&\qquad\qquad \mathtt{idp}_{\mathtt{baser}}) \\
\mathsf{ap}^{+}_{\sigma\text{-}\mathtt{nat}_{f,g}}(\mathtt{pushr}(b)) &:= \blacksquare^2 : [\ldots]
\end{aligned}$$

More rewriting!

$\mathsf{ap}_{\sigma_{A',B'}\circ(f\wedge g)}(\mathtt{pushl}(a))$

More rewriting!

$$\mathsf{ap}_{\sigma_{A',B'}\circ(f\wedge g)}(\mathtt{pushl}(a))$$
$$\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{f\wedge g}(\mathtt{pushl}(a)))$$

More rewriting!

$$\begin{aligned}
&\mathsf{ap}_{\sigma_{A',B'}\circ(f\wedge g)}(\texttt{pushl}(a))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{f\wedge g}(\texttt{pushl}(a)))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{\texttt{proj}(f(a),-)}(\star_g)\bullet\texttt{pushl}(f(a)))
\end{aligned}$$

More rewriting!

$$
\begin{aligned}
&\mathsf{ap}_{\sigma_{A',B'}\circ(f\wedge g)}(\mathtt{pushl}(a))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{f\wedge g}(\mathtt{pushl}(a)))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g)\bullet\mathtt{pushl}(f(a)))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g))\bullet\mathsf{ap}_{\sigma_{A',B'}}(\mathtt{pushl}(f(a)))
\end{aligned}
$$

More rewriting!

$$
\begin{aligned}
&\mathsf{ap}_{\sigma_{A',B'}\circ(f\wedge g)}(\mathtt{pushl}(a))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{f\wedge g}(\mathtt{pushl}(a)))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g)\bullet \mathtt{pushl}(f(a)))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g))\bullet \mathsf{ap}_{\sigma_{A',B'}}(\mathtt{pushl}(f(a)))\\
&\rightsquigarrow \mathsf{ap}_{\mathtt{proj}(-,f(a))}(\star_g)\bullet \mathtt{pushr}(f(a))
\end{aligned}
$$

More rewriting!

$$
\begin{aligned}
&\mathsf{ap}_{\sigma_{A',B'}\circ(f\wedge g)}(\mathtt{pushl}(a))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{f\wedge g}(\mathtt{pushl}(a)))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g)\bullet\mathtt{pushl}(f(a)))\\
&\rightsquigarrow \mathsf{ap}_{\sigma_{A',B'}}(\mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g))\bullet\mathsf{ap}_{\sigma_{A',B'}}(\mathtt{pushl}(f(a)))\\
&\rightsquigarrow \mathsf{ap}_{\mathtt{proj}(-,f(a))}(\star_g)\bullet\mathtt{pushr}(f(a))
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{ap}_{(g\wedge f)\circ\sigma_{A,B}}(\mathtt{pushl}(a))\\
&\rightsquigarrow \mathsf{ap}_{g\wedge f}(\mathsf{ap}_{\sigma_{A,B}}(\mathtt{pushl}(a)))\\
&\rightsquigarrow \mathsf{ap}_{g\wedge f}(\mathtt{pushr}(a))\\
&\rightsquigarrow \mathsf{ap}_{\mathtt{proj}(-,f(a))}(\star_g)\bullet\mathtt{pushr}(f(a))
\end{aligned}
$$

More rewriting rules

$$
\begin{aligned}
\mathsf{ap}_{\sigma_{A,B}}(\mathtt{pushl}(a)) &\rightsquigarrow \mathtt{pushr}(a) && \text{and other } \beta\text{-reduction rules for HITs} \\
\mathsf{ap}_{\lambda x.x}(p) &\rightsquigarrow p \\
\mathsf{ap}_g(\mathsf{ap}_f(p)) &\rightsquigarrow \mathsf{ap}_{g\circ f}(p) \\
\mathsf{ap}_{g\circ f}(p) &\rightsquigarrow \mathsf{ap}_g(p') && (\text{if } \mathsf{ap}_f(p) \rightsquigarrow p') \\
\mathsf{ap}_f(u \bullet v) &\rightsquigarrow \mathsf{ap}_f(u) \bullet \mathsf{ap}_f(v) \\
u \bullet v &\rightsquigarrow u' \bullet v' && (\text{if } u \rightsquigarrow u' \text{ and } v \rightsquigarrow v', \\
& && \quad \text{via horizontal composition})
\end{aligned}
$$

Example 4: associativity

$$\alpha_{A,B,C} : (A \wedge B) \wedge C \to A \wedge (B \wedge C),$$

$$\alpha_{A,B,C}(\mathtt{proj}(x, c)) :\equiv \alpha^{\mathtt{proj}}_{A,B,C}(x, c),$$

$$\alpha_{A,B,C}(\mathtt{basel}) :\equiv \blacksquare^{0},$$

$$\alpha_{A,B,C}(\mathtt{baser}) :\equiv \blacksquare^{0},$$

$$\mathsf{ap}_{\alpha_{A,B,C}}(\mathtt{pushl}(x)) :\equiv \alpha^{\mathtt{pushl}}_{A,B,C}(x),$$

$$\mathsf{ap}_{\alpha_{A,B,C}}(\mathtt{pushr}(c)) :\equiv \blacksquare^{1}.$$

Example 4: associativity

$$
\begin{aligned}
\alpha_{A,B,C} &: (A \wedge B) \wedge C \to A \wedge (B \wedge C),\\
\alpha_{A,B,C}(\mathtt{proj}(x, c)) &:= \alpha^{\mathtt{proj}}_{A,B,C}(x, c),\\
\alpha_{A,B,C}(\mathtt{basel}) &:= \blacksquare^{0},\\
\alpha_{A,B,C}(\mathtt{baser}) &:= \blacksquare^{0},\\
\mathsf{ap}_{\alpha_{A,B,C}}(\mathtt{pushl}(x)) &:= \alpha^{\mathtt{pushl}}_{A,B,C}(x),\\
\mathsf{ap}_{\alpha_{A,B,C}}(\mathtt{pushr}(c)) &:= \blacksquare^{1}.
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{\mathtt{proj}}_{A,B,C} &: A \wedge B \to C \to A \wedge (B \wedge C),\\
\alpha^{\mathtt{proj}}_{A,B,C}(\mathtt{proj}(a, b), c) &:= \mathtt{proj}(a, \mathtt{proj}(b, c)),\\
&\quad [\blacksquare^{0} \dots \blacksquare^{0} \dots \blacksquare^{1} \dots \blacksquare^{1}]
\end{aligned}
$$

Example 4: associativity

$$\begin{aligned}
\alpha_{A,B,C} &: (A \wedge B) \wedge C \to A \wedge (B \wedge C), \\
\alpha_{A,B,C}(\mathtt{proj}(x, c)) &:= \alpha^{\mathtt{proj}}_{A,B,C}(x, c), \\
\alpha_{A,B,C}(\mathtt{basel}) &:= \blacksquare^0, \\
\alpha_{A,B,C}(\mathtt{baser}) &:= \blacksquare^0, \\
\mathsf{ap}_{\alpha_{A,B,C}}(\mathtt{pushl}(x)) &:= \alpha^{\mathtt{pushl}}_{A,B,C}(x), \\
\mathsf{ap}_{\alpha_{A,B,C}}(\mathtt{pushr}(c)) &:= \blacksquare^1.
\end{aligned}$$

$$\begin{aligned}
\alpha^{\mathtt{proj}}_{A,B,C} &: A \wedge B \to C \to A \wedge (B \wedge C), \\
\alpha^{\mathtt{proj}}_{A,B,C}(\mathtt{proj}(a, b), c) &:= \mathtt{proj}(a, \mathtt{proj}(b, c)), \\
& [\blacksquare^0 \ldots \blacksquare^0 \ldots \blacksquare^1 \ldots \blacksquare^1]
\end{aligned}$$

$$\begin{aligned}
\alpha^{\mathtt{pushl}}_{A,B,C} &: (x : A \wedge B) \to \alpha^{\mathtt{proj}}_{A,B,C}(x, \star_C) = \alpha_{A,B,C}(\mathtt{basel}), \\
& [\blacksquare^1 \ldots \blacksquare^1 \ldots \blacksquare^1 \ldots \blacksquare^2 \ldots \blacksquare^2]
\end{aligned}$$

Hexagon

$$
\begin{tikzcd}
 & (A \wedge B) \wedge C \arrow[r, "\alpha_{A,B,C}"] \arrow[dl, "\sigma_{A,B} \wedge \mathrm{id}_C"'] & A \wedge (B \wedge C) \arrow[dr, "\sigma_{A,B\wedge C}"] & \\
(B \wedge A) \wedge C \arrow[dr, "\alpha_{B,A,C}"'] & & & (B \wedge C) \wedge A \arrow[dl, "\alpha_{B,C,A}"] \\
 & B \wedge (A \wedge C) \arrow[r, "\mathrm{id}_B \wedge \sigma_{A,C}"'] & B \wedge (C \wedge A) &
\end{tikzcd}
$$

Example 5: hexagon

$$\texttt{hexagon}_{A,B,C} : (x : (A \wedge B) \wedge C) \to \texttt{Id}(\dots,\dots)$$
$$[\texttt{hexagon}^{\texttt{proj}}_{A,B,C} \dots \blacksquare^{1} \dots \blacksquare^{1} \dots \texttt{hexagon}^{\texttt{pushl}}_{A,B,C} \dots \blacksquare^{2}]$$

Example 5: hexagon

$$\texttt{hexagon}_{A,B,C} : (x : (A \wedge B) \wedge C) \to \texttt{Id}(\ldots, \ldots)$$
$$[\texttt{hexagon}^{\texttt{proj}}_{A,B,C} \ldots {\color{blue}\blacksquare}^{\color{blue}1} \ldots {\color{blue}\blacksquare}^{\color{blue}1} \ldots \texttt{hexagon}^{\texttt{pushl}}_{A,B,C} \ldots {\color{red}\blacksquare}^{\color{red}2}]$$

$$\texttt{hexagon}^{\texttt{proj}}_{A,B,C} : (x : A \wedge B) \to C \to \texttt{Id}(\ldots, \ldots)$$
$$[\texttt{idp} \ldots {\color{blue}\blacksquare}^{\color{blue}1} \ldots {\color{blue}\blacksquare}^{\color{blue}1} \ldots {\color{red}\blacksquare}^{\color{red}2} \ldots {\color{red}\blacksquare}^{\color{red}2}]$$

Example 5: hexagon

$$\texttt{hexagon}_{A,B,C} : (x : (A \wedge B) \wedge C) \to \texttt{Id}(\ldots, \ldots)$$
$$[\texttt{hexagon}^{\texttt{proj}}_{A,B,C} \ldots {\color{blue}\blacksquare^{1}} \ldots {\color{blue}\blacksquare^{1}} \ldots \texttt{hexagon}^{\texttt{pushl}}_{A,B,C} \ldots {\color{red}\blacksquare^{2}}]$$

$$\texttt{hexagon}^{\texttt{proj}}_{A,B,C} : (x : A \wedge B) \to C \to \texttt{Id}(\ldots, \ldots)$$
$$[\texttt{idp} \ldots {\color{blue}\blacksquare^{1}} \ldots {\color{blue}\blacksquare^{1}} \ldots {\color{red}\blacksquare^{2}} \ldots {\color{red}\blacksquare^{2}}]$$

$$\texttt{hexagon}^{\texttt{pushl}}_{A,B,C} : (x : A \wedge B) \to C \to \texttt{Square}(\ldots, \ldots, \ldots, \ldots)$$
$$[{\color{red}\blacksquare^{2}} \ldots {\color{red}\blacksquare^{2}} \ldots {\color{red}\blacksquare^{2}} \ldots \blacksquare^{3} \ldots \blacksquare^{3}]$$

Example 6: pentagon

$$\texttt{pent} : (x : ((A \wedge B) \wedge C) \wedge D) \to \texttt{Id}(\ldots, \ldots)$$

Example 6: pentagon

$$
\begin{aligned}
\texttt{pent} &: (x : ((A \wedge B) \wedge C) \wedge D) \to \texttt{Id}(\dots, \dots) \\
\texttt{pent} &: [\texttt{pent}^{\texttt{proj}} \dots \blacksquare^{1} \dots \blacksquare^{1} \dots \texttt{pent}^{\texttt{pushl}} \dots \blacksquare^{2}] \\
\texttt{pent}^{\texttt{proj}} &: [\texttt{pent}^{\texttt{proj},\texttt{proj}} \dots \blacksquare^{1} \dots \blacksquare^{1} \dots \texttt{pent}^{\texttt{proj},\texttt{pushl}} \dots \blacksquare^{2}] \\
\texttt{pent}^{\texttt{proj},\texttt{proj}} &: [\texttt{idp} \dots \blacksquare^{1} \dots \blacksquare^{1} \dots \blacksquare^{2} \dots \blacksquare^{2}] \\
\texttt{pent}^{\texttt{proj},\texttt{pushl}} &: [\blacksquare^{2} \dots \blacksquare^{2} \dots \blacksquare^{2} \dots \blacksquare^{3} \dots \blacksquare^{3}] \\
\texttt{pent}^{\texttt{pushl}} &: [\texttt{pent}^{\texttt{pushl},\texttt{proj}} \dots \blacksquare^{2} \dots \blacksquare^{2} \dots \texttt{pent}^{\texttt{pushl},\texttt{pushl}} \dots \blacksquare^{3}] \\
\texttt{pent}^{\texttt{pushl},\texttt{proj}} &: [\blacksquare^{2} \dots \blacksquare^{2} \dots \blacksquare^{2} \dots \blacksquare^{3} \dots \blacksquare^{3}] \\
\texttt{pent}^{\texttt{pushl},\texttt{pushl}} &: [\blacksquare^{3} \dots \blacksquare^{3} \dots \blacksquare^{3} \dots \blacksquare^{4} \dots \blacksquare^{4}]
\end{aligned}
$$

Cubical proof-relevant rewriting

Definition

Given $f : A \to B$ and $sq : \mathtt{Square}_A(p, q, r, s)$, we have

$$\mathsf{ap}_f^2(sq) : \mathtt{Square}_B(\mathsf{ap}_f(p), \mathsf{ap}_f(q), \mathsf{ap}_f(r), \mathsf{ap}_f(s)).$$

Cubical proof-relevant rewriting

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Given $f : A \to B$ and $sq : \mathtt{Square}_A(p, q, r, s)$, we have

$$\mathsf{ap}^2_f(sq) : \mathtt{Square}_B(\mathsf{ap}_f(p), \mathsf{ap}_f(q), \mathsf{ap}_f(r), \mathsf{ap}_f(s)).$$

We want a reduction rule

$$\mathsf{ap}^2_{\lambda x.x}(sq) \rightsquigarrow sq$$

Cubical proof-relevant rewriting

Definition

Given $f : A \to B$ and $sq : \mathtt{Square}_A(p, q, r, s)$, we have

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Cubical proof-relevant rewriting:
If s and s' are two squares, we say

$$s \rightsquigarrow s' \quad \text{via} \quad c$$

if c is a cube with s and s' as two of its opposite faces.

More variants of ap

	$p : \mathtt{Id}_A$	$p : \mathtt{Square}_A$	$p : \mathtt{Cube}_A$
$f : A \to B$	$\mathsf{ap}_f(p)$	$\mathsf{ap}^2_f(p)$	
$f : A \to \mathtt{Id}_B$	$\mathsf{ap}^+_f(p)$		
$f : A \to \mathtt{Square}_B$			
$f : A \to \mathtt{Cube}_B$			

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$f : A \to \mathtt{Square}_B$	$\mathsf{ap}^{++}_f(p)$	$\mathsf{ap}^{2,++}_f(p)$	
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$f : A \to \mathtt{Cube}_B$	$\mathsf{ap}^{+++}_f(p)$		

They interact in various ways, for instance

$$
\begin{aligned}
\mathsf{ap}^2_g(\mathsf{ap}^+_f(p)) &\rightsquigarrow \mathsf{ap}^+_{\mathsf{ap}_g \circ f}(p) \\
\mathsf{ap}^+_g(\mathsf{ap}_f(p)) &\rightsquigarrow \mathsf{ap}^+_{g \circ f}(p) \\
\mathsf{ap}^+_f(p \bullet q) &\rightsquigarrow \mathsf{ap}^+_f(p) \diamond \mathsf{ap}^+_f(q) \\
\mathsf{ap}^+_{\lambda x.p(x) \bullet q(x)}(r) &\rightsquigarrow \mathsf{ap}^+_{\lambda x.p(x)}(r) \blacklozenge \mathsf{ap}^+_{\lambda x.q(x)}(r)
\end{aligned}
$$

Globular coherences

We can construct any map of the form:

$$\begin{aligned}\mathtt{coh} : (X : \mathtt{Type})&(a : X)\\ &[\ldots]\\ &(x_n : T_n)(p_n : x_n = u_n)\\ &[\ldots]\\ &\to T\end{aligned}$$

where T_n, u_n and T are built only from previous variables and other coherences, and T is an identity type.

Idea: path induction on all of the p_n, then give `idp`.

Use: p_n represents a rewriting rule, and x_n the term being rewritten.

Cubical coherences

We also need to allow pairs of arguments of the form

$$(x_n : T_n)(p_n : \mathtt{Square}(x_n, u_n, v_n, w_n))$$

$$(x_n : T_n)(p_n : \mathtt{Cube}(x_n, u_n, v_n, w_n, r_n, s_n))$$

We can still construct all such coherences, using a generalized version of J where three sides of a square are fixed and one side is free.

Algorithm for building the proof

In order to fill a hole ($\blacksquare^1$, $\blacksquare^2$, $\blacksquare^3$ or $\blacksquare^4$) we proceed as follows.
The variables are ℓ_1 a list of terms and ℓ_2 a list of pairs of terms.

- We start with ℓ_1 consisting of all the faces (in every dimension) of the hole, and ℓ_2 is empty.
- Take the first element t of ℓ_1.
- If it is the base point, or is already present in ℓ_2, discard it.
- Otherwise, reduce it (it gives an n-cube s which has t as one of its faces), add (t, s) to ℓ_2 and all the other faces of s to ℓ_1.
- Repeat until ℓ_1 is empty.
- Build a cubical coherence out of ℓ_2.
- Use that coherence to fill the hole.

Example

We want to prove $\mathtt{proj}(f(a), g(\star_B)) = \mathtt{basel}$.

$\ell_1 = [\mathtt{proj}(f(a), g(\star_B)), \mathtt{basel}]$
$\ell_2 = []$

$$\mathtt{proj}(f(a), g(\star_B)) \rightsquigarrow \mathtt{proj}(f(a), \star_{B'}) \text{ via } \mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g)$$

$\ell_1 = [\mathtt{proj}(f(a), \star_{B'}), \mathtt{basel}]$
$\ell_2 = [(\mathtt{proj}(f(a), g(\star_B)), \mathsf{ap}_{\mathtt{proj}(f(a),-)}(\star_g))]$

Example

$$\ell_1 = [\texttt{proj}(f(a), \star_{B'}), \texttt{basel}]$$
$$\ell_2 = [(\texttt{proj}(f(a), g(\star_B)), \mathsf{ap}_{\texttt{proj}(f(a),-)}(\star_g))]$$

$$\texttt{proj}(f(a), \star_{B'}) \rightsquigarrow \texttt{basel} \text{ via } \texttt{pushl}(f(a))$$

$$\ell_1 = [\texttt{basel}, \texttt{basel}]$$
$$\ell_2 = [(\texttt{proj}(f(a), g(\star_B)), \mathsf{ap}_{\texttt{proj}(f(a),-)}(\star_g)), (\texttt{proj}(f(a), \star_{B'}), \texttt{pushl}(f(a)))]$$

Example

$$\ell_1 = [\texttt{basel}, \texttt{basel}]$$
$$\ell_2 = [(\texttt{proj}(f(a), g(\star_B)), \mathsf{ap}_{\texttt{proj}(f(a),-)}(\star_g)),$$
$$(\texttt{proj}(f(a), \star_{B'}), \texttt{pushl}(f(a)))]$$

$$\texttt{basel} \rightsquigarrow \texttt{proj}(\star_{A'}, \star_{B'}) \text{ via } \texttt{pushl}(\star_{A'})$$

$$\ell_1 = [\texttt{proj}(\star_{A'}, \star_{B'}), \texttt{basel}]$$
$$\ell_2 = [(\texttt{proj}(f(a), g(\star_B)), \mathsf{ap}_{\texttt{proj}(f(a),-)}(\star_g)),$$
$$(\texttt{proj}(f(a), \star_{B'}), \texttt{pushl}(f(a)))]$$
$$(\texttt{basel}, \texttt{pushl}(\star_{A'}))]$$

We're done, as everything in ℓ_1 is either in ℓ_2 or the base point.

Example

$$\ell_2 = [(\texttt{proj}(f(a), g(\star_B)), \mathsf{ap}_{\texttt{proj}(f(a),-)}(\star_g)),$$
$$(\texttt{proj}(f(a), \star_{B'}), \texttt{pushl}(f(a)))]$$
$$(\texttt{basel}, \texttt{pushl}(\star_{A'}))]$$

$$\begin{array}{llll}
\texttt{coh} : & (X : \texttt{Type}) & & (A' \wedge B') \\
& (a : X) & & (\texttt{proj}(\star_{A'}, \star_{B'})) \\
& (x_0 : X) & & (\texttt{basel}) \\
& (p_0 : a = x_0) & & (\texttt{pushl}(\star_{A'})) \\
& (x_1 : X) & & (\texttt{proj}(f(a), \star_{B'})) \\
& (p_1 : x_1 = x_0) & & (\texttt{pushl}(f(a))) \\
& (x_2 : X) & & (\texttt{proj}(f(a), g(\star_B))) \\
& (p_2 : x_2 = x_1) & & (\mathsf{ap}_{\texttt{proj}(f(a),-)}(\star_g)) \\
& \quad \to x_2 = x_0 & &
\end{array}$$

The result is the desired term of type $\texttt{proj}(f(a), g(\star_B)) = \texttt{basel}$.

Metaprogramming

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The generated proof is written in the proof assistant Agda, and the generating program is also written in Agda, used as a programming language.

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Workflow

```
$ agda --compile SmashGenerate.agda
                            # generate the executable
$ ./SmashGenerate > Result.agda  # generate the proof
$ agda Result.agda               # check the proof
```

Results

The current version can prove almost everything except for the pentagon. In particular it can construct/prove

- $f \wedge g$, compatibility with identities
- σ, involutivity, naturality
- α, α^{-1}, inverses to each other, naturality (takes 10 minutes and 25 GB of memory)
- the hexagon (takes 7 minutes and 8 GB of memory)

Future directions

- Finish the pentagon and the few other things missing.
- Get a full meta-theoretic proof that it does work.
- Prove that the smash product is ∞-coherent (externally).
- Can this idea of higher dimensional rewriting be applied in other situations?

In topology

In topology, $A \wedge B$ is defined as a quotient.

- We identify points with each other, instead of adding paths between them.
- It is easy to define, e.g., $\alpha_{A,B,C} : (A \wedge B) \wedge C \to A \wedge (B \wedge C)$.
- The pentagon is trivial.
- It is *not* easy to prove that $\alpha_{A,B,C}$ is continuous!

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- There are some propositional equalities that we would like to pretend are reduction rules.
- Cubical type theory does turn many of them into reduction rules, but we can't really hope for, e.g., $f(\star_A) \rightsquigarrow \star_{A'}$ or $\mathtt{proj}(a, \star_B) \rightsquigarrow \mathtt{basel}$ to ever be an actual reduction rule.
- Can we find an automated way to handle such propositional reduction rules?
- To a user of the proof assistant, it would look like things reduce, in reality the proof assistant is doing all the work behind the scenes.